chapter 3 Vector space

Introduction

$M_2(\mathbb{R})$

(+) sum of matrices
1. closed $A+B \in M_2(\mathbb{R})$
2. identity (0)
3. additive inverse $-A \in M_2(\mathbb{R})$
4. associative $(A+B)+C = A+(B+C)$
5. Commutative $A+B = B+A$

(λA) scalar product
1. $\lambda \cdot A \in M_2(\mathbb{R})$
2. $(\lambda_1+\lambda_2)A = \lambda_1 A + \lambda_2 A$
3. $(\lambda_1 \cdot \lambda_2)A = \lambda_1(\lambda_2 A)$
4. $\lambda(A+B) = \lambda A + \lambda B$
5. $1 \cdot A = A$

Notice that we can define two operations over M_2 which are (1) sum of matrices
(2) Scalar product over $\mathbb{R}$.

as, all 10 conditions are satisfied, $M_2(\mathbb{R})$ is called Vector space or Linear space.

Definition Let V be a set of vectors. The system $V(\mathbb{R})$ with two operations: ① sum of vectors, ② scalar product; is called vector space (Linear space) if satisfied the following conditions

(1) if $v_1, v_2 \in V$ then $v_1+v_2 \in V$

(2) $0 \in V$

(3) If $v \in V$ then $-v \in V$ where $v+(-v) = 0$

(4) $v_1+(v_2+v_3) = (v_1+v_2)+v_3$

(5) $v_1+v_2 = v_2+v_1$

(6) $\lambda \cdot v \in V$

(7) $(\lambda_1+\lambda_2)v = \lambda_1 v + \lambda_2 v$

(8) $(\lambda_1 \cdot \lambda_2)v = \lambda_1(\lambda_2 v)$

(9) $\lambda(v_1+v_2) = \lambda v_1 + \lambda_2 v$

(10) $1 \cdot v = v$

(Ex) Does $\mathbb{Z}(\mathbb{R})$ be a vector space ?

Notice that $\underbrace{1/3}_{\in \mathbb{R}} \underbrace{(2)}_{\in V = \mathbb{Z}} = 2/3 \notin \mathbb{Z} = V$

Hence $\mathbb{Z}(\mathbb{R})$ is Linear space

(Ex) Does $\mathbb{R} \times \mathbb{Z}(\mathbb{R})$ be a vector space ?

Notice that $\mathbb{R} \times \mathbb{Z} = \{(a,b) : a \in \mathbb{R} \wedge b \in \mathbb{Z}\}$

Now $\underbrace{\frac{1}{3}}_{\in \mathbb{R}} \underbrace{(2,5)}_{\in \mathbb{R} \times \mathbb{Z} = V} = (2/3, 5/3) \notin \mathbb{R} \times \mathbb{Z} = V$

Hence, $\mathbb{R} \times \mathbb{Z}$ is not vector space over $\mathbb{R}$.

(Ex) $\mathbb{R}(\mathbb{R})$ is Linear space.

(Ex) $\mathbb{R}^2(\mathbb{R})$ is Linear space where $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a,b) : a \wedge b \in \mathbb{R}\}$
and $(a,b) + (c,d) = (a+c, b+d)$
$\lambda(a,b) = (\lambda a, \lambda b)$

Remark :

1. In general $\mathbb{R}^n(\mathbb{R})$ is Linear space where
$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(a_1, a_2, \cdots, a_n) : a_1, a_2, -, a_n \in \mathbb{R}\}$
   i. $(a_1, \cdots + a_n) + (b_1, \cdots, b_n) = (a_1 + b_1, \cdots, a_n + b_n) \in \mathbb{R}^n$
   ii. $\lambda(a_1, \cdots, a_n) = (\lambda a_1, \cdots + \lambda a_n) \in \mathbb{R}^n$

   $\mathbb{R}^n(\mathbb{R})$ is called <u>Eucleaden space</u>

2. If we wanted to study $V(\mathbb{R})$, we should know :
   (i) The set of vectors V
   (ii) The sum of vectors ; $v_1 + v_2 \in V$
   (iii) The scaler product ; $\lambda v \in V$

3. from now on, If we said V is vector space, we mean that $V(\mathbb{R})$.

$F_5(\mathbb{R})$ be a vector space?

SoL Let $v_1 = x^5 + 4x^4 + 1 \in F_5$

$v_2 = -x^5 + 2x^3 + x \in F_5$

but $v_1 + v_2 = 4x^4 + 2x^3 + x + 1 \notin F_5$

Hence, $F_5(\mathbb{R})$ is not vector space.

Definition $P_n(x)$ is the set of all Polynomials of degree $\leq n$.

for example $x^3 + x + 1 \in P_3(x)$

$x + 1 \in P_3(x)$

$5 \in P_3(x)$

Remark $P_n(x)(\mathbb{R})$ is vector space with the following two operations:

(1) $(a_0 + a_1 x + \cdots + a_n x^n) + (b_0 + b_1 x + \cdots + b_n x^n)$

$= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \in P_n(x)$

(2) $\lambda(a_0 + a_1 x + \cdots + a_n x_n) = \lambda a_0 + \lambda a_1 x + \cdots + \lambda a_n x^n \in P_n(x)$

The famous classes of vector spaces

$M_n(\mathbb{R})$ such as

$M_2(\mathbb{R})$

$M_3(\mathbb{R})$

$\mathbb{R}^n(\mathbb{R})$

Eucleaden space

such as

$\mathbb{R}^2(\mathbb{R})$

$\mathbb{R}^3(\mathbb{R})$

$\mathbb{R}(\mathbb{R})$

$P_n(x)(\mathbb{R})$

such as

$P_1(x)$

$P_2(x)$

$P_3(x)$

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Linear Subspace:

Let V be a vector space and $\phi \neq W \subseteq V$. We say that W is vector subspace of V ($W \trianglelefteq V$) iff $W(\mathbb{R})$ is vector space.

Remark:

(1) $W \trianglelefteq V \iff$ (1) $W \subseteq V$
(2) $W(\mathbb{R})$ is vector space.

(2) For any vector space V, there are two trivial subspace $W_1 = \{0\}$
$W_2 = V$

Examples:

(1) $\mathbb{R}$ is vector space and $\mathbb{Z} \subseteq \mathbb{R}$, but $\mathbb{Z} \ntrianglelefteq \mathbb{R}$ because $\mathbb{Z}(\mathbb{R})$ is not vector space.

(2) $\left.\begin{array}{l} P_2(x) \subseteq P_5(x) \\ P_2(x) \text{ is vectorspace} \end{array}\right\} \Rightarrow P_2(x) \trianglelefteq P_5(x)$

(3) $\mathbb{R}^2$ is vector space, $\mathbb{R}^3$ is vector space, but $\mathbb{R}^2 \ntrianglelefteq \mathbb{R}^3$ because $\mathbb{R}^2 \not\subseteq \mathbb{R}^3$.

Theorem: Let V be a vector space and $W \subseteq V$. Then

$W \trianglelefteq V$ iff ① $0 \in W$
② $v_1, v_2 \in W \Rightarrow v_1 + v_2 \in W$
③ $v \in W \wedge \lambda \in \mathbb{R} \Rightarrow \lambda v \in W$

** The above criteria is used to show whether a subset of vector space is linear subspace or not.

(Ex) Let $W = \{(a,1) : a \in \mathbb{R}\}$. Does $W \triangleleft \mathbb{R}^2$?

<u>Sol</u> $0 = (0,0) \notin W$. So, $W \ntriangleleft \mathbb{R}^2$.

(Ex) Let $W = \{(a,0) : a \in \mathbb{R}\}$. Does $W \triangleleft \mathbb{R}^2$?

(1) $0 = (0,0) \in W$

(2) Let $v_1 = (a,0) \wedge v_2 = (b,0) \in W$. Then

$v_1 + v_2 = (a+b, 0) \in W$.

