Differential and Integral Calculus (Math 203)

BEN AMIRA Aymen

Department of Mathematics

College of Sciences

King Saud University

Chapter 3: Double Integrals

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Definition 1.1 (Riemann sum)

Let f be a function of two variables that is defined on a region R and let $P = \{R_k\}$ be an inner partition of R. A **Riemann sum of f for P** *is any sum of the form*

$$\sum_k f(u_k, v_k)\Delta A_k$$

where (u_k, v_k) is a point in R_k and ΔA_k is the area of R_k. The summation extends over all the subregions $R_1, R_2, \ldots, R_n$ of P.

Double integral

Definition 1.2

Let f be a function of two variables that is defined on a region R. The **double integral of f over R**, *denoted by* $\iint_R f(x, y)\, dA$, *is*

$$\iint\limits_R f(x, y)\, dA = \lim_{\|p\| \to 0} \sum_k f(u_k, v_k) \Delta A_k,$$

provided the limit exists.

Example 1.1

$\iint_R \cos(x^2y+1)\, dA$ for the rectangle R with $-1 \leq x \leq 1$ and $0 \leq y \leq \frac{\pi}{2}$ using Riemann sums with regular partitions of each side of the rectangle and evaluation points where each coordinate is as small as possible.

Double integral

Solution

For positive integers n and m, let the regular partitions be

$$-1 = x_0 < x_1 < x_2 < \cdots < x_n, \; \Delta x = \frac{2}{n}$$

$$0 = y_0 < y_1 < y_2 < \cdots < y_m, \; \Delta x = \frac{\pi}{2m}$$

These regular partitions divide the rectangle R into mn subrectangles each of area $\Delta x \Delta y$. The vertices of each subrectangle are of the form (x_{i-1}, y_{j-1}), (x_{i-1}, y_j), (x_i, y_{j-1}), (x_i, y_j) for $1 \leq i \leq n$, $1 \leq j \leq m$, so the point with the smallest coordinates is (x_{i-1}, y_{j-1}). To add up the terms of the Riemann sum in a systematic way, we first fix a value of i and sum over the rectangles that lie above the subinterval $[x_{i-1}, x_i]$ This gives the sum

$$\sum_{j=1}^{m} f(x_{i-1}, y_{j-1})\Delta x \Delta y = \sum_{j=1}^{m} \cos(x_{i-1}^2 y_{j-1} + 1)\frac{2}{n}\frac{\pi}{2m}$$

Double integral

Next we add up these sums as i varies from 1 to n, which yields a Riemann sum I given by

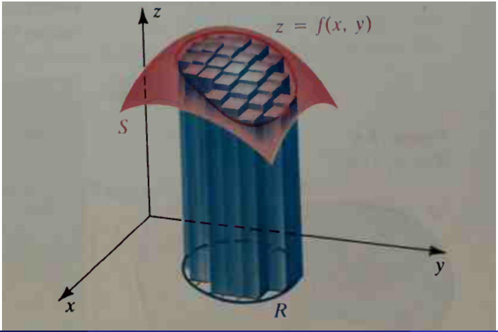
$$\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\cos(x_{i-1}^2 y_{j-1}+1)\frac{2}{n}\frac{\pi}{2m}\right)=\frac{\pi}{nm}\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\cos(x_{i-1}^2 y_{j-1}+1)\right)$$

n	m	I
10	10	0.996659
30	20	0.974834
50	30	0.959850

Definition 1.3

Let f be a continuous function of two variables such that $f(x, y)$ is nonnegative for every (x, y) in a region R. The **volume** *V of the solid that lies under the graph of $z = f(x, y)$ and over R is*

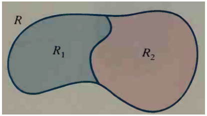
$$V = \iint_R f(x, y)\, dA$$



Double integral

Theorem 1.1

1. $\iint\limits_R cf(x,y)\ dA = c\iint\limits_R f(x,y)\ dA$, *for every real number* c.
2. $\iint\limits_R [f(x,y)+g(x,y)]\ dA = \iint\limits_R f(x,y) + \iint\limits_R g(x,y)\ dA$
3. *If* R *is the union of two nonoverlapping regions* R_1, *and* R_2,
$$\iint\limits_R f(x,y)\ dA = \iint\limits_{R_1} f(x,y) + \iint\limits_{R_2} f(x,y)\ dA$$
4. *If* $f(x,y) \geq 0$ *throughout* R, *then* $\iint\limits_R f(x,y)\ dA \geq 0$



