Chapter 4: Function of Several Variables

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Function of several variables

Definition 1.1 (function of two variables)

A **function** f **of two variables** *is a rule that assigns to each ordered pair of real numbers* (x, y) *in a set* D *a unique real number denoted by* $f(x, y)$*. The set* D *is the* **domain** *of* f *and its* **range** *is the set of values that* f *takes on, that is,* $\{f(x, y)|(x, y) \in D\}$*.*

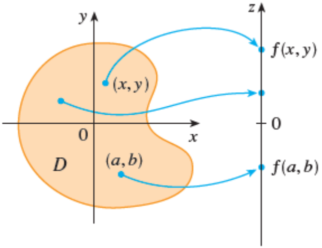


Figure 1: Domain of f

Example 1.1

For each of the following functions, evaluate $f(3,2)$ and find and sketch the domain.

1. $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$
2. $f(x,y) = x\ln(y^2 - x)$

Solution

1. $f(3,2) = \dfrac{\sqrt{3+2+1}}{3-1} = \dfrac{\sqrt{6}}{2}$
   The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is
   $$D = \{(x,y) | x+y+1 \geq 0, x \neq 1\}$$
   The inequality $x+y+1 \geq 0$, or $y \geq -x-1$, describes the points that lie on or above the line $y = -x-1$, while $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain.

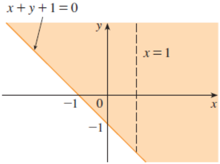


Figure 2: Domain

2. $f(3,2) = 3\ln(1) = 0$
   Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$, the domain of f is $D = \{(x,y) | x < y^2\}$. This is the set of points to the left of the parabola $x = y^2$.

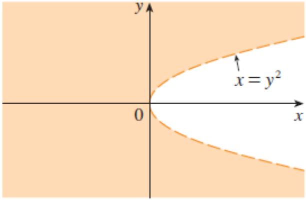


Figure 3: Domain

Example 1.2

Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution

The domain of g is

$$D = \{(x, y)|9 - x^2 - y^2 \geq 0\} = \{(x, y)|x^2 + y^2 \leq 9\}$$

which is the disk with center $(0, 0)$ and radius 3. (See Figure 4.)

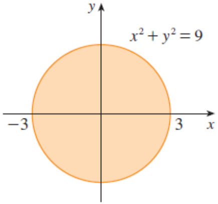


Figure 4: Domain of $g(x, y)$

The range of g is

$$\{z | z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\}$$

Since z is a positive square root, $z \geq 0$. Also, because $9 - x^2 - y^2 \leq 9$, we have

$$\sqrt{9 - x^2 - y^2} \leq 3$$

So the range is

$$\{z | -3 \leq z \leq 3\} = [-3, 3]$$

Definition 1.2 (Graph)

*If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.*

Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$.
We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane (see Figure 5).

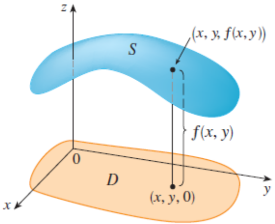


Figure 5:

Example 1.3

Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

Solution

The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts. Putting $y = z = 0$ in the equation, we get $x = 2$ as the x-intercept. Similarly, the y-intercept is $y = 3$ and the z-intercept is $z = 6$. This helps us sketch the portion of the graph that lies in the first octant in Figure 6.

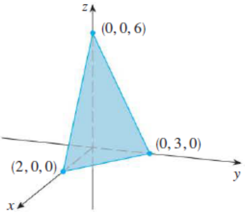


Figure 6:

Example 1.4

Sketch the graph of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution

The graph has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geq 0$, the graph of g is just the top half of this sphere (see Figure 7).

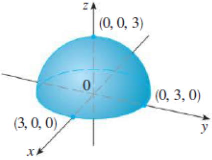


Figure 7: Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Example 1.5

Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

Solution

Notice that $h(x, y)$ is defined for all possible ordered pairs of real numbers (x, y), so the domain is $\mathbb{R}^2$, the entire xy-plane. The range of h is the set $[0, \infty)$ of all non-negative real numbers. The graph of h has the equation $z = 4x^2 + y^2$ which is the elliptic paraboloid. Horizontal traces are ellipses and vertical traces are parabolas (see Figure 8).