(3) Let $v = (a,0) \in W$ and $\lambda \in \mathbb{R}$. Then

$\lambda v = \lambda(a,0) = (\lambda a, 0) \in W$.

Hence $W \triangleleft V$.

(Ex) Let $W = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} ; a+b=0 \right\}$. Does $W \triangleleft M_2(\mathbb{R})$?

<u>Sol</u> (1) clearly, $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$.

(2) Let $v_1 = \underbrace{\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}}_{\boxed{a+b=0}}$ and $v_2 = \underbrace{\begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}}_{\boxed{c+d=0}} \in W$

our goal is to show that $v_1 + v_2 = \begin{bmatrix} 0 & a+c \\ b+d & 0 \end{bmatrix} \in W$.

which means $(a+c) + (b+d) = 0$. For that

$L.H.S = (a+c) + (b+d) = (a+b) + (c+d) = 0 + 0 = 0 = R.H.S$

(3) Let $\lambda \in \mathbb{R}$ and $v = \underbrace{\begin{bmatrix} a & a \\ b & 0 \end{bmatrix}}_{\boxed{a+b=0}} \in W$. Then

$\lambda v = \begin{bmatrix} 0 & \lambda a \\ \lambda b & 0 \end{bmatrix} \in W$ because $\lambda a + \lambda b = \lambda(a+b) = \lambda(0) = 0$

Hence, $W \triangleleft V$.

(Ex) Let $W = \left\{ \begin{bmatrix} x & y \\ z & f \end{bmatrix} : x+y = z+f \right\}$. Show that $W \trianglelefteq M_2(\mathbb{R})$?

Sol (i) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{R})$

(2) Let $v_1 = \begin{bmatrix} x & y \\ z & f \end{bmatrix}$ and $v_2 = \begin{bmatrix} x' & y' \\ z' & f' \end{bmatrix} \in W$

$\boxed{x+y = z+f}$ $\boxed{x'+y' = z'+f'}$

our goal is to show that

$v_1 + v_2 = \begin{bmatrix} x+x' & y+y' \\ z+z' & f+f' \end{bmatrix} \in W$; i.e. to show that

$(x+x') + (y+y') = (z+z') + (f+f')$

$$\begin{aligned} L.H.S &= (x+x') + (y+y') \\ &= (x+y) + (x'+y') \\ &= (z+f) + (z'+f') \\ &= (z+z') + (f+f') = R.H.S. \end{aligned}$$

(3) Let $\lambda \in \mathbb{R}$ and $\begin{bmatrix} x & y \\ z & f \end{bmatrix} \in W$. Now

$\boxed{x+y = z+f}$

$\lambda v = \lambda \begin{bmatrix} x & y \\ z & f \end{bmatrix} = \begin{bmatrix} \lambda x & \lambda y \\ \lambda z & \lambda f \end{bmatrix} \in W$ because

$\lambda x + \lambda y = \lambda(x+y) = \lambda(z+f) = \lambda z + \lambda f$ ▨

Linear Combination

Let $S = \{v_1, \cdots, v_n\} \subseteq V$. We say that $v \in V$ is a Linear Combination of S iff $v = \underset{\in\mathbb{R}}{\square} v_1 + \underset{\in\mathbb{R}}{\square} v_2 + \cdots + \underset{\in\mathbb{R}}{\square} v_n$

For example

(1) In space $\mathbb{R}$, $v = 7$ is a L.C. of $S = \{3, 5\}$

because $7 = \boxed{\frac{7}{3}}(3) + \boxed{0}(5)$

(Ex) In $\mathbb{R}^2$, $(3,5)$ is a Linear combination of $S = \{(1,0), (0,1)\}$ because $(3,5) = \underbrace{\boxed{3}}_{\in \mathbb{R}}(1,0) + \underbrace{\boxed{5}}_{\in \mathbb{R}}(0,1)$

(Ex) In $P_2(x)$, $2-x^2$ is a L.C. of $S = \{1, x, x^2\}$ because $2-x^2 = \underbrace{\boxed{2}}_{\in \mathbb{R}}(1) + \underbrace{\boxed{0}}_{\in \mathbb{R}}(x) \underbrace{\boxed{-1}}_{\in \mathbb{R}}(x^2)$

Remark:

To Prove whether v is a L.C of $\{v_1, v_2, \dots, v_n\}$:

STEP1: Suppose $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$

STEP2: Deduce non-Homo system in $\lambda_1, \lambda_2, \dots, \lambda_n$

STEP3: Study the system → Consistent → L.C.
or → in consistent → not L.C.

(Ex) Does $(2,1,0)$ be a L.C. of $\{(1,0,3), (2,1,5)\}$?

Sol: suppose $(2,1,0) = \lambda_1 (1,0,3) + \lambda_2 (2,1,5)$

$$\Rightarrow \quad \begin{aligned} 2 &= \lambda_1 + 2\lambda_2 \\ 1 &= 2\lambda_2 \\ 0 &= 3\lambda_1 + 5\lambda_2 \end{aligned}$$

To Solve L.S. $\Rightarrow$

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 3 & 5 & 0 \end{array}\right] \xrightarrow{-3R_1+R_3} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & -1 & -6 \end{array}\right] \xrightarrow{\frac{1}{2}R_2}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & 1/2 \\ 0 & -1 & -6 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -11/6 \end{array}\right]$$

So, the system has no SoL

$\Rightarrow$ $(2,1,0)$ is not L.C. of $\{(1,0,3), (2,1,5)\}$

(Ex) Does $(5,1,0)$ be a L.C. of $\{(1,1,1),(2,3,5),(1,0,1)\}$ If yes, write $(5,1,0)$ as L.C. ?

Sol Suppose $(5,1,0) = \lambda_1(1,1,1) + \lambda_2(2,3,5) + \lambda_3(1,0,1)$

Then
$$\begin{aligned} \lambda_1 + 2\lambda_2 + \lambda_3 &= 5 \\ \lambda_1 + 3\lambda_2 \quad\quad &= 1 \\ \lambda_1 + 5\lambda_2 + \lambda_3 &= 0 \end{aligned} \qquad \left(\begin{array}{c} \text{square system} \\ + \\ \text{Non Homo} \end{array} \right)$$

we will examine
$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 5 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$
$$= 2 + 1 = 3 \neq 0$$

So, we have a unique sol

$\Rightarrow (5,1,0)$ is a L.C. of S.

Now To write $(5,1,0)$ as a L.C. of S, we will use Jauss ;

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 1 & 3 & 0 & 1 \\ 1 & 5 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 3 & 0 & -5 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 3 & 7 \end{array}\right]$$

So, $3\lambda_3 = 7$

$\lambda_2 - \lambda_3 = -4$

$\lambda_1 + 2\lambda_2 + \lambda_3 = 5$

$\Rightarrow \lambda_3 = 7/3$

$\Rightarrow \lambda_2 = \boxed{} \to$ Complete

$\lambda_1 = \boxed{} \to$ Complete

Hence $(5,1,0) = \underset{\lambda_1}{\boxed{7/3}}(1,1,1) + \underset{\lambda_2}{\boxed{}}(2,3,5) + \underset{\lambda_3}{\boxed{}}(1,0,1)$

(Ex) Does $1-2x+3x^2$ a Linear combination of
$S = \{1-x^2,\ x+2x^2,\ 3-x\}$

Sol suppose $1-2x+3x^2 = \lambda_1(1-x^2) + \lambda_2(x+2x^2) + \lambda_3(3-x)$

So, we can deduce the following system :

$$\begin{aligned} \lambda_1 + 0\lambda_2 + 3\lambda_3 &= 1 \\ 0\lambda_1 + \lambda_2 - \lambda_3 &= -2 \\ -\lambda_1 + 2\lambda_2 + 0\lambda_3 &= 3 \end{aligned} \qquad \left(\begin{array}{c} \text{square} \\ + \\ \text{Non Homo} \end{array} \right)$$

So, will examine $|A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ -1 & 2 & 0 \end{vmatrix}$

$$= 1\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= 2 + 3 = 5 \neq 0$$

So, the system has unique SoL
$\Rightarrow$ the system is Consistent
$\Rightarrow$ $1-2x+3x^2$ is a L.C. of S.

