Integral Calculus (Math 228)

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Chapter 1: Integrals

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Substitution Method (changing variable method)

Theorem (Substitution)

If F is an antiderivative of f , then $f(g(x))g'(x)$ has antiderivative $F(g(x))$. Or,

$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

This is obvious. It is called "substitution" since it can be obtained by substituting $u = g(x)$ and $du = g'(x)dx$ into $\int f(u)du = F(u) + c$

Substitution Method (changing variable method)

Example

Solve $\int (4x+1)^2 dx.$

Put $u = 4x+1$ then $du = 4dx$ hence $\frac{1}{4}du = dx$

$$\int (4x+1)^2 dx = \int u^2 \frac{1}{4} du = \frac{1}{4} \int u^2 du = \frac{1}{4} \frac{u^3}{3} + C$$

$$= \frac{1}{4} \frac{(4x+1)^3}{3} + C$$

Example

1. $\displaystyle\int (x^2+1)^n 2x dx \overset{u=x^2+1}{=} \int u^n du = \frac{u^{n+1}}{n+1} = \frac{(x^2+1)^{n+1}}{n+1} + c.$
2. $\displaystyle\int \sin(2x+3) dx \overset{u=2x+3}{=} \frac{1}{2}\int \sin u du = -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos(2x+3) + c.$
3. $\displaystyle\int \frac{1}{\cos^2(\pi x)} dx \overset{u=\pi x}{=} \frac{1}{\pi}\int \frac{1}{cos^2(u)} du = \frac{1}{\pi}\tan(\pi x) + c.$

Example

$$\int (20x+30)(x^2+3x-5)^9\ dx = \int 10(2x+3)(x^2+3x-5)^9\ dx$$
$$= \int 10(\underbrace{x^2+3x-5}_{u})^9\ \underbrace{(2x+3)\ dx}_{du}$$
$$= \int 10u^9\ du$$
$$= u^{10}+C \quad (\text{replace } u \text{ with } x^2+3x-5)$$
$$= (x^2+3x-5)^{10}+C$$

Substitution Method (changing variable method)

Theorem (Substitution Rule for Definite Integrals)

If g' is continuous on $[a, b]$, and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

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Integration By Parts

It is used to solve integration of a product of two functions using the formula:

$$\int u\ dv = uv - \int v\ du$$

1. $\displaystyle\int xe^x dx$, We put, $u = x \quad dv = e^x\ dx$, Then $du = dx \quad v = e^x$
$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$$

2. $\displaystyle\int_0^\pi x \sin x\ dx$, We put $u = x \quad dv = \sin x\ dx$, Then,
$du = dx \quad v = -\cos x$
$$\int_0^\pi x \sin x\ dx = [-x \cos x]_0^\pi + \int_0^\pi \cos x dx = [-x \cos x]_0^\pi + [\sin x]_0^\pi$$
$$[(-\pi \cos \pi) - (-(0) \cos 0)] + [\sin \pi - \sin 0] = \pi$$

Integration By Parts: Examples

- $\displaystyle\int xe^x \, dx = (x-1)e^x + c$
- $\displaystyle\int x^2e^x \, dx = (x^2 - 2x + 2)e^x + c$
- $\displaystyle\int x^3e^x \, dx = (x^3 - 3x^2 + 6x - 6)e^x + c$

Integration By Parts: Examples

- $\displaystyle\int x\cos x\ dx = x\sin x + \cos x + c$
- $\displaystyle\int x^2\cos x\ dx = (x^2-2)\sin x + 2x\cos x + c$
- $\displaystyle\int x^3\cos x\ dx = (x^3-6x)\sin x + (3x^2-6)\cos x + c$
- $\displaystyle\int x^4\cos x\ dx = (x^4-12x^2+24)\sin x + (4x^3-24x)\cos x + c$

Integration By Parts: Examples

- $\int x \sin x \; dx = -x \cos x + \sin x + c$
- $\int x^2 \sin x \; dx = (-x^2 + 2) \cos x + 2x \sin x + c$
- $\int x^3 \sin x \; dx = (-x^3 + 6x) \cos x + (3x^2 - 6) \sin x + c$
- $\int x^4 \sin x \; dx = (-x^4 + 12x^2 - 24) \cos x + (4x^3 - 24x) \sin x + c$

Evaluate $\int \cos(\ln(x))\ dx$.
Letting: $u = \ln(x)$, we have $du = 1/x\ dx$.