Function of several variables

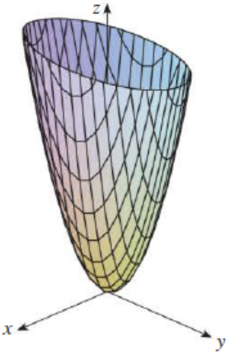


Figure 8: Graph of $h(x, y) = 4x^2 + y^2$

Definition 1.3 (Level curves)

The **level curves** *of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).*

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it shows where the graph of f has height k. You can see from Figure 9 the relation between level curves and horizontal traces. The level curves $f(x, y) = k$ are just the traces of the graph of f $z = k$ projected down to the xy-plane.

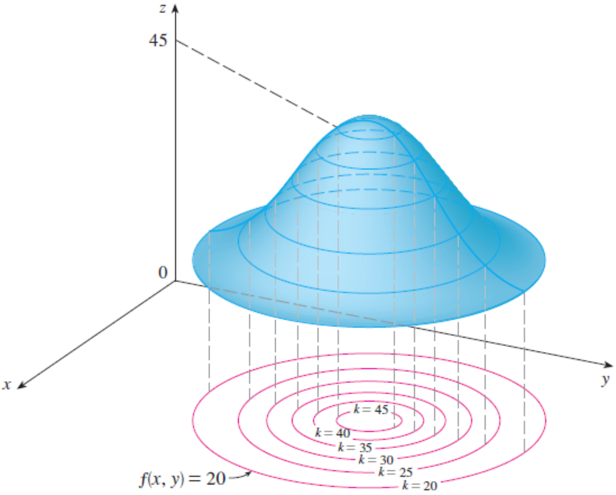


Figure 9: level

Example 1.6

A contour map for a function f is shown in Figure 10. Use it to estimate the values of $f(1,3)$ and $f(4,5)$.

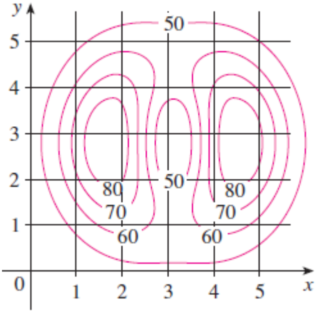


Figure 10: level

Solution

The point $(1,3)$ lies partway between the level curves with z-values 70 and 80. We estimate that $f(1,3) \approx 73$
Similarly, we estimate that $f(4,5) \approx 56$.

Example 1.7

Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

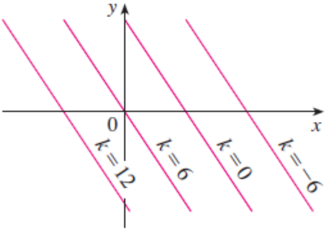


Figure 11: Contour map of $f(x, y) = 6 - 3x - 2y$

Solution

The level curves are

$$6 - 3x - 2y = k \text{ or } 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope $-\dfrac{3}{2}$. The four particular level curves with $k = -6, 0, 6$ and 12 are $3x + 2y - 12 = 0$, $3x + 2y - 6 = 0$, $3x + 2y = 0$, and $3x + 2y + 6 = 0$. They are sketched in Figure 11. The level curves are equally spaced parallel lines because the graph of f is a plane (see Figure 6).

Example 1.8

Sketch the level curves of the function

$$g(x,y)=\sqrt{9-x^2-y^2} \text{ for } k=0,1,2,3$$

Solution

The level curves are

$$\sqrt{9-x^2-y^2}=k \text{ or } 9-x^2-y^2=k^2$$

This is a family of concentric circles with center $(0,0)$ and radius $\sqrt{9-k^2}$. The cases $k=0,1,2,3$ are shown in Figure 12. Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure 7.