Try to solve the following question :

Does $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ be a Linear combination of

$S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix} \right\}$?

Spanning set (generating set)

Let $S = \{v_1, v_2, \dots, v_n\}$ be a set of vectors in V. We say that S generates (spans) V iff any vector v of V is a Linear combination of S which means: $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$ for some $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$.

In this case $v_1, v_2, \dots, v_n$ are called generators of the space V.

Example $\{(1,0), (0,1)\}$ is spanning set of $\mathbb{R}^2$ because for any $(a,b) \in \mathbb{R}^2$; $(a,b) = a(1,0) + b(0,1)$ ∎

Example $\{1, x, x^2\}$ is spanning set of $P_2(x)$ because for any $a + bx + cx^2 \in P_2(x)$;

$$a + bx + cx^2 = a(1) + b(x) + c(x^2)$$ ∎

Remark To examine $S = \{v_1, \dots, v_n\}$ is spanning set of V:

STEP 1: Take the general form of the vector $u \in V$

STEP 2: Study u is L.C. of S.

Example: Does $(1,2)$, $(3,-1)$ be generators of $\mathbb{R}^2$?

Sol suppose $\lambda_1(1,2) + \lambda_2(3,-1) = (a,b)$

$$\Rightarrow \begin{aligned} \lambda_1 + 3\lambda_2 &= a \\ 2\lambda_1 - \lambda_2 &= b \end{aligned} \quad \text{(square + Non Homo)}$$

$|A| = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -7 \neq 0 \Rightarrow$ the system has unique sol (i.e. consistent) $\Rightarrow \{(1,2), (3,-1)\}$ spans $\mathbb{R}^2$ ∎

(Exc) Show that whether $\mathbb{R}^3$ is generated by $S = \{(1,1,1), (2,0,1), (1,3,0)\}$?

Sol Take $u = (a,b,c) \in \mathbb{R}^3$, and suppose that

$$\lambda_1 (1,1,1) + \lambda_2 (2,0,1) + \lambda_3 (1,3,0) = (a,b,c)$$

$$\Rightarrow \quad \begin{aligned} \lambda_1 + 2\lambda_2 + \lambda_3 &= a \\ \lambda_1 + \quad + 3\lambda_3 &= b \\ \lambda_1 + \lambda_2 + \quad &= c \end{aligned} \qquad \left(\begin{gathered} \text{Square} \\ + \\ \text{Non Homo} \end{gathered} \right)$$

we will examine $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 0 \end{vmatrix} = 1 \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$

$= 6 - 2 = 4 \neq 0$

So, the system has unique sol (Consistent)

$\Rightarrow$ S generates $\mathbb{R}^3$.

The space span(s) [The space generated by S]

Given $S = \{v_1, v_2, \dots v_n\} \subseteq V$. we can induce a set S' which is the set of all vectors can by written by S. So,

$$S' = \{ \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n : \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R} \}$$

$\boxed{1}$ S' is a Linear space which denoted by <u>span(s)</u>

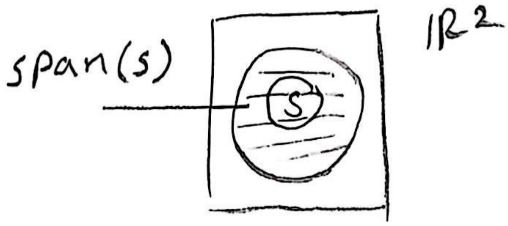
$\boxed{2}$ S spans span(s)

$\boxed{3}$ span(s) is the smallest linear space contains S.

To clarify : Let $V = \mathbb{R}^3$, $S = \{(1,1,1), (1,0,1)\}$

Then $\text{Span}(S) = \{ u = \lambda_1 (1,1,1) + \lambda_2 (1,0,1) : \lambda_1, \lambda_2 \in \mathbb{R} \}$

$\nabla\ \mathbb{R}^2$



Remarks:

(1) Let V be a L.S. and $S \subseteq V$. Then span(S) is the smallest space that contains S. (we should examine the zero vector $\in$ span(S)).

(2) span$(S) = V$ where V is L.S. iff S spans V.

(S is called <u>The set of generators</u>)

(3) In some types of showing whether W is L. subspace of V or not, we will show $W = $ span(S) for some $S \subseteq V$. See the following questions.

(Ex) Show that

$W = \{(a,b,c) : a = 2x+y,\ b = x-y$ and $c = x+3y$ where $x,y \in \mathbb{R}\}$

is Linear subspace of $\mathbb{R}^3$?

<u>Sol</u>: Let $v \in W$. Then

$$v = (2x+y,\ x-y,\ x+3y)$$
$$= (2x, x, x) + (y, -y, 3y)$$
$$= x(2,1,1) + y(1,-1,3)$$

So, $W = $ span(S) where

$S = \{(2,1,1), (1,-1,3)\}$.

Hence $W \trianglelefteq V$ because $0 \in$ span(S).

(Ex) study whether

$W = \{(a+b-1, 2a, 3b-a) : a, b \in \mathbb{R}\}$

$\triangleleft \mathbb{R}^2$, or not ?

Sol: Let $v \in W$. Then

$$v = (a+b-1, 2a, 3b-a)$$
$$= (a, 2a, -a) + (b, 0, 3b) + (-1, 0, 0)$$
$$= a(1, 2, -1) + b(1, 0, 3) + (-1, 0, 0)$$

So, $W = \text{span}(S) \not\triangleleft \mathbb{R}^3$ because $(0,0,0) \notin \text{span}(S)$ where

$S = \{(1, 2, -1), (1, 0, 3), (-1, 0, 0)\}$.

(Ex) study whether

$W = \{a+b + (2a-b)x + 3ax^2 :$

$a, b \in \mathbb{R}\} \triangleleft P_2(x)$, or not ?

Sol: Let $v \in W$. Then

$$v = (a+b) + (2a-b)x + 3ax^2$$
$$= (a + 2ax + 3ax^2) + (b - bx)$$
$$= a(\underbrace{1 + 2x + 3x^2}_{\in P_2(x)}) + b(\underbrace{1-x}_{\in P_2(x)})$$

Hence, $W = \text{span}(S)$ where

$S = \{1 + 2x + 3x^2, 1-x\}$. As

$0 \in \text{span}(S)$, $W = \text{span}(S) \triangleleft P_2(x)$.

(Ex) study whether

$$W = \left\{ \begin{bmatrix} a-2b & a+1 \\ b-2 & a+b \end{bmatrix} : a,b \in \mathbb{R} \right\} \triangleleft M_2(\mathbb{R})$$

or not ?

Sol : Take any $v \in W$. Then

$$v = \begin{bmatrix} a-2b & a+1 \\ b-2 & a+b \end{bmatrix}$$

$$= \begin{bmatrix} a & a \\ 0 & a \end{bmatrix} + \begin{bmatrix} -2b & 0 \\ b & b \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

$$= a \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{v_1} + b \underbrace{\begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}}_{v_2} + \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}}_{v_3}$$

Clearly, $W = \text{span}(S)$ where $S = \{v_1, v_2, v_3\}$

Since $0 \notin \text{span}(S)$, $W \ntriangleleft M_2(\mathbb{R})$.