$$du = \frac{1}{x}\ dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln(x)$, we can use inverse functions and conclude that $e^{\ln(x)} = e^u \Rightarrow x = e^u$. therefore we have that $dx = x \cdot du = e^u\ du$.

$$\begin{aligned}\int \cos(\ln(x))\ dx &= \int e^u \cos(u)\ du \\ &= \frac{1}{2} e^u \big(\sin(u) + \cos(u) \big) + C \\ &= \frac{1}{2} e^{\ln(x)} \big(\sin(\ln(x)) + \cos(\ln(x)) \big) + C \\ &= \frac{1}{2} x \big(\sin(\ln(x)) + \cos(\ln(x)) \big) + C.\end{aligned}$$

$$\int e^x \cos x \, dx$$

$$\begin{array}{ll} u = \cos x & dv = e^x \ dx \\ du = -\sin x \ dx & v = e^x \end{array}$$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

Now to solve $\int e^x \sin x \, dx$

$$\begin{array}{ll} u = \sin x & dv = e^x \ dx \\ du = \cos x \ dx & v = e^x \end{array}$$

Therefore , $\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

$$\int e^x \cos x \, dx = \frac{1}{2} \left[e^x \cos x + e^x \sin x \right] + c \ .$$

Integration By Parts: Examples

$$\int \ln|x|\,dx$$

$$u = \ln|x| \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \qquad v = x$$

$$\int \ln|x|\,dx = x\ln|x| - \int x\,\frac{1}{x}\,dx = x\ln|x| - \int dx = x\ln|x| - x + c$$

Integration By Parts: Examples

$$\int \tan^{-1} x \, dx$$

$$u = \tan^{-1} x \qquad dv = dx$$

$$du = \frac{1}{1+x^2} \, dx \qquad v = x$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int x \, \frac{1}{1+x^2} \, dx$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c$$

$$\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx$$

$u = \sec x \qquad\qquad dv = \sec^2 x\ dx$

$du = \sec x \tan x\ dx \qquad v = \tan x$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right)\ dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + c$$

Integration By Parts: Examples

$$\int \ln(1+x^2)\,dx$$

$$u = \ln(1+x^2) \qquad dv = dx$$

$$du = \frac{2x}{1+x^2}\,dx \qquad v = x$$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{(2x^2+2)-2}{1+x^2}\,dx$$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2(x^2+1)}{1+x^2}\,dx + 2\int \frac{1}{1+x^2}\,dx$$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - 2x + 2\tan^{-1}x + c$$

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Integrals Involving Trigonometric Functions

First form

Integrals of the forms

$$\int \sin ax \cos bx \ dx, \quad \int \sin ax \sin bx \ dx, \quad \int \cos ax \cos bx \ dx$$

Where $a, b \in \mathbb{Z}$

1. The integral $\int \sin ax \cos bx \ dx$ can be solved using the formula $\sin ax \cos bx = \frac{1}{2}[\sin(ax + bx) + \sin(ax - bx)]$
2. The integral $\int \sin ax \sin bx \ dx$ can be solved using the formula $\sin ax \sin bx = \frac{1}{2}[\cos(ax - bx) - \cos(ax + bx)]$
3. The integral $\int \cos ax \cos bx \ dx$ can be solved using the formula $\cos ax \cos bx = \frac{1}{2}[\cos(ax + bx) + \cos(ax - bx)]$

Integrals Involving Trigonometric Functions (Examples)

1. $\int \sin 3x \cos 2x \; dx = \frac{1}{2} \int [\sin 5x + \sin x] dx =$
$\frac{1}{2} \int \sin 5x \; dx + \frac{1}{2} \int \sin x \; dx = -\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + c$
2. $\int \sin x \sin 3x \; dx = \frac{1}{2} \int [\cos 2x - \cos 4x] dx =$
$\frac{1}{2} \int \cos 2x \; dx - \frac{1}{2} \int \cos 4x \; dx = \frac{1}{4} \sin 2x - \frac{1}{8} \sin 4x + c$
3. $\int \cos 5x \cos 2x \; dx = \frac{1}{2} \int [\cos 7x + \cos 3x] dx =$
$\frac{1}{2} \int \cos 7x \; dx + \int \cos 3x \; dx = \frac{1}{4} \sin 7x + \frac{1}{6} \sin 3x + c$

Integrals Involving Trigonometric Functions

Second form

Integrals of the forms

$$\int \sin^n x \cos^m x \; dx, \quad \int \sinh^n x \cosh^m x \; dx, \qquad Where\ n, m \in \mathbb{N}$$

The above two integrals can be solved by substitution if n or m is odd.