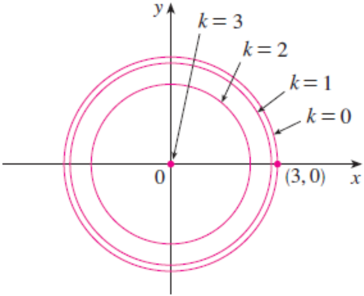


Figure 12: Contour map of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Example 1.9

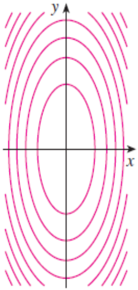
Sketch some level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

Solution

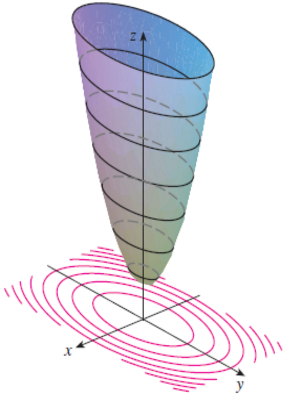
The level curves are

$$4x^2 + y^2 + 1 = k \text{ or } \frac{x^2}{\frac{1}{4}(k-1)} + \frac{y^2}{(k-1)} = 1$$

which, for $k > 1$, describes a family of ellipses with semiaxes $\sqrt{k-1}$ and $\frac{1}{2}\sqrt{k-1}$. Figure 13(a) shows a contour map of h drawn by a computer. Figure 13(b) shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces. We see from Figure 13 how the graph of h is put together from the level curves.



(a) Contour map



(b) Horizontal traces are raised level curves

Figure 13: The graph of $h(x, y) = 4x^2 + y^2 + 1$

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Definition 1.4 (function of three variables)

A function of three variables, f *, is a rule that assigns to each ordered triple* (x, y, z) *in a domain* $D \subset \mathbb{R}^3$ *a unique real number denoted by* $f(x, y, z)$*. For instance, the temperature* T *at a point on the surface of the earth depends on the longitude* x *and latitude* y *of the point and on the time* t*, so we could write* $T = f(x, y, t)$*.*

Example 1.10

Find the domain of f if

$$f(x,y,z) = \ln(z-y) + xy\sin(z)$$

Solution

The expression for $f(x,y,z)$ is defined as long as $z-y>0$, so the domain of f is

$$D = \{(x,y,z) \in \mathbb{R}^3 | z > y\}$$

This is a half-space consisting of all points that lie above the plane $z=y$

Example 1.11

Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

Solution

The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$. These form a family of concentric spheres with radius k . (See Figure 14.) Thus, as (x, y, z) varies over any sphere with center O, the value of $f(x, y, z)$ remains fixed.

Function of several variables

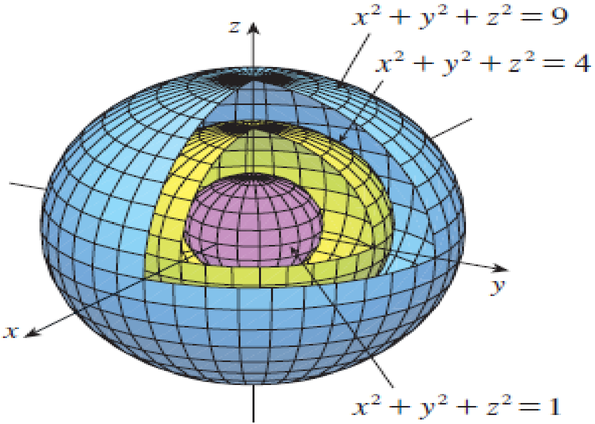

Figure 14: The graphs of $x^2 + y^2 + z^2 = k$

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Definition 2.1 (Partial derivative at (a, b) with respect to x)

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the partial derivative of f with respect to x at (a, b) and denote it by $f_x(a, b)$. Thus

$$f_x(a, b) = g'(a) \text{ where } g(x) = f(x, b)$$

Definition 2.2 (Partial derivative at (a, b) with respect to y)

In general, if f is a function of two variables x and y, suppose we let only y vary while keeping x fixed, say $x = a$, where a is a constant. Then we are really considering a function of a single variable y, namely, $u(y) = f(a, y)$. If h has a derivative at b, then we call it the partial derivative of f with respect to y at (a, b) and denote it by $f_y(a, b)$. Thus

$$f_y(a, b) = u'(b) \text{ where } u(y) = f(a, y)$$

Results

1. By the definition of a derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

and so

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

2. By the definition of a derivative, we have

$$u'(b) = \lim_{h \to 0} \frac{u(b+h) - u(b)}{h}$$

and so

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Definition 2.3

If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notations for Partial Derivatives