(Ex) (To try) : Study whether

$S = \{(2a-1, b+2c, 2a+b) : a,b,c \in \mathbb{R}\}$

is Linear subspace of $\mathbb{R}^3$ or not ??

standered spanning sets :

Linear space	standered spanning set
$\mathbb{R}$	$S = \{1\}$
$\mathbb{R}^2$	$S = \{(1,0), (0,1)\}$
$\mathbb{R}^3$	$S = \{(1,0,0), (0,1,0), (0,0,1)\}$
$P_1(x)$	$S = \{1, x\}$
$P_2(x)$	$S = \{1, x, x^2\}$
$M_2(\mathbb{R})$	$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

The set of Sol of Homo-System $AX=0$

Theorem: Let $AX=0$ be a homo-L.S. Then the set of sol is always Linear subspace of $\mathbb{R}^n$.

Proof

Let S be a set of Sol of $AX=0$.
Then $S = \{ v \in \mathbb{R}^n : A \cdot v = 0 \}$.
The goal is to show $S \trianglelefteq \mathbb{R}^n$.

(1) $0 \in S$ because $A(0) = 0$.

(2) Let $v_1, v_2 \in S$ which means that $Av_1 = 0$ and $Av_2 = 0$. our goal is to show that $v_1 + v_2 \in S$ (i.e. $A(v_1+v_2) = 0$). For that

$L.H.S = A(v_1+v_2)$
$= Av_1 + Av_2 = 0 + 0 = 0 = R.H.S.$

(3) Let $v \in S$ and $\lambda \in \mathbb{R}$. our goal is to show that $\lambda v \in S$ (i.e. $A(\lambda v) = 0$)

$L.H.S = A(\lambda v)$
$= \lambda(Av) = \lambda(0) = 0 = R.H.S.$

(Ex) Does a set of sol of $AX = B$ be a Linear subspace of $\mathbb{R}^n$?

(Ex) Study whether
$W = \{(a,b,c) : 2a+b-c = 0,$
$3a-b+c = 0$ and $b-2a+c = 0\}$
be a L-subspace of $\mathbb{R}^3$ or not?

Sol: notice that W is the set of sol of the following ystem

$2x + y - z = 0$
$3x - y + z = 0$
$-2x + y + z = 0$

Homo-system

By theorem, $W \trianglelefteq \mathbb{R}^3$.

(EX) Let

$W = \{(a,b,c) : 3a - 2b + c = 0$ and $a + b + c = 1\}$. Study whether $W \trianglelefteq \mathbb{R}^3$ or not?

<u>Sol</u>: clearly, W is the set of sol of the following system:

$$3x - 2y + z = 0$$
$$x + y + z = 1$$
(Non-Homo system)

So, $W \ntrianglelefteq \mathbb{R}^3$.

(EX) Let $W = \{(a,b,c) : 2a - b + c = 0\}$. Show that $W \trianglelefteq \mathbb{R}^3$ and find the set of generators of W.

<u>Sol</u>: clearly, W is the set of sol of the following system:

$$2x - y + z = 0 \quad \text{(Homo-system)}$$

By theorem, $W \trianglelefteq \mathbb{R}^3$.

To find the set of the generators, we should find the set of sol of the system:

$$2x - y + z = 0 \quad \text{(Use Gauss)}$$

$|E| = 1 \quad |V| = 3 \Rightarrow |P| = 2$ (s,t)

Put $x = s$

$y = t$ then $z = y - 2x = t - 2s$.

So, $S = \left\{ \begin{bmatrix} s \\ t \\ t-2s \end{bmatrix} : s, t \in \mathbb{R} \right\}$

To find the geneators, take any vector of S

$$\begin{bmatrix} s \\ t \\ t-2s \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ -2s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

$$= s \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

So, $S' = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is the set of generators of the space S.

(Ex) (To try):

(1) Show that

$W = \{ (a, b, c) : 2a - b + c = 0$ and $3a - b - c = 0 \}$ is Linear space. Find its set of generators.

(2) Show that

$W = \{ (2a-b, b+c, c-a) : a, b, c \}$ is Linear space. Find its set of generators.

Remark

To show $W \triangledown V$,
we have 4 types of questions

(1) By definition

for example

$P_2(x) \triangledown P_3(x)$

(2) By criteria

e.g.

$W = \{(a,0) : a \in \mathbb{R}\} \triangledown \mathbb{R}^2$

(3) $W = \text{span}(S)$

for example

$W = \{(a+b, a-b) : a,b \in \mathbb{R}\}$

(4) set of sol of $AX = 0$

for example

$W = \{(a,b,c) : 2a-b+c=0,\ a+b-2c=0\}$.

Summary !!

(Ex) Let W_1, W_2 be two Linear subspaces of V. Show that

(1) $W_1 \cap W_2 \triangleleft V$

(2) $W_1 \cap W_2 \triangleleft W_1$

(3) Does $W_1 \cup W_2 \triangleleft V$?

(4) If $W_3 \triangleleft W_1$, show that $W_3 \triangleleft V$.

<u>Solution</u>

(1) To show $W_1 \cap W_2 \triangleleft V$,

(i) as $0 \in W_1$ and $0 \in W_2$ then $0 \in W_1 \cap W_2$.

(ii) Let v_1 and $v_2 \in W_1 \cap W_2$ then

$v_1, v_2 \in W_1$ and $v_1, v_2 \in W_2$

$\downarrow$ as $W_1 \triangleleft V$ $\qquad$ $\downarrow$ as $W_2 \triangleleft V$

$$\underbrace{v_1 + v_2 \in W_1 \qquad v_1 + v_2 \in W_2}$$

$$v_1 + v_2 \in W_1 \cap W_2$$

(iii) Let $v \in W_1 \cap W_2$ and $\lambda \in \mathbb{R}$

$\Rightarrow$ $v \in W_1$ and $\lambda \in \mathbb{R}$ $\qquad$ $v \in W_2$ and $\lambda \in \mathbb{R}$

$$\underbrace{\lambda v \in W_1 \qquad \lambda v \in W_2}$$

$$\lambda v \in W_1 \cap W_2$$

By (i), (ii) and (iii)

$$W_1 \cap W_2 \triangleleft V.$$

(2) As $W_1 \cap W_2 \triangledown V$, $W_1 \cap W_2$ is Linear space. So, we have

$$\left.\begin{array}{l} W_1 \cap W_2 \text{ L.S.} \\ (W_1 \cap W_2) \subseteq W_1 \end{array}\right\} \Rightarrow W_1 \cap W_2 \triangledown W_1$$

(3) Let $W_1 = \{(a,0) : a \in \mathbb{R}\} \triangledown \mathbb{R}^2$

$W_2 = \{(0,a) : a \in \mathbb{R}\} \triangledown \mathbb{R}^2$

$W_1 \cup W_2 = \{(x,y) : \text{either } x=0 \text{ or } y=0\}$

Now,

$v_1 = (1,0) \in W_1 \cup W_2$

$v_2 = (0,1) \in W_1 \cup W_2$

But $v_1 + v_2 = (1,1) \notin W_1 \cup W_2$

Hence $W_1 \cup W_2 \not\triangledown V$.

(4) As $W_3 \triangledown W_1$, $W_3 \subseteq W_1$ and W_3 is Linear space.

Now we have :

$$\left.\begin{array}{l} W_3 \subseteq W_1 \subseteq V \\ W_3 \text{ L-S} \end{array}\right\} \Rightarrow W_3 \triangledown V.$$

(Ex) Let $v \in V$ where V is L.S. Prove that $0 \cdot v = 0$?

Sol

$$0 \cdot v = (0+0)\, v$$

$$\Leftrightarrow \quad 0 \cdot v = 0 \cdot v + 0 \cdot v$$

add $-0 \cdot v$

$$\Leftrightarrow \quad (-)\, 0 \cdot v + 0 \cdot v = 0 \cdot v + 0 \cdot v + (-) 0 \cdot v$$

$$\Leftrightarrow \quad 0 = 0 \cdot v + 0$$

$$\Leftrightarrow \quad 0 = 0 \cdot v$$

3 **Basis and Dimension**

Definition Let $S = \{v_1, \dots, v_n\}$ be a set of vectors of linear space V. Then

S is a basis of V iff $\begin{cases} ① \ S \text{ spans } V. \\ ② \ S \text{ is linear independent.} \end{cases}$

Remarks

① If S is a basis of a linear space V, then $|S|$ is called the dimension of V, $\text{Dim}(V)$, .