To evaluate the double integral, we begin by the case when the function f is continuous on a closed rectangular region R of the type illustrated in Figure 1.1

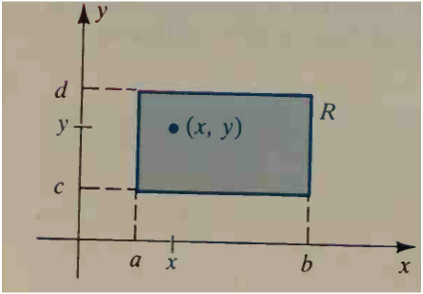


figure 1.1: Rectangular region R

Double integral

It is shown in advanced calculus that the double integral $\iint\limits_R f(x,y)\ dA$ can be evaluated by using an iterated integral of the following type:

Definition 1.4

1. $$\int_a^b \int_b^c f(x,y)\ dy\ dx = \int_a^b \left[\int_b^c f(x,y)\ dy \right] dx$$
2. $$\int_b^c \int_a^b f(x,y)\ dx\ dy = \int_b^c \left[\int_a^b f(x,y)\ dx \right] dy$$

Example 1.2

Evaluate $\displaystyle\int_1^4 \int_{-1}^2 (2x + 6x^2 y)\ dy\ dx.$

Double integral

Solution

$$\begin{aligned}
\int_1^4 \int_{-1}^2 (2x + 6x^2 y)\ dy\ dx &= \int_1^4 \left[2xy + 6x^2 \frac{y^2}{2} \right]_{-1}^2 dx \\
&= \int_1^4 \left[(4x + 12x^2) - (-2x + 3x^2) \right] dx \\
&= \int_1^4 (6x + 9x^2) dx \\
&= \left[3x^2 + 3x^3 \right]_1^4 = 234
\end{aligned}$$

Example 1.3

Evaluate $\int_{-1}^{2}\int_{1}^{4}(2x+6x^2y)\;dx\;dy$.

Solution

$$\begin{aligned}
\int_{-1}^{2}\int_{1}^{4}(2x+6x^2y)\;dx\;dy &= \int_{-1}^{2}\left[x^2+2x^3y\right]_1^4 dy\\
&= \int_{-1}^{2}\left[(16+128y)-(1+2y)\right]dy\\
&= \int_{-1}^{2}(15+126y)dy\\
&= \left[15y+63y^2\right]_{-1}^{2} = 234
\end{aligned}$$

The fact that the iterated integrals in Examples 1.2 and 1.3 are equal is no accident. If f is continuous, then the two iterated integrals defined in definition 1.4 are always equal. We say that the order of integration is immaterial.

An iterated double integral may be defined (in definition 1.5) over an R_x or R_y region of the type shown in the Figure 1.2.

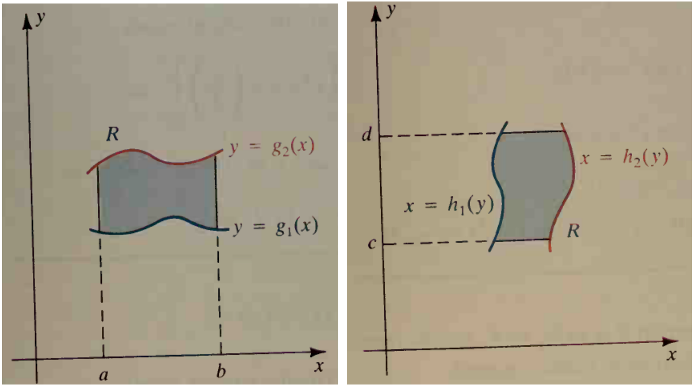


figure 1.2: R_x and R_y

Definition 1.5

1. $$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\right] dx$$
2. $$\int_b^c \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy = \int_b^c \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\right] dy$$