If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(x,y) = \frac{\partial z}{\partial x}$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(x,y) = \frac{\partial z}{\partial y}$$

Rule for Finding Partial Derivatives of $z = f(x,y)$

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Example 2.1

If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

Solution

Holding y constant and differentiating with respect to x, we get

$$f_x(x,y) = 3x^2 + 2xy^3$$

and so

$$f_x(2,1) = 3 \times 4 + 2 \times 2 \times 1 = 12 + 4 = 16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x,y) = 3x^2y^2 - 4y$$

and so

$$f_y(2,1) = 3 \times 4 \times 1 - 4 \times 1 = 12 - 4 = 8$$

Example 2.2

If $f(x,y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and interpret these numbers as slopes.

Solution

We have

$$f_x(x,y) = -2x \qquad f_y(x,y) = -4y$$

$$f_x(1,1) = -2 \qquad f_y(1,1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y - 1$. (As in the preceding discussion, we label it C_1 in Figure 15). The slope of the tangent line to this parabola at the point $(1,1,1)$ is $f_x(1,1) = -2$. Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z - 3 - 2y^2$, $x - 1$, and the slope of the tangent line at $(1,1,1)$ is $f_y(1,1) = -4$. (See Figure 15.)

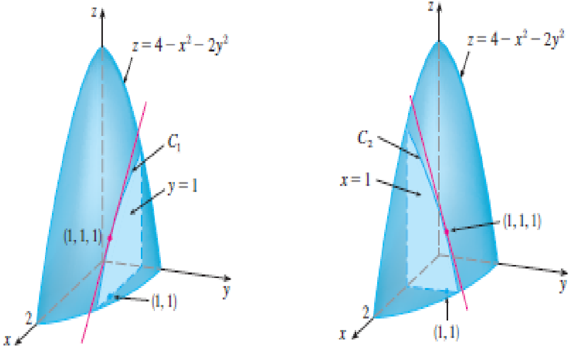

Figure 15: Curves for $z = 4 - x^2 - 2y^2$

Example 2.3

If $f(x,y)=\sin\left(\frac{x}{y+1}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Solution

Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x}=\cos\left(\frac{x}{y+1}\right)\frac{\partial}{\partial x}\left(\frac{x}{y+1}\right)=\cos\left(\frac{x}{y+1}\right)\frac{1}{y+1}$$

$$\frac{\partial f}{\partial y}=\cos\left(\frac{x}{y+1}\right)\frac{\partial}{\partial y}\left(\frac{x}{y+1}\right)=-\cos\left(\frac{x}{y+1}\right)\frac{1}{(y+1)^2}$$

Example 2.4

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Solution

To find $\dfrac{\partial z}{\partial x}$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Solving this equation for $\dfrac{\partial z}{\partial x}$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial x} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Partial Derivatives

Definition 2.4 (Higher derivatives)

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the second partial derivatives of f. If $z = f(x, y)$, we use the following notation:

1. $(f_x)_x = f_{xx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 z}{\partial x^2}$
2. $(f_x)_y = f_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial y \partial x}$
3. $(f_y)_x = f_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial x \partial y}$
4. $(f_y)_y = f_{yy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial^2 z}{\partial y^2}$

Example 2.5

Find the second partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

Solution

In Example 2.1 we found that
$f_x(x,y) = 3x^2 + 2xy^3$ and $f_y(x,y) = 3x^2y^2 - 4y$.
Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^2 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

Theorem 2.1 (Clairaut's Theorem)

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Definition 2.5 (Partial Differential Equations)

Partial derivatives occur in partial differential equations that express certain physical laws. For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation** *after Pierre Laplace (1749–1827). Solutions of this equation are called* **harmonic functions***; they play a role in problems of heat conduction, fluid flow, and electric potential.*

Example 2.6

Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

Solution

We first compute the needed second-order partial derivatives:

$$u_x = e^x \sin y \qquad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \qquad u_{yy} = -e^x \sin y$$

So

$$u_{xx} + uyy = e^x \sin y - e^x \sin y = 0$$

Therefore u satisfies Laplace's equation.