② If $\text{Dim}(V) < \infty$, then V is finite dimensional.
If $\text{Dim}(V) = \infty$, then V is infinite dimensional.

③ Every vector space has atleast a basis which is not necessarily unique

Standered basis of some famous Linear spaces :

Linear space	Basis	Dim
$\mathbb{R}^2$	$B = \{(1,0), (0,1)\}$	2
$\mathbb{R}^3$	$B = \{(1,0,0), (0,1,0), (0,0,1)\}$	3
$M_2(\mathbb{R})$	$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$	4
$P_1(x)$	$B = \{1, x\}$	2
$P_2(x)$	$B = \{1, x, x^2\}$	3

(ex) Let $S = \{ (\frac{3}{4}, \frac{5}{2}, \frac{3}{2}), (3,4,7/2), (-\frac{3}{2}, 6, 2) \}$ be a set of vectors of $\mathbb{R}^3$. Determine, if S is Linear independent or dependent ?

<u>Solution</u>

suppose that $\lambda_1(\frac{3}{4}, \frac{5}{2}, \frac{3}{2}) + \lambda_2(3,4,7/2) + \lambda_3(-\frac{3}{2}, 6, 2) = (0,0,0)$

then,

$$\left.\begin{array}{lcl} \frac{3}{4}\lambda_1 + 3\lambda_2 - \frac{3}{2}\lambda_3 = 0 & \Leftrightarrow & 3\lambda_1 + 12\lambda_2 - 6\lambda_3 = 0 \\ \frac{5}{2}\lambda_1 + 4\lambda_2 + 6\lambda_3 = 0 & \Leftrightarrow & 5\lambda_1 + 8\lambda_2 + 12\lambda_3 = 0 \\ \frac{3}{2}\lambda_1 + \frac{7}{2}\lambda_2 + 2\lambda_3 = 0 & \Leftrightarrow & 3\lambda_1 + 7\lambda_2 + 4\lambda_3 = 0 \end{array}\right\}$$

It is square - Homogenous system.

$$\text{So, } |A| = \begin{vmatrix} 3 & 12 & -6 \\ 5 & 8 & 12 \\ 3 & 7 & 4 \end{vmatrix} = 3 \begin{vmatrix} 1 & 4 & -2 \\ 5 & 8 & 12 \\ 3 & 7 & 4 \end{vmatrix} \;\; \frac{-5R_1 + R_2}{-3R_1 + R_3}$$

$$= 3 \begin{vmatrix} 1 & 4 & -2 \\ 0 & -12 & 22 \\ 0 & -5 & 10 \end{vmatrix}$$

$$= 3(1) \begin{vmatrix} -12 & 22 \\ -5 & 10 \end{vmatrix} \neq 0$$

So, A^{-1} is existed $\Rightarrow$ the zero - solution is the unique solution

$\Rightarrow$ S is Linear independent.

(ex) Let $S = \{ 1 + x^2 , 2 + x + x^2 \}$ be a set of vectors of $P_2(x)$. Determine, if S is Linear independent or not ?

<u>Solution</u> suppose that $\lambda_1(1+x^2) + \lambda_2(2+x+x^2) = 0 + 0x + 0x^2$

$$\Rightarrow \left.\begin{array}{r} \lambda_1 + 2\lambda_2 = 0 \\ \lambda_2 = 0 \\ \lambda_1 + \lambda_2 = 0 \end{array}\right\}$$

It is clear that $\lambda_1 = \lambda_2 = 0$ is the unique solution.

Hence, S is Linear independent $\blacksquare$

1

** Linear Dependency

Definition

Let $\{v_1, v_2, \dots, v_n\}$ be a set of vectors of a Linear space.

(i) If $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$ iff $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ then $\{v_1, v_2, \dots, v_n\}$ is Linear independent where $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$

(ii) If $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$ has a non-zero values of $\lambda_1, \dots, \lambda_n$ then $\{v_1, \dots, v_n\}$ is Linear dependent

Remark

To examine $\{v_1, \dots, v_n\}$ is Linear independent or not,

STEP1 : suppose $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$.

STEP2 : Deduce Homogenous system

STEP3 : If it has unique (zero) solution, i.e., $\lambda_1 = \dots = \lambda_n$ then $\{v_1, \dots, v_n\}$ is Linear independent. otherwise, The system has non-zero solution, and then it is Linear Dependent.

(Ex) Let $\{(6,2,1), (-1,3,2)\}$ be a set of vectors of $\mathbb{R}^3$. Does the set be Linear independent?

Solution suppose that $\lambda_1(6,2,1) + \lambda_2(-1,3,2) = (0,0,0)$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Then

$$\left.\begin{aligned} 6\lambda_1 - \lambda_2 &= 0 \\ 2\lambda_1 + 3\lambda_2 &= 0 \\ \lambda_1 + 2\lambda_2 &= 0 \end{aligned}\right\}$$ rectangular - Homogenous system

$$\left[\begin{array}{cc|c} 6 & -1 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow{\text{REEF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has unique solution $\lambda_1 = \lambda_2 = 0$

$\therefore \{(6,2,1), (-1,3,2)\}$ is Linear independent

Some Properties of basis of L-S.

(Prop 1)

If B_1 and B_2 are two basis of V, then $|B_1| = |B_2| = Dim(V)$.

(The converse is not true)

(Ex) Let $S = \{v_1, v_2, v_3\}$ be a set of vectors in $\mathbb{R}^2$. S is not basis because $|S| = 3 \neq Dim(\mathbb{R}^2)$

(Prop 2)

Let $0 \in S \subseteq V$. Then S is L-Dep, and therefore, S is not basis.

why?

Let $S = \{v_1, v_2, \dots, v_n, 0\}$, and suppose that

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n + \lambda_{n+1}(0) = 0$$

we have non-zero solution

$\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ and $\lambda_{n+1} = 10$

Hence, S is L-Dep.

(Prop 3)
Let S be a set of vectors in V.

S is L.Dep $\iff$ there is a vector of S can be written as a Linear combination of others vectors in S.

(Ex) Let $S = \{(1,1), (2,1)\}$ then S is L. indep because there is no λ where $(1,1) = \lambda(2,1)$.

(Ex) Let $S = \{5, 1-x, x^2\}$.
Notice that $5 \neq \lambda(1-x) + \lambda'(x^2)$
$1-x \neq \lambda(5) + \lambda'(x^2)$
$x^2 \neq \lambda(5) + \lambda'(1-x)$
Hence, S is L. indep.

(Ex) Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ where $2v_1 - v_2 = 3v_3 + 5v_5$.
Then S is L. Dep ... why?
since $2v_1 - v_2 = 3v_3 + 5v_5$
$\Rightarrow v_2 = 2v_1 - 3v_3 - 5v_5$
$\Rightarrow v_2 = 2v_1 - 3v_3 + (0)v_4 - 5v_5$

Therefore,

v_2 can be written by others vectors in S

$\Rightarrow S$ is L-Dep.

(Ex) Let $S = \{v, 3v, u\}$ be a set in $\mathbb{R}^3$. Does S be a basis of $\mathbb{R}^3$?

Sol Notice that

$$3v = 3(v) + 0(u)$$

So, $3v$ can be written by other vector in $S \Rightarrow S$ is L-Dep

$\Rightarrow S$ is not basis ▨

(Prop 4)

Let $W \trianglelefteq V$. Then $\text{Dim}(W) \leq \text{Dim}(V)$

(Ex) Let W be a Proper Linear subspace of $\mathbb{R}^3$. Then $\text{Dim}(W) = 0$ or 1 or 2.