1. If n is odd: The substitution $u = \cos x$ can be used to solve $\int \sin^n x \cos^m x \; dx$
   The substitution $u = \cosh x$ can be used to solve $\int \sinh^n x \cosh^m x \; dx$
2. If m is odd: The substitution $u = \sin x$ can be used to solve $\int \sin^n x \cos^m x \; dx$
   The substitution $u = \sinh x$ can be used to solve $\int \sinh^n x \cosh^m x \; dx$

1. Evaluate $I = \int \sin^5 x \cos^4 x dx$

$$\int \sin^5 x \cos^4 x dx = \int (\sin^2 x)^2 \cos^4 x \sin x dx$$
$$= \int (1 - \cos^2 x) \cos^4 x \sin x \ dx$$

to solve this integral put

$$u = \cos x \Rightarrow -du = \sin x \ dx$$

$$I = -\int (1 - u^2)^2 u^4 du = -\int u^4 - 2u^6 + u^8 du$$
$$= -\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right] + c = -\left[\frac{\cos^5 x}{5} - \frac{2\cos^7 x}{7} + \frac{\cos^9 x}{9}\right] + c$$

2. Evaluate $I = \int \sin^7 \cos^3 x \ dx$

$\int \sin^7 \cos^3 x \ dx = \int \sin^7 x(1 - \sin^2 x) \cos x \ dx$

to solve this integral put

$$u = \sin x \Rightarrow du = \cos x \ dx$$

$$I = \int u^7(1 - u^2) \ du = \int u^7 - u^9 \ du = \frac{u^8}{8} - \frac{u^{10}}{10} + c$$

$$= \frac{\sin^8 x}{8} - \frac{sin^{10}}{10} + c$$

3. $\int \sinh^3 x \cosh^2 x \; dx$ to solve this integral put
$u = \cosh x \Rightarrow du = \sinh x$
$\int \sinh^3 x \cosh^2 x \; dx = \int (\cosh^2 x - 1) \cosh^2 x \sinh x \; dx =$
$\int (u^2 - 1) u^2 du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + c$

$$= \frac{\cosh^5 x}{5} - \frac{\cosh^3 x}{3} + c$$

$$\int \sqrt{\sin x}\ \cos^3 x\, dx = \int \sqrt{\sin x}\ \cos^2 x\ \cos x\, dx$$

$$= \int (\sin x)^{\frac{1}{2}}\ \left(1 - \sin^2 x\right)\ \cos x\, dx$$

Put $u = \sin x \Rightarrow du = \cos x\ dx$

$$\int \sqrt{\sin x}\ \cos^3 x\, dx = \int u^{\frac{1}{2}}(1 - u^2)\ du = \int \left(u^{\frac{1}{2}} - u^{\frac{5}{2}}\right)\ du$$

$$= \frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{7}{2}}}{7} + c = \frac{2\,(\sin x)^{\frac{3}{2}}}{3} - \frac{2\,(\sin x)^{\frac{7}{2}}}{7} + c$$

$$\int \frac{\sin^3 x}{\cos^2 x}\,dx = \int \sin^2 x\ \cos^{-2} x\ \sin x\,dx = \int \left(1-\cos^2 x\right)\ \cos^{-2} x\ \sin x\,dx$$

Put $u = \cos x \Rightarrow -du = \sin x\ dx$

$$\int \frac{\sin^3 x}{\cos^2 x}\,dx = -\int \left(1-u^2\right) u^{-2}\ du = -\int \left(u^{-2}-1\right) du$$

$$= -\frac{u^{-1}}{-1} + u + c = \frac{1}{u} + u + c = \sec x + \cos x + c$$

Integrals Involving Trigonometric Functions (Examples)

$$\int \sin^7 x \ \cos^3 x \, dx = \int \sin^7 x \ \cos^2 x \ \cos x \, dx$$

$$= \int \sin^7 x \ (1 - \sin^2 x) \ \cos x \, dx$$

Put $u = \sin x \Rightarrow du = \cos x \ dx$

$$\int \sin^7 x \ \cos^3 x \, dx = \int u^7 \ (1 - u^2) \ du = \int (u^7 - u^9) \ du$$

$$= \frac{u^8}{8} - \frac{u^{10}}{10} + c = \frac{\sin^8 x}{8} - \frac{\sin^{10} x}{10} + c$$

Special cases :

1. $$\int \sin^2 x \; dx = \frac{1}{2}\int [1 - \cos 2x] \; dx = \frac{1}{2}[x - \frac{\sin 2x}{2}] + c$$
2. $$\int \cos^2 x \; dx = \frac{1}{2}\int [1 + \cos 2x] \; dx = \frac{1}{2}[x + \frac{\sin 2x}{2}] + c$$

Third form

Integrals of the forms

$$\int \sec^n x \tan^m x \; dx, \quad \int csc^n x \cot^m x \; dx,$$

$$\int sech^n x \tanh^m x \; dx, \quad \int csch^n x \coth^m x \; dx$$

The above four integrals can be solved by substitution if n is even or m is odd.