Double integral

Example 1.4

Evaluate the following integrals

1. $\displaystyle\int_0^2 \int_{x^2}^{2x} (x^3 + 4y)\ dy\ dx$
2. $\displaystyle\int_1^3 \int_{\frac{\pi}{6}}^{y^2} 2y \cos x\ dx\ dy$

Solution

1. $$\int_0^2 \int_{x^2}^{2x} (x^3 + 4y)\ dy\ dx = \int_0^2 \left[yx^3 + 2y^2\right]_{x^2}^{2x}\ dx$$
$$= \int_0^2 \left[(2x^4 + 8x^2) - (x^5 + 2x^4)\right]\ dx$$
$$= \int_0^2 (-x^5 + 8x^2) dx = \left[\frac{8x^3}{3} - \frac{x^6}{6}\right]_0^2 = \frac{32}{3}$$

2. $$\int_1^3 \int_{\frac{\pi}{6}}^{y^2} 2y \cos x\ dx\ dy = \int_1^3 2y \left[\sin x\right]_{\frac{\pi}{6}}^{y^2}\ dy$$
$$= \int_1^3 2y(\sin y^2 - \frac{1}{2}) dy$$
$$= \int_1^3 (2y \sin y^2 - y) dy = \left[-\cos y^2 - \frac{y^2}{2}\right]_1^3$$
$$= (-\cos 9 - \frac{9}{2}) + (\cos 1 + \frac{1}{2}) \approx -2.55$$

Double integral

Theorem 1.2 (Evaluation Theorem for Double Integrals)

1. *Let R be the R_x region shown in the left of Figure 1.2. If f is continuous on R, then*
$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$
2. *Let R be the R_y region shown in the right of Figure 1.2. If f is continuous on R, then*
$$\iint_R f(x,y)\,dA = \int_b^c \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

For more complicated regions, we divide R into R_x or R_y subregions, apply Theorem 1.2 to each, and add the values of the resulting integrals.

Example 1.5

Let R be the region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 2x$. Evaluate $\displaystyle\iint\limits_R (x^3 + 4y)\, dA$ when

1. $R = R_x$.
2. $R = R_y$.

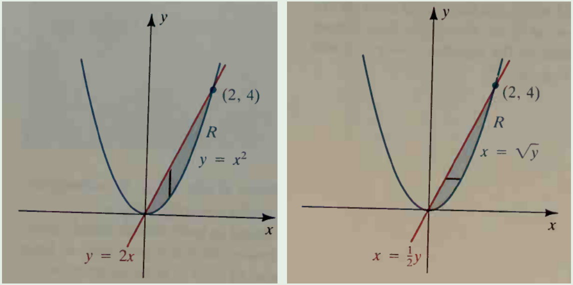


figure 1.3: R_x and R_y region for R

Solution

The region R is sketched in Figure 1.3. Note that R is both an R_x region and an R_y region.

1. Let us regard R as an R_x region having lower boundary $y = x^2$ and upper boundary $y = 2x$, with $0 \leq x \leq 2$ drawn We have a vertical line segment between these boundaries to indicate that the first integration is with respect to y (from the lower boundary to the upper boundary). By Theorem 1.2,

$$\iint_R (x^3 + 4y)\ dA = \int_0^2 \int_{x^2}^{2x} (x^3 + 4y)\ dy\ dx$$

From Example 1.4, we know that the last integral equals to $\dfrac{32}{3}$

Double integral

The region R is sketched in Figure 1.3. Note that R is both an R_x region and an R_y region.

2. Let us regard R as an R_y region having lower boundary $x = \dfrac{y}{2}$ and upper boundary $x = \sqrt{y}$, with $2 \leq y \leq 4$. The horizontal line segment in Figure 1.3 extends from the left boundary to the right boundary, indicating that the first integral is with respect to x. By Theorem 1.2 (2)

$$\iint_R (x^3 + 4y)\ dA = \int_2^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^3 + 4y)\ dx\ dy$$

$$= \int_2^4 \left[\frac{x^4}{4} + 4xy\right]_{\frac{y}{2}}^{\sqrt{y}}\ dy$$

$$= \int_2^4 \left[(\frac{y^2}{4} + 4y^{\frac{3}{2}}) - (\frac{y^4}{64} + 2y^2)\right]\ dy = \frac{32}{3}$$

Example 1.6

Let R be the region bounded by the graphs of the equations $y = \sqrt{x}$, $y = \sqrt{3x - 18}$ and $y = 0$. If f is an arbitrary continuous function on R, express the double integral $\displaystyle\iint_R f(x, y)\, dA$ in terms of iterated integrals using only

1. $R = R_x$.
2. $R = R_y$.