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Definition 3.1

A function of two variables has a local maximum at (a, b) *if* $f(x, y) \leqslant f(a, b)$ *when* (x, y) *is near* (a, b). *[This means that* $f(x, y) \leqslant f(a, b)$ *for all points* (x, y) *in some disk with center* (a, b).*] The number* $f(a, b)$ *is called a local maximum value. If* $f(x, y) \geqslant f(a, b)$ *when* (x, y) *is near* (a, b), *then* f *has a local minimum at* (a, b) *and* $f(a, b)$ *is a local minimum value.*

Theorem 3.1

If f has a local maximum or minimum at (a, b) *and the first-order partial derivatives of* f *exist there, then* $f_x(a, b) = 0$ *and* $f_y(a, b) = 0$.

Example 3.1

Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x,y) = 2x - 2 \qquad f_y(x,y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1,3)$. By completing the square, we find that

$$f(x,y) = 4 + (x-1)^2 + (y-3)^2$$

Since $(x-1)^2 \geqslant 0$ and $(y-3)^2 \geqslant 0$, we have $f(x,y) \geqslant 4$ for all values of x and y. Therefore $f(1,3) = 4$ is a local minimum, and in fact it is the absolute minimum of f. This can be confirmed geometrically from the graph of f, which is the elliptic paraboloid with vertex $(1,3,4)$ shown in Figure 16.

Maximum and Minimum Values

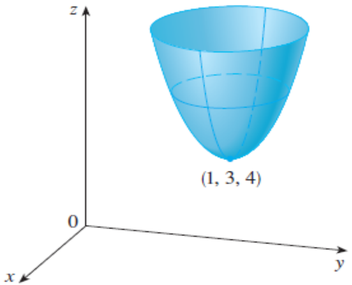

Figure 16: $z = x^2 + y^2 - 2x - 6y + 14$

Example 3.2

Find the extreme values of $f(x,y) = y^2 - x^2$.

Solution

Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0,0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x,y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y-axis we have $x = 0$, so $f(x,y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0,0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0,0) = 0$ can't be an extreme value for f , so f has no extreme value.

Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
3. If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Remark 3.1

Note 1 In case (3) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

Note 2 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

Note 3 To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

Example 3.3

-Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

Solution

We first locate the critical points:

$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \quad \text{and} \quad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives
$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$
so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0, 0), (1, 1)$, and $(-1, -1)$.

Next we calculate the second partial derivatives and $D(x,y)$:

$$f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 12y^2$$

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0,0) = -16 < 0$, it follows from case (3) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0,0)$.
Since $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$, we see from case (1) of the test that $f(1,1) = -1$ is a local minimum. Similarly, we have $D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$, so $f(-1,-1) = -1$ is also a local minimum. The graph of f is shown in Figure 17.

Maximum and Minimum Values

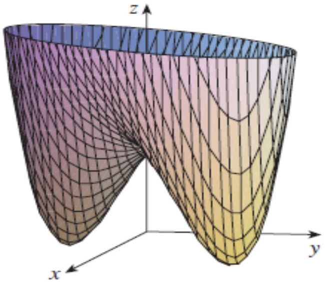

Figure 17: $z = x^4 + y^4 - 4xy + 1$

Example 3.4

Find and classify the critical points of the function

$$f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also find the highest point on the graph of f.

Solution

The first-order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \qquad f_y = 10x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

$$2x(10y - 5 - 2x^2) = 0 \tag{1}$$

$$5x^2 - 4y - 4y^3 = 0 \tag{2}$$

From Equation 1 we see that either

$$x = 0 \quad \text{or} \quad 10y - 5 - 2x^2 = 0$$

In the first case $(x = 0)$, Equation 2 becomes $-4y(1 + y^2) = 0$, so $y = 0$ and we have the critical point $(0, 0)$.
In the second case $(10y - 5 - 2x^2)$, we get

$$x^2 = 5y - 2.5 \tag{3}$$

and, putting this in Equation 2, we have $25y - 12.5 - 4y - 4y^3 = 0$. So we have to solve the cubic equation