1. If n is even:
   The substitution $u = \tan x$ can be used to solve $\int \sec^n x \tan^m x \; dx$.
   The substitution $u = \cot x, \; u = \tanh x$ and $u = \coth x$ can be used to solve the other three integrals respectively.
2. If m is odd:
   The substitution $u = \sec x$ can be used to solve $\int \sec^n x \tan^m x \; dx$.
   The substitutions $u = csc \;\, x, u = sech \;\, x$ and $u = csch \;\, x$ can be used to solve the other theree integrals respictively.

Integrals Involving Trigonometric Functions (Examples)

1. Evaluate $I = \int \tan^3 x \sec^3 x \; dx$

   to solve this integral put: $u = \sec x \Rightarrow du = \sec x \tan x \; dx$

   $I = \int \tan^3 x \sec^3 x \; dx = \int (\sec^2 x - 1) \sec^2 x \sec x \tan x \; dx$

   $$= (u^2 - 1)u^2 \; du = \int u^4 - u^2 \; du$$

   $$= \frac{u^5}{5} - \frac{u^3}{3} + c$$

   $$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c$$

2. Evaluate $I = \int tanh^3 x\ sech\ x\ dx$

   to solve this integral put: $u = sech\ x \Rightarrow -du = sech\ x \tanh x\ dx$

$$I = \int tanh^3 x\ sech\ x\ dx = \int (1 - sech^2 x) sech\ x\ tanh\ x\ dx$$

$$= -\int (1 - u^2) du = -u + \frac{u^3}{3} + c$$

$$= -sech\ x + \frac{sech^3 x}{3} + c$$

$$\int \csc^4 x \ \cot^4 x \, dx$$

$$= \int \csc^2 x \ \cot^4 x \ \csc^2 x \, dx = \int (1 + \cot^2 x) \cot^4 x \ \csc^2 x \, dx$$

Put $u = \cot x \Rightarrow -du = \csc^2 x \ dx$

$$\int \csc^4 x \ \cot^4 x \, dx = -\int (1 + u^2) u^4 \ du = -\int (u^4 + u^6) \ du$$

$$= -\frac{u^5}{5} - \frac{u^7}{7} + c = -\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + c$$

Integrals Involving Trigonometric Functions (Examples)

$$\int \frac{\sec^4 x}{\sqrt{\tan x}}\, dx$$

$$\int \sec^2 x \ (\tan x)^{-\frac{1}{2}} \ \sec^2 x\, dx = \int (1+\tan^2 x) \ (\tan x)^{-\frac{1}{2}} \ \sec^2 x\, dx$$

Put $u = \tan x \Rightarrow du = \sec^2 x \ dx$

$$\int \frac{\sec^4 x}{\sqrt{\tan x}}\, dx = \int (1+u^2)u^{-\frac{1}{2}} \ du = \int \left(u^{-\frac{1}{2}} + u^{\frac{3}{2}}\right) \ du$$

$$= 2u^{\frac{1}{2}} + \frac{2u^{\frac{5}{2}}}{5} + c = 2(\tan x)^{\frac{1}{2}} + \frac{2(\tan x)^{\frac{5}{2}}}{5} + c$$

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Trigonometric Substitutions

If the integrand contains a term of the form
$\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ where $a > 0$,
then trigonometric substitutions can be used to solve the integral.

1. An integral involving $\sqrt{a^2 - x^2}$ use the substitution $x = a \sin \theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to solve the integral.
2. An integral involving $\sqrt{a^2 + x^2}$ use the substitution $x = a \tan \theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to solve the integral.
3. An integral involving $\sqrt{x^2 - a^2}$ use the substitution $x = a \sec \theta$ where $0 \leq \theta < \frac{\pi}{2}$ to solve the integral.