Solution

The graphs of the equations $y = \sqrt{x}$ and $y = \sqrt{3x - 18}$ are the top halves of the parabolas $y^2 = x$ and $y^2 = 3x - 18$. The region R is sketched in Figure 1.4

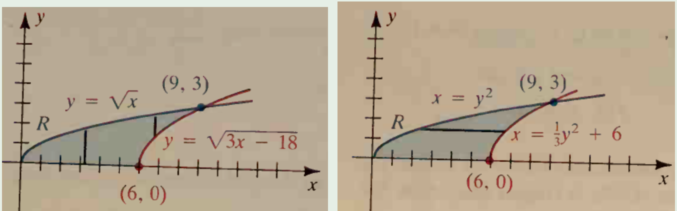


figure 1.4: R_x and R_y region for R

1. If we wish to use only Theorem 1.2 (1), then it is necessary to employ two iterated integrals, because if $0 \leq x \leq 6$, the lower boundary of the region is the graph of $y = 0$, and if $6 \leq x \leq 9$, the lower boundary is the graph of $y = \sqrt{3x - 18}$. If R_1, denotes the part of the region R that lies between $x = 0$ and $x = 6$ and if R_2 denotes the part between $x = 6$ and $x = 9$, then both R_1 and R_2 are R_x regions. Hence,

$$\iint_R f(x,y)\ dA = \iint_{R_1} f(x,y)\ dA + \iint_{R_2} f(x,y)\ dA$$
$$= \int_0^6 \int_0^{\sqrt{x}} f(x,y)\ dy\ dx + \int_6^9 \int_{\sqrt{3x-18}}^{\sqrt{x}} f(x,y)\ dy\ dx$$

2. To use Theorem 1.2 (2), we must solve each of the given equations for x in terms of y, obtaining $x = y^2$ and $x = \dfrac{1}{3}y^2 + 6$, with $0 \leq y \leq 3$ Only one iterated integral is required in this case, since R is an R_y region. Thus,

$$\iint_R f(x,y)\ dA = \int_0^3 \int_{x^2}^{\frac{1}{3}y^2+6} f(x,y)\ dx\ dy$$

Remark 1.1

Generally the choice. of the order of integration $dy\ dx$ or $dx\ dy$ depends on the form of $f(x, y)$ and the region R. Sometimes it is extremely difficult, or even impossible to evaluate a given iterated double integral. However, by reversing the order of integration from $dy\ dx$ to $dx\ dy$, or vice versa, it may be possible to find an equivalent iterated double integral that can be easily evaluated. This technique is illustrated in the next example.

Double integral

Example 1.7

Given $\int_0^4 \int_{\sqrt{y}}^2 y \cos x^5 \; dx \; dy$, reverse the order of integration and evaluate the resulting integral.

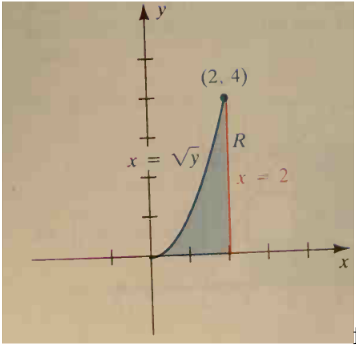


figure 1.5: graph

Solution

The given order of integration, $dx\ dy$, indicates that the region is an R region. As illustrated in Figure 1.5, the left and right boundaries are the graphs of the equations $x = \sqrt{y}$ and $x = 2$ respectively, with $0 \leq y \leq 4$. Note that R is also an R_x region whose lower and upper boundaries are given by $y = 0$ and $y = x^2$ respectively, with $0 \leq x \leq 2$. Hence by Theorem 1.2 (1) the integral can be evaluated as follows:

$$\begin{aligned}
\int_0^4 \int_{\sqrt{y}}^2 y \cos x^5 \ dx\ dy &= \iint_R y \cos x^5 \ dA = \int_0^2 \int_0^{x^2} y \cos x^5 \ dy\ dx \\
&= \int_0^2 \left[\frac{y^2}{2} \cos x^5 \right]_0^{x^2} \ dx = \int_0^2 \left(\frac{x^4}{2} \cos x^5 \right) \ dx \\
&= \frac{1}{10} \int_0^2 \left(5x^4 \cos x^5 \right) \ dx = \frac{1}{10} \left[\sin x^5 \right]_0^2 \\
&= \frac{\sin(32)}{10} \approx 0.055
\end{aligned}$$