$$4y^3 - 21y + 12.5 = 0 \tag{4}$$

Using a graphing calculator or computer to graph the function

$$t(y) = 4y^3 - 21y + 12.5$$

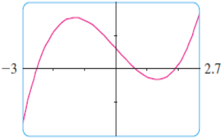

Figure 18: Cure of $t(y) = 4y^3 - 21y + 12.5$

as in Figure 18, we see that Equation 4 has three real roots. By zooming in, we can find the roots to four decimal places:

$$y \approx -2.5452 \quad y \approx 0.6468 \quad y \approx 1.8984$$

Critical point	Value of f	f_{xx}	D	Conclusion
$(0, 0)$	0.00	−10.00	80.00	local maximum
$(\pm 2.64, 1.90)$	8.50	−55.93	2488.72	local maximum
$(\pm 0.86, 0.65)$	−1.48	−5.87	−187.64	saddle point

Absolute Maximum and Minimum Values

For a functions of two variables, the Extreme Value Theorem says that if f is continuous on a closed set in $\mathbb{R}^2$, then f has an absolute minimum value and an absolute maximum value.
A **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.
A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk. In other words, it is finite in extent. Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

Theorem 3.2 (Extreme Value Theorem for Functions of Two Variables)

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Maximum and Minimum Values

Theorem 3.3 (Method to find Extreme Value)

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. *Find the values of f at the critical points of f in D.*
2. *Find the extreme values of f on the boundary of D.*
3. *The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.*

Example 3.5

Find the absolute maximum and minimum values of the function $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y) | 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.

Solution

Since f is a polynomial, it is continuous on the closed, bounded rectangle D, f have both an absolute maximum and an absolute minimum. According to step 1 in Theorem (3.3), we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

so the only critical point is $(1,1)$, and the value of f there is $f(1,1) = 1$.

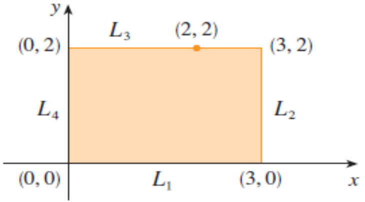

Figure 19: Rectangle $D = \{(x,y)|0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$

Maximum and Minimum Values

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 19. On L_1 we have $y = 0$ and

$$f(x,0) = x^2 \qquad 0 \leqslant x \leqslant 3$$

This is an increasing function of x, so its minimum value is $f(0,0) = 0$ and its maximum value is $f(3,0) = 9$. On L_2 we have $x = 3$ and

$$f(3,y) = 9 - 4y \qquad 0 \leqslant y \leqslant 2$$

This is a decreasing function of y, so its maximum value is $f(3,0) = 9$ and its minimum value is $f(3,2) = 1$. On L_3 we have $y = 2$ and

$$f(x,2) = x^2 - 4x + 4 \qquad 0 \leqslant x \leqslant 3$$

By observing that $f(x,2) = (x-2)^2$, we see that the minimum value of this function is $f(2,2) = 0$ and the maximum value is $f(0,2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0,y) = 2y \qquad 0 \leqslant y \leqslant 2$$

with maximum value $f(0,2) = 4$ and minimum value $f(0,0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

Maximum and Minimum Values

In step 3 we compare these values with the value $f(1,1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3,0) = 9$ and the absolute minimum value is $f(0,0) = f(2,2) = 0$. Figure 20 shows the graph of f .

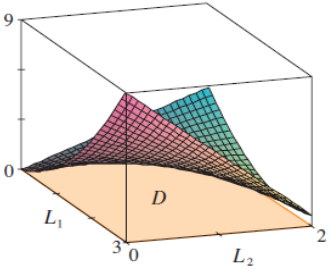

Figure 20: $f(x,y) = x^2 - 2xy + 2y$

Exercise 3.1

Find the local minimum and maximum values and the saddle points of the functions.

1. $f(x,y) = x^2 + xy + y^2 + y.$
2. $f(x,y) = xy - 2x - 2y2 - x^2 - y^2.$
3. $f(x,y) = y(e^x - 1).$
4. $f(x,y) = 2 - x^4 + 2x^2 - y^2.$

Exercise 3.2

Find the absolute maximum and minimum values of f on the set D.

1. $f(x, y) = x^2 + y^2 - 2x$, D is the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$.
2. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1$, $D = \{(x, y) | 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 3\}$