Remark:

If $W \trianglelefteq V$ and $\dim(W) = \dim(V)$, then $W = V$.

(Ex) Let $S = \{v\} \subseteq V$ where V is L. space. Study S ?

Solution

Suppose $\lambda v = 0$ then

either $v = 0$ or $\lambda = 0$

$\Downarrow$ S is L. Dep $\qquad$ $\Downarrow$ S is L. indep.

for example:

$S = \{(0,0)\}$ is L. Dep.

$S = \{(1,3)\}$ is L. indep.

(Ex) Let $S = \{u, v\}$ is L. indep.

Study $S^* = \{2u - v, u + v\}$.

Sol:

Suppose $\lambda_1 (2u - v) + \lambda_2 (u + v) = 0$

$\Rightarrow 2\lambda_1 u - \lambda_1 v + \lambda_2 u + \lambda_2 v = 0$

$\Rightarrow (2\lambda_1 + \lambda_2) u + (-\lambda_1 + \lambda_2) v = 0$

as $\{u, v\}$ L indep, we have

$$\left.\begin{aligned} 2\lambda_1 + \lambda_2 &= 0 \\ -\lambda_1 + \lambda_2 &= 0 \end{aligned}\right\} \text{ Linear system}$$

$\Rightarrow \lambda_1 = \lambda_2 = 0$

$\Rightarrow \{2u - v, u + v\}$ L-indep $\blacksquare$

(Prop 5)

Let $S \subset V$ where V is L-space.
Then S is a **basis** $\Longleftrightarrow$
(1) S L indep (2) S spans V
$\Longleftrightarrow$
(1) $|S| = Dim(V)$ (2) S is L indep.
$\Longleftrightarrow$
(1) $|S| = Dim(V)$ (2) S spans V.

~~Notice that the definition of basis~~

Remark :
To prove S is a basis, we can use one of the above 3 definitions. The best one is :
(1) $|S| = Dim(V)$ (2) S is L indep.

(Ex) Show that $S = \{(1,1), (2,0)\}$ is a basis of $\mathbb{R}^2$?

Sol (i) $|S| = Dim(\mathbb{R}^2) = 2$
(ii) $(1,1) \neq \lambda(2,0)$ for all $\lambda \in \mathbb{R}$
$\Rightarrow$ S is L indep
By (i), (iii), S is a basis ∎

(Prop 6)

Let $S \subseteq V$ be Linear indep. Then we have 2 Possibilities

either | or

$|S| = \text{Dim } V$ $\Downarrow$ S a basis

$|S| < \text{Dim } V$ $\Downarrow$ $\exists B$ basis where $S \subseteq B$

(How to find B?)

(STEP1) Form a matrix A where columns of A are S and standered basis of V.

(STEP2) Do REF.

(STEP3) Columns that meet columns of 1 leaders is the basis B.

for example: (1) show that $S = \{(1,1,1), (2,1,0)\}$ is L. indep in $\mathbb{R}^3$. (2) find a basis B where $S \subseteq B$?

Sol: (1) since $(1,1,1) \neq \lambda(2,1,0)$ forall $\lambda \in \mathbb{R}$, S is L. indep.

(2) To find basis B where $S \subseteq B$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & -2 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_2} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -2 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{2R_2+R_3} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

So, $B = \{(1,1,1), (2,1,0), (1,0,0)\}$.

(Ex) Put (T) or (F) :

(1) Let $S \subseteq P_2(x)$ where $|S| = 4$ then S is L. Dep ( )

(2) Let $S \subseteq P_2(x)$ where $|S| = 3$ then S is basis ( )

(3) Let $S \subseteq P_2(x)$ where $|S| = 2$ then S is L. indep ( )

(Prop 7) Let $S \subseteq V$ be a spanning set of V. Then we have 2 Possibilites regarding S

either: $|S| = \text{Dim}(V)$ $\Rightarrow$ S is a basis

or: $|S| > \text{Dim}(V)$ $\Rightarrow$ $\exists B$ basis of V where $B \subset S$

Remmark

If we know generators, we could find the basis.

(Ex) Let $W = \text{span}(S)$ where

$$S = \{(1,2,1,1),\ (0,1,3,2),\ (0,0,1,5),\ (1,0,1,1),\ (1,1,1,1)\}$$

find the basis of W?

Sol:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 2 & 5 & 1 & 1 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix}$$

So, $B = \{(1,2,1,1),\ (0,1,3,2),\ (0,0,1,5),\ (1,0,1,1)\}$

Hence $\text{Dim}(W) = 4$ ∎

(Ex) Let $S = \{0,\ 1-x,\ 2-2x,\ x^2\}$ spans W which $W \trianglelefteq P_2(x)$.

find basis of W?

Sol: we will exclude 0.

Notice that $2-2x = 2(1-x)$.

So, $2-2x$ and $1-x$ are L. Dep.

Hence, we will exclude $2-2x$.

Now $S' = \{1-x,\ x^2\}$ generates W and L. indep (why?). So, S' is a basis of W.

Remark :

Let $AX=0$ be a homo Linear system. We have shown that the set of Sol of $AX=0$ is Linear space.

(Ex) Suppose $S = \left\{ \begin{bmatrix} t \\ 2t+s \\ s \\ s-t \end{bmatrix} ; s,t \in \mathbb{R} \right\}$

be a set of Sol of $AX=0$. find the basis of S.

Sol : Let $v = \begin{bmatrix} t \\ 2t+s \\ s \\ s-t \end{bmatrix}$

$$= \begin{bmatrix} t \\ 2t \\ 0 \\ -t \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ s \\ s \end{bmatrix}$$

$$= t \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

So, $B = \{(1,2,0,-1), (0,1,1,1)\}$ is a set of generators of $S \Rightarrow B$ spans S

Notice that $(1,2,0,-1) \neq \lambda (0,1,1,1)$ forall $\lambda \Rightarrow B$ is L indep

Hence, B is a basis of S

$\Rightarrow \text{Dim}(S) = 2$ ▨

Q) Let $W = \{(a-b, 2a+b, 3b) : a, b \in \mathbb{R}\}$ be a L-subspace of $\mathbb{R}^3$. Find basis of W?

Sol

Let $v \in W \Rightarrow$

$$v = (a-b, 2a+b, 3b)$$
$$= (a, 2a, 0) + (-b, b, 3b)$$
$$= a(1,2,0) + b(-1,1,3)$$

Hence $S = \{(1,2,0), (-1,1,3)\}$ spans W and $(1,2,0) \neq \lambda(-1,1,3)$ for all $\lambda \in \mathbb{R}$

$\Rightarrow$ S is L indep

Therefore, S is a basis of W and hence, $\text{Dim}(W) = 2$ ■

(Ex) 1. Prove that $W = \{(a,b,c) : 2a - c = 0, a+b-c = 0 \text{ and } 2a+b+3c = 0\}$ is L-subspace of $\mathbb{R}^3$. Find $\text{Dim}(W)$?

(Ex) Let $W = \{(a+b) + (2a-b)x + 3bx^2 : a, b \in \mathbb{R}\}$. Prove that $W \trianglelefteq P_2(x)$. Find $\text{Dim}(W)$?

(Ex) Let $W = \{(3a, a+b-c, 2c-b) : a, b, c \in \mathbb{R}\}$. Show that $W \trianglelefteq \mathbb{R}^3$. Find $\text{Dim}(W)$?

Theorem A squar matrix is invertible iff it is columns are linear independent. iff it is rows are linear independent

Theorem If A is row-echelon form matrix then the non-zero rows is linear independent

for example

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is row-echelon form

Then, the set $\{(1,2,3,4), (0,0,1,1)\}$ is

[illegible] in $\mathbb{R}^4$.