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Area and Volume

Volume

Previously, we saw that, if $f(x,y) \geq 0$ and f is continuous, then the volume of the solid that lies under the graph of $z = f(x,y)$ and over a region R in the xy-plane, is given by

$$V = \iint\limits_R f(x,y)\, dA$$

Area

In the volume formula, if we take $f(x,y) = 1$, we find the area of the region R given by

$$\iint\limits_R dA$$

Definition 2.1

If $R = R_x = \{(x,y) \in \mathbb{R}^2 | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, *then*

$$\iint\limits_R dA = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\ dx = \int_a^b [g_2(x) - g_1(x)]\ dx$$

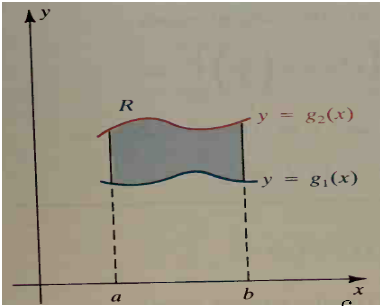


figure 2.1: $R = R_x$

Definition 2.2

If $R = R_x = \{(x,y) \in \mathbb{R}^2 | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$, *then*

$$\iint\limits_R dA = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy = \int_c^d [h_2(y) - h_1(y)]\, dy$$

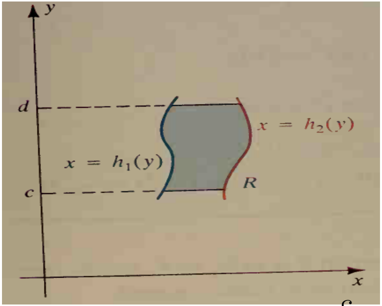


figure 2.2: $R = R_y$

Example 2.1

Consider the triangle $T = \left\{(x, y) \in [0, 1]^2 | x + y \leq 1\right\}$. We have:

$$\mathrm{Area}(T) = \int_0^1 \left(\int_0^{1-x} 1dy\right) dx = \int_0^1 (1 - x)dx = \frac{1}{2}.$$

Example 2.2

An artificially created lake is bordered on one side by a straight dam. The shape of the lake's surface is that of a region in the xy-plane bounded by the graphs of $2y = 16 - x^2$ and $x + 2y = 4$. Find the area A of the surface of the lake.

Solution

we sketch the region and a typical rectangle of area $dy\ dx$, as in Figure 2.3. We solve the equations for y in terms of x, and we label the boundaries of the region $y = 2 - \dfrac{x}{2}$ and $y = 8 - \dfrac{x^2}{2}$.

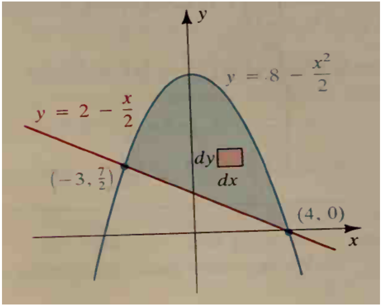


figure 2.3:

Area and Volume

$$
\begin{aligned}
A = \int_{-3}^{4}\int_{2-\frac{x}{2}}^{8-\frac{x^2}{2}} \, dy\, dx &= \int_{-3}^{4} [y]_{2-\frac{x}{2}}^{8-\frac{x^2}{2}} \, dx \\
&= \int_{-3}^{4} \left[\left(8-\frac{x^2}{2}\right) - \left(2-\frac{x}{2}\right)\right] \, dx \\
&= \int_{-3}^{4} \left(6+\frac{x}{2}-\frac{x^2}{2}\right) \, dx \\
&= \left[+\frac{x^2}{4}-\frac{x^3}{6}\right]_{-3}^{4} \\
&= \frac{343}{12} \approx 28.6
\end{aligned}
$$

Example 2.3

If the depth of the lake described in Example 2.2 at the point (x, y) is given by $f(x, y) = x^2 + y$, find the volume of water in the lake.

Solution

Since the depth is nonnegative for all points in the region, the volume V of water is the double integral over the region of the depth function. Thus we have

$$V = \int_{-3}^{4}\int_{2-\frac{x}{2}}^{8-\frac{x^2}{2}} f(x,y)\ dy\ dx$$

$$= \int_{-3}^{4}\int_{2-\frac{x}{2}}^{8-\frac{x^2}{2}} (x^2+y)\ dy\ dx$$

$$= \int_{-3}^{4}\left[x^y + \frac{y^2}{2}\right]_{2-\frac{x}{2}}^{8-\frac{x^2}{2}}\ dx =$$

$$\int_{-3}^{4}\left\{x^2\left(8-\frac{x^2}{2}\right)+\frac{1}{2}\left(8-\frac{x^2}{2}\right)^2-\left[x^2\left(2-\frac{x}{2}\right)+\frac{1}{2}\left(2-\frac{x}{2}\right)^2\right]\right\}\ dx$$

Area and Volume

$$\begin{aligned} V &= \int_{-3}^{4} \left(-\frac{3}{8}x^4 + \frac{1}{2}x^3 + \frac{15}{8}x^2 + x + 30 \right)\, dx \\ &= \left[-\frac{3}{40}x^5 + \frac{1}{8}x^4 + \frac{5}{8}x^3 + \frac{1}{2}x^2 + 30x \right]_{-3}^{4} \\ &= \frac{7889}{40} \\ &= 197.225 \end{aligned}$$

Example 2.4

Find the area A of the region in the xy-plane bounded by the graphs of $x = y^3$, $x + y = 2$, and $y = 0$.

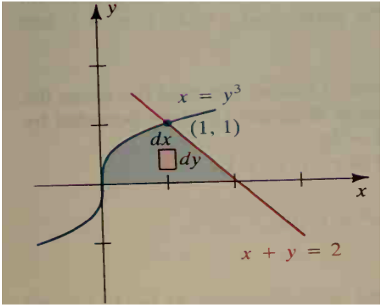


figure 2.4:

Solution

$$\int_0^1 \int_{y^3}^{2-y} \, dx \, dy = \int_0^1 \left[\int_{y^3}^{2-y} \, dx \right] \, dy = \int_0^1 [x]_{y^3}^{2-y} \, dy$$

$$= \int_0^1 (2 - y - y^3) \, dy = \left[2y - \frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = \frac{5}{4}$$

Example 2.5

Consider the triangle $T = \left\{(x,y) \in [0,1]^2 | x+y \leq 1\right\}$. We have:

$$\text{Area}(T) = \int_0^1 \left(\int_0^{1-x} 1dy\right) dx = \int_0^1 (1-x)dx = \frac{1}{2}.$$

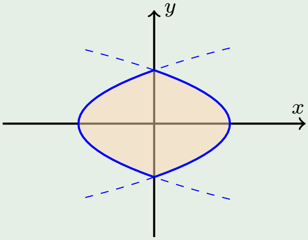
Example 2.6

Consider the disc D of center 0 and radius 1. Then

$$\begin{aligned}
\mathrm{Area}(D) &= \int_{x=-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1dy\right)dx \\
&= 2\int_{x=-1}^{1}\sqrt{1-x^2}dx = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2(\theta)d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos(2\theta))d\theta = \pi
\end{aligned}$$

Example 2.7

Consider the domain $\Omega = \{-\sqrt{2} \leq y \leq \sqrt{2},\ -2 + y^2 \leq x \leq 2 + y^2\}$.



$$\begin{aligned} \iint_{\Omega} dxdy &= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\int_{-2+y^2}^{2-y^2} 1\, dx \right) dy \\ &= \frac{16}{3}\sqrt{2}. \end{aligned}$$

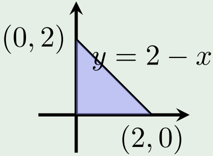
Example 2.8

The volume of the solid that lies under the graph of the function $f(x,y) = 4x^2 + y^2$ and over the region in the $xy-$plane bounded by the polygon with vertices $(0,0), (0,1)$ and $(2,1)$.