** Rank and Nullity :-

Let $\underset{m\times n}{A}$ be a matrix

Column space $Col(A)$ = the space that generated by columns $\subseteq \mathbb{R}^n$

To find basis of $Col(A)$

1. Do $REF(A)$
2. Columns of 1 leaders form basis

row space $Row(A)$ = the space that generated by rows $\subseteq \mathbb{R}^m$

To find basis of $Row(A)$

1. Do $REF(A)$
2. Rows of 1 leaders form basis

null space $N(A)$ = the set of sol of $AX=0$ $\subseteq \mathbb{R}^n$

To find basis of $N(A)$

1. find Sol $AX=0$
2. find basis of the Sol

$$\boxed{Dim(N(A)) = nullity(A)}$$

$$\boxed{Dim(Col(A)) = Dim(Row(A)) = Rank(A)}$$

** Theorem

1. $Rank(A) = Rank(A^t)$
2. $Rank(A) + nullity(A) =$ number of Columns of A

(Ex)

Let $A = \begin{bmatrix} 1 & 2 & 3 & 5 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Clearly, A in the form of REF

$B_{col(A)} = \{(1,0,0), (3,1,0), (2,1,1)\} \subseteq \mathbb{R}^3$

$B_{Row(A)} = \{(1,2,3,5,2), (0,0,1,0,1), (0,0,0,0,1)\}$

$Rank(A) = 3$

$Nullity(A) = 5 - 3 = 2$

Theorem: Let $AX = B$ be a Linear system

1. If $Rank(A) = Rank([A|B])$ then the system $AX = B$ is consistent
2. If $Rank(A) \neq Rank([A|B])$ then the system $AX = B$ is inconsistent

for example:

Suppose L.S. $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 3 \end{array}\right]$

$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow rank(A) = 1$

$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 3 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \Rightarrow rank(A|B) = 2$

So, the system $AX = B$ inconsistent.

Remark :

Suppose $\underset{n\times m}{A}$ and $AX=B$ consistent.

then we have 2 cases :

unique SoL $\iff$ $\text{rank}(A) = n$

∞ many SoL $\iff$ $\text{rank}(A) < n$

Example : Suppose the system

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice that

$\text{Rank}([A|B]) = \text{Rank}(A) = 2$

$\Rightarrow$ the system is Consistent

Now : $\underset{3\times 3}{A}$

$\text{Rank}(A) = 2 < 3 \Rightarrow$ The system has ∞ many SoL. $\boxtimes$

(Ex) Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. find $N(A)$, nullity of A ? find Rank(A) ?

Sol

To find $N(A)$, we should solve the system $AX = 0 \iff$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$ The matrix is eliminted

then

$$\left.\begin{array}{l} x + z + f = 0 \\ y + 3z + 2f = 0 \end{array}\right\} \Rightarrow \begin{array}{l} |E| = 2 \\ |V| = 4 \end{array} \Rightarrow \begin{array}{l} |P| = 2 \\ (t, s) \end{array}$$

put $f = t$, $z = s$

$\Rightarrow$ $x = -t - s$ and $y = -2t - 3s$

Hence

$$S = N(A) = \left\{ \begin{bmatrix} -t-s \\ -2t-3s \\ s \\ t \end{bmatrix} ; \ s, t \in \mathbb{R} \right\}$$

to find the basis :

$$\begin{bmatrix} -t-s \\ -2t-3s \\ s \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} -s \\ -3s \\ s \\ 0 \end{bmatrix}$$

$$= t \cdot \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + s \cdot \begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $W = \{(-1, -2, 0, 1), (-1, -3, 1, 0)\}$ spans $N(A)$

and clearly, W is L. indep.

$\Rightarrow$ W is a basis of $N(A) \Rightarrow \text{Dim } N(A)$

Therefore

$\text{nullity}(A) = \text{Dim}(N(A)) = 2$

$\Rightarrow$

$\text{rank}(A) = \text{number of col of } A - \text{nullity}(A)$

$= 4 - 2 = 2$

(Ex) Let $A = \begin{bmatrix} 1 & 1 & 5 & 1 & 7 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. find

$\text{nullity}(A)$?

Sol : clearly, A in the form REF.

So, $\text{Rank}(A) = 2$

Therefore, $\text{nullity}(A) = 5 - 2 = 3$

(Ex) Let $\underset{7\times 10}{A}$ and $\text{rank}(A) = 3$

find $\text{nullity}(A^t)$?

Sol

$\text{nullity}(A^t) = \text{col of } (A^t) - \text{rank}(A^t)$

$= 7 - 3$ (because $\text{rank}(A) = \text{rank}(A^t)$)

$= 4$

** Change of basis and the transition matrix

(1) Coordinate of a vector regarding a basis ($[V]_B$) :-

Let $B = \{u_1, u_2, \dots, u_n\}$ be ~~and~~ a basis
$v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$
then

$$[v]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

for example $S = \{(1,1), (2,0)\}$ is a basis of $\mathbb{R}^2$ (why?) . find $[(1,3)]_S$?

Sol : Firstly, we should write $(1,3)$ as L.C. of S .

Let $(1,3) = \lambda_1 (1,1) + \lambda_2 (2,0)$

$$\Rightarrow \lambda_1 + 2\lambda_2 = 1$$

$$\lambda_1 + 0\lambda_2 = 3 \Rightarrow \lambda_1 = 3$$

$$\lambda_2 = -1$$

Hence $[(1,3)]_S = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(Ex) Let $B = \{(1,0,1), (2,1,1), (1,3,0)\}$ be a basis of $\mathbb{R}^3$ and $[v]_B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

find the vector v?

Sol: As $[v]_B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$;

$$v = 2(1,0,1) + (-1)(2,1,1) + 1(1,3,0)$$
$$= (1,2,1)$$

Theorem (The fundemintal Theorem)

Let $v \in V$ and $B = \{u_1, u_2, \dots, u_n\}$ be a basis of V. Then v has a unique represention of B (i.e. $v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$ in the unique form).

Proof: we will divide the Proof into 2 Parts:

(I) as B a basis $\Rightarrow$ B spans V

$\Rightarrow$ $v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$ for some $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$

(II) to show the uniqueness, suppose v has another representation

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_n u_n$$

Now

$0 = v - v$

$\Leftrightarrow 0 = (\alpha_1 - \lambda_1) u_1 + \dots + (\alpha_n - \lambda_n) u_n$

But $\{u_1, \dots, u_n\}$ is L-indep

$$\Rightarrow \begin{cases} \alpha_1 - \lambda_1 = 0 & \Rightarrow \alpha_1 = \lambda_1 \\ \alpha_2 - \lambda_2 = 0 & \Rightarrow \alpha_2 = \lambda_2 \\ \vdots & \\ \alpha_n - \lambda_n = 0 & \Rightarrow \alpha_n = \lambda_n \end{cases}$$

Hence, v has unique rep ∎

Remark:

1. Any vector space has at least a basis B. And, any vector v can be written (in one represention) by B ∎

2. ★ B basis + vector ⇒ we can get $[v]_B$

★ $[v]_B$ + basis B ⇒ we can get v

(2) Transition matrix ${}_{B_2}P_{B_1}$:

Suppose V is a linear space with two bases
$B_1 = \{u_1, u_2\}$
$B_2 = \{v_1, v_2\}$. Then we can deduce

$${}_{B_2}P_{B_1} = \begin{bmatrix} [u_1]_{B_2} & [u_2]_{B_2} \end{bmatrix}$$

$${}_{B_1}P_{B_2} = \begin{bmatrix} [v_1]_{B_1} & [v_2]_{B_1} \end{bmatrix}$$

★Theorem :-

① ${}_{B_2}P_{B_1} = \left({}_{B_1}P_{B_2}\right)^{-1}$

② $[v]_{B_2} = {}_{B_2}P_{B_1} \, [v]_{B_1}$

(Ex) Let ${}_{B_2}P_{B_1} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$

and $B_1 = \{(1,0), (2,1)\}$. find B_2.