$$V = \int_0^2 \int_0^1 (4x^2 + y^2)dydx = \frac{34}{3}.$$

Example 2.9

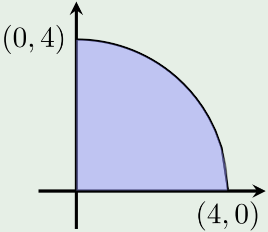
The volume of the solid in the first octant bounded by the graphs of equations
$z = 4 - x^2$, $x + y = 2$, $x = 0$, $y = 0$, $z = 0$.



$$V = \int_0^2 \int_0^{2-x} (4 - x^2)\,dy\,dx = \frac{20}{3}.$$

Example 2.10

The volume of the solid in the first octant bounded by the graphs of equations $z = x$, $x^2 + y^2 = 16$, $x = 0$, $y = 0$.



$$V = \int_0^4 \int_0^{\sqrt{16-x^2}} x\,dy\,dx = \frac{64}{3}.$$

Example 2.11

Find the volume V of the solid in the first octant bounded by the coordinate planes, the paraboloid $z = x^2 + y^2 + 1$ and the plane $2x + y = 2$.

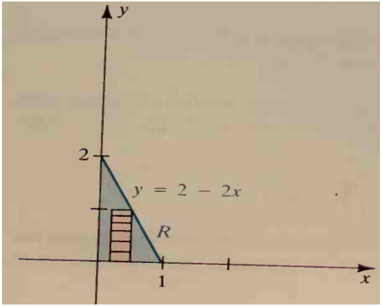


figure 2.5:

Solution

$$V = \int_0^1 \int_0^{2-2x} (x^2 + y^2 + 1)\ dy\ dx \quad = \int_0^1 \left[x^2 y + \frac{y^3}{3} + y\right]_0^{2-2x}\ dx$$
$$= \int_0^1 \left(-\frac{14}{3}x^3 + 10x^2 - 10x + \frac{14}{3}\right)\ dx$$
$$= \left[-\frac{7}{6}x^4 + \frac{10}{3}x^3 - 5x^2 + \frac{14}{3}x\right]_0^1 = \frac{11}{6}$$

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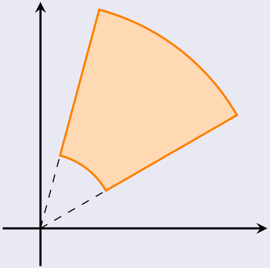
Double Integrals in Polar Coordinates

Definition 3.1

Polar coordinates are defined by $x = r\cos\theta, y = r\sin\theta$. *The area of the shaded region* $R = \{(r,\theta) : a \le r \le b, \quad \alpha \le \theta \le \beta\}$.



The integral of a continuous function $f(x,y)$ *over a polar rectangle* R *given by* $a \le r \le b$, $\alpha \le r \le \beta$, *is*

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example 3.1

1. Find $\iint_R (2x-y)\,dA$ if R is the region in the first quadrant bounded by the circle $x^2+y^2=4$ and the lines $x=0$ and $y=x$.
2. Find $\iint_R e^{-x^2-y^2}\,dA$ if D is the region bounded by the semicircle $x=\sqrt{4-y^2}$ and the y-axis.

Double Integrals in Polar Coordinates

Definition 3.2

If f is continuous over a polar region of the form

$$D = \{(r,\theta) : \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

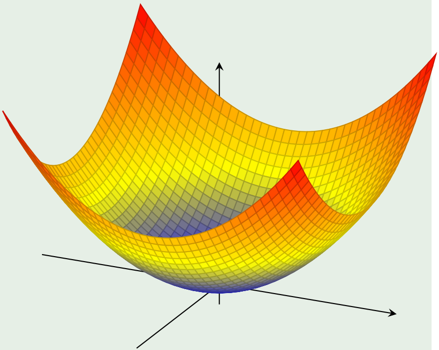
$$\iint_D f(x,y)\,dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Double Integrals in Polar Coordinates