Sol. Notice that ${}_{B_2}P_{B_1} = \begin{bmatrix} [(1,0)]_{B_2} & [2,1]_{B_2} \end{bmatrix}$

It is not useful !! So, we need to find

$$ {}_{B_1}P_{B_2} = {}_{B_2}P_{B_1}^{-1} $$

$$ = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}^{-1} $$

$$ = \begin{bmatrix} 0 & -1/2 \\ -1/2 & -1/4 \end{bmatrix} $$

$$ = \begin{bmatrix} [v_1]_{B_1} & [v_2]_{B_1} \end{bmatrix} $$

Hence $v_1 = 0(1,0) + (-1/2)(2,1)$

$v_2 = -1/2(1,0) + (-1/4)(2,1)$

$\Rightarrow \quad v_1 = (-1, -1/2)$

$v_2 = (-1, -1)$

Hence, $B_2 = \{(-1,-1/2), (-1,-1)\}$.

(Ex) Let $V = \mathbb{R}^3$, B_1 standard basis, and ~~[illegible]~~

If $_{B_2}P_{B_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. find $[(1,5,1)]_{B_2}$

Sol $[(1,5,1)]_{B_2} = {}_{B_2}P_{B_1} [v]_{B_1}$

$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 1 \end{bmatrix} $$

Example: Let $B = \{(2,1), (0,3)\}$ and $B` = \{(-1,0), (3,3)\}$ be two basis of $\mathbb{R}^2$. Find the transmission matrix from $B$ into $B`$?

Solution step 1: find $[(2,1)]_{B`}$. For that suppose that $(2,1) = \lambda_1(-1,0) + \lambda_2(3,3)$.

Then
$$\left.\begin{aligned} -\lambda_1 + 3\lambda_2 &= 2 \\ 3\lambda_2 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda_2 &= 3 \\ \lambda_1 &= -1 \end{aligned}$$

Hence
$$[(2,1)]_{B`} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

step 2: find $[(0,3)]_{B`}$. By the same method, we will deduce that

$$[(0,3)]_{B`} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Therefore,

$$\underset{B`B}{P} = \begin{array}{l}\text{transmission matrix} \\ \text{from } B \text{ into } B`\end{array} = \begin{bmatrix} -1 & 3 \\ \frac{1}{3} & 1 \end{bmatrix}$$

Rules

① $P_{s's^*} \cdot P_{s^*s} = P_{s's}$

② $P_{s^*s} \cdot P_{ss^*} = I \iff P_{ss^*} = \left(P_{s^*s} \right)^{-1}$

III Some Excersices :

① Let $M = \{ (x,y,z) : 2x - y + z = 0 \}$.

(i) Prove M is Linear subspace of $\mathbb{R}^3$?

(ii) find $Dim(M)$

solution

(i) Homework !

(ii) notice that every $(x,y,z) \in M$ is a solution of $2x - y + z = 0$

$$\left.\begin{matrix} N(E) = 1 \\ N(V) = 3 \end{matrix}\right\} \Rightarrow N(P) = 2$$

Let $x = t$, $y = s$

then $z = -2t + s$

So, $M = \left\{ \begin{bmatrix} t \\ s \\ -2t+s \end{bmatrix} ; s, t \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} t \\ 0 \\ -2t \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ s \end{bmatrix} ; s, t \in \mathbb{R} \right\}$

$= \left\{ t \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} ; s, t \in \mathbb{R} \right\}$

Hence $B := \{ (1,0,-2) , (0,1,1) \}$ spans M

since $(1,0,-2) \neq \lambda (0,1,1) \quad \forall \lambda \in \mathbb{R}$

then B is Linear independant

Therefore, B is basis to $M \Rightarrow Dim(M) = 2$

② Let $M = \{ a + bx + cx^2 + dx^3 : a + b = c - 2d = 0 \}$

(i) Prove M is Linear subspace ?

(ii) find $Dim(M)$?

Solution

(i) Homework!

(ii) We have $a + b = 0 \Rightarrow a = -b$ and

$c - 2d = 0 \Rightarrow c = 2d$

So, any Polynomial belongs to M will be written as :

$$-b + bx + 2dx^2 + dx^3$$
$$= b\underbrace{(1-x)}_{\boxed{v_1}} + d\underbrace{(2x^2 - x^3)}_{\boxed{v_2}}$$

So, $B = \{ v_1 = -1 + x \ , \ v_2 = 2x^2 + x^3 \}$ spans M

Notice that $v_1 \neq \lambda v_2 \quad \forall \lambda \in \mathbb{R}$

Hence, B is Linear independent

Therefore, B is basis of $M \Rightarrow Dim(M) = 2$

③ Does $S = \{ (1,3,-1), (0,1,5), (2,2,3) \}$ be a basis of $\mathbb{R}^3$?

<u>Solution</u>

Notice that $|S| = 3 = Dim(\mathbb{R}^3)$. So, it is enough to study if S is Linear independent or not.

suppose that

$\lambda_1 (1,3,-1) + \lambda_2 (0,1,5) + \lambda_3 (2,2,3) = (0,0,0)$. Then, we have

$$\left.\begin{aligned} \lambda_1 + \quad\quad + 2\lambda_3 &= 0 \\ 3\lambda_1 + \lambda_2 + 2\lambda_3 &= 0 \\ -\lambda_1 + 5\lambda_2 + 3\lambda_3 &= 0 \end{aligned}\right\}$$ square + Homogenous system

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & 2 \\ -1 & 5 & 3 \end{vmatrix} \xrightarrow[R_1 + R_3]{-3R_1 + R_2} \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 5 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -4 \\ 5 & 5 \end{vmatrix} = 25 \neq 0$$

So, we have <u>only</u> the zero solution

Hence, S is Linear independent $\Rightarrow$ S is a basis of $\mathbb{R}^3$.

Example Let $\{(1,2),(-1,4)\} = S$ be a set of vectors of $\mathbb{R}^2$. (1) Prove that S is a basis?

(2) find $[(5,6)]_B$?

Solution 1. Homework!

2. suppose that $(5,6) = \lambda_1 (1,2) + \lambda_2 (-1,4)$

then, we have $\lambda_1 - \lambda_2 = 5$

$$2\lambda_1 + 4\lambda_2 = 6$$

Therefore $\lambda_1 = \frac{13}{3}$, $\lambda_2 = -\frac{2}{3}$

Hence $[(5,6)]_B = \begin{bmatrix} \frac{13}{3} \\ \frac{-2}{3} \end{bmatrix}$

Example Let $B=\{ v_1 = 3 , v_2 = -1+x , v_3 = x^2 \}$ be a set of vetors of $P_2(x)$. (1) Prove that B is basis.

(2) find $[1-x^2]_B$?

Solution (1) Notice that $|B| = 3 = \text{Dim}(P_2(x))$

So, It is enough to Prove that B is Linear indep. to show that B is basis. For that

$\lambda_1 (3) + \lambda_2 (-1+x) + \lambda_3 (x^2) = 0 + 0x + 0x^2$

Then, we have

$$\left.\begin{aligned} 3\lambda_1 - \lambda_2 &= 0 \\ \lambda_2 &= 0 \\ \lambda_3 &= 0 \end{aligned}\right\}$$

Hence
$\lambda_1 = \lambda_2 = \lambda_3 = 0$
Therefore, B is Linear indep- which implies that B is basis.

2. suppose that $1-x^2 = \lambda_1 (3) + \lambda_2 (-1+x) + \lambda_3 (x^2)$

Then

$$\left.\begin{aligned} 3\lambda_1 - \lambda_2 &= 1 \\ \boxed{\lambda_2 = 0} \\ \boxed{\lambda_3 = -1} \end{aligned}\right\} \Rightarrow \boxed{\lambda_1 = \frac{1}{3}}$$

Hence $[(1-x^2)]_B = \begin{bmatrix} 1/3 \\ 0 \\ -1 \end{bmatrix}$