Example 3.2

The volume under the paraboloid $z = f(x, y) = x^2 + y^2$ and above the disc $x^2 + y^2 < 3$.
In polar coordinates, the disc is parameterized by: $x = r\cos\theta$, $y = r\sin\theta$, with $r \in [0, 3]$ and $\theta \in [0, 2\pi]$. Since polar coordinates $f(x, y)$ is transformed to $g(r, \theta) = r^2$. Hence the volume under the paraboloid $z = x^2 + y^2$ and above the disc $x^2 + y^2 < 3$ is

$$V = \int_0^3 \int_0^{2\pi} r^2 dr d\theta = 18\pi.$$



Exercise 3.1

Use polar coordinate to evaluate

1. $$\int_0^a \int_0^{\sqrt{a^2-y^2}} y\,dx\,dy$$
2. $$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)^{\frac{3}{2}}\,dx\,dy$$
3. $I = \iint_R x^2(x^2+y^2)^3\,dA$, with R is the region bounded by the semi-circle $y=\sqrt{1-x^2}$ and x-axis.
4. $J = \iint_R (x+y)\,dA$, with R is the region bounded by the circle $x^2+y^2 \leqslant 2y$.
5. $K = \iint_R \sqrt{x^2+y^2}\,dA$, with R is the region bounded by the triangle with vertices $(0,0), (3,0)$ and $(3,3)$.

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Surface Area

Consider a surface S defined by $z = f(x, y)$, for (x, y) in a closed region $R \in \mathbb{R}^2$. We assume that $f(x, y) \geq 0$ and f is continuously differentiable. We assume also that no normal vector to S is parallel to the $xy-$plane. $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \neq (0, 0)$ for all $(x, y) \in R$. The surface area of S is

$$A = \iint_R \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} dxdy.$$

Example 4.1

Consider the surface $z = 1 - x^2 - y^2$, for $z \geqslant 0$.
We have $\dfrac{\partial f}{\partial x} = -2x$ and $\dfrac{\partial f}{\partial y} = -2y$. Then

$$\begin{aligned} A &= \iint_{x^2+y^2<1} \sqrt{1+4x^2+4y^2}\,dxdy \\ &= \int_0^{2\pi}\int_0^1 r\sqrt{1+4r^2}\,drd\theta \\ &= 2\pi\frac{1}{12}\left[(1+4r^2)^{\frac{3}{2}}\right]_0^1 = \frac{\pi.(5^{\frac{3}{2}}-1)}{6} \end{aligned}$$

Surface Area

Example 4.2

Consider the sphere $S = \{(x,y,z) \in \mathbb{R}^3 : \; x^2+y^2+z^2 = R^2\}$. The area of S is the double of the area of the surface of the upper half sphere $S' = \{(x,y,z) \in \mathbb{R}^3 : \; z = \sqrt{1-x^2-y^2}\}$.
We have $\dfrac{\partial z}{\partial x} = -\dfrac{x}{\sqrt{R^2-x^2-y^2}}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{y}{\sqrt{R^2-x^2-y^2}}$.
The area is

$$\begin{aligned} A &= 2\iint_{x^2+y^2<R^2} \sqrt{1+\frac{x^2+y^2}{R^2-x^2-y^2}}\,dxdy \\ &= 2R\int_0^{2\pi}\int_0^R \frac{r}{\sqrt{R^2-r^2}}\,drd\theta = 4\pi R^2. \end{aligned}$$

Example 4.3

Consider the surface $S = \{(x, y, z) \in \mathbb{R}^3 : \; z = 13 - 4x^2 - 4y^2\}$ on the domain $z = 1$, $x > 0$ and $y < 0$.

We have $\dfrac{\partial z}{\partial x} = -8x$ and $\dfrac{\partial z}{\partial y} = -8y$. Then the area of S is

$$\begin{aligned} A &= \iint_{x^2+y^2<3, x>0, y<0} \sqrt{1+16(x^2+y^2)}\,dxdy \\ &= \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{3}} \sqrt{1+16r^2}\,rdrd\theta \\ &= \frac{\pi}{2}\frac{1}{48}\left[(1+16r^2)^{\frac{3}{2}}\right]_0^{\sqrt{3}} = \frac{\pi}{2}. \end{aligned}$$

Exercise 4.1

Find the area of the surface S if S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that is inside the cylinder $x^2 + y^2 = 2y$.