Trigonometric Substitutions (examples)

Solve the following integral: $\int \frac{1}{x^2\sqrt{16-x^2}}dx$

$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \int \frac{1}{x^2\sqrt{(4)^2-x^2}}dx,$$

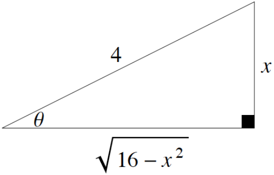
Put $x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta$

$$I = \int \frac{4\cos\theta}{16\sin^2\theta\sqrt{16-16\sin\theta}}d\theta$$

$$= \int \frac{4\cos\theta}{16\sin^2\theta\ 4\cos\theta}d\theta$$

$$= \frac{1}{16}\int \frac{1}{\sin^2\theta}d\theta = \frac{1}{16}\int \csc^2\theta d\theta =$$

$$-\frac{1}{16}\cot\theta + c$$



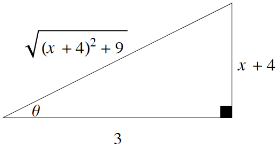
$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = -\frac{1}{16}\frac{\sqrt{16-x^2}}{x} + c$$

Solve the following integral:

$$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx$$

$$I = \int \frac{1}{[(x+4)^2+3^2]^{\frac{3}{2}}}\ dx.$$

Put $x+4 = 3\tan\theta \Rightarrow dx = 3\sec^2\theta\ d\theta$



$$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx = \int \frac{3\sec^2\theta}{(9\tan^2\theta+9)^{\frac{3}{2}}}\ d\theta$$

$$= \int \frac{3\sec^{\theta}}{(9\sec^2\theta)\frac{3}{2}}\ d\theta = \int \frac{3\sec^2\theta}{27\sec^3\theta} d\theta$$

$$= \tfrac{1}{9}\int \frac{1}{\sec\theta} d\theta = \frac{1}{9}\sin\theta + c$$

$$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx = \frac{1}{9}\frac{x+4}{\sqrt{x^2+8x+25}} + c$$

Solve the following integral: $\displaystyle\int \frac{\sqrt{x^2-4}}{x^2}\, dx$

Put $x = 2\sec\theta \Rightarrow dx = 2\sec\theta\tan\theta\ d\theta$

$$\int \frac{\sqrt{x^2-4}}{x^2}\, dx =$$

$$\int \frac{\sqrt{4\sec^2\theta - 4}\ 2\sec\theta\tan\theta}{4\sec^2\theta} d\theta$$

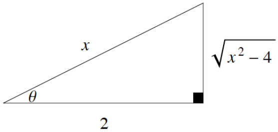
$$= \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)}{4\sec^2\theta} d\theta$$

$$= \int \frac{\tan^2\theta}{\sec\theta} d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta} d\theta$$

$$= \int \frac{\sec^2\theta}{\sec\theta} d\theta - \int \frac{1}{\sec\theta} d\theta$$

$$= \int \sec\theta d\theta - \int \cos\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + c$$



$$\int \frac{\sqrt{x^2-4}}{x^2}\, dx = \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| - \frac{\sqrt{x^2-4}}{x} + c$$

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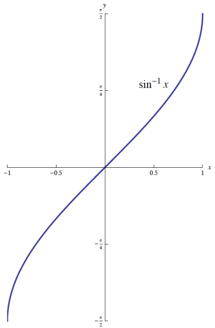
Definition (Inverse if sine)

The inverse sine function is denoted by $\sin^{-1}$ and it is defined as
$y = \sin^{-1} x \Leftrightarrow x = \sin y$, where $x \in [-1, 1]$ and $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
The domain of the inverse sine function is $[-1, 1]$
The range of the inverse sine function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

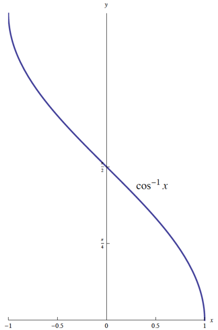


Graph of $\sin^{-1} x$

Inverse Trigonometeric

Definition (Inverse of cosine)

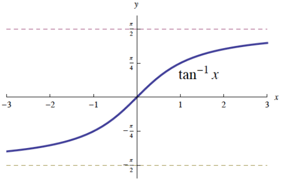
The inverse cosine function is denoted by $\cos^{-1}$ and it is defined as $y = \cos^{-1} x \Leftrightarrow x = \cos y$, where $x \in [-1, 1]$ and $y \in [0, \pi]$
The domain of the inverse cosine function is $[-1, 1]$
The range of the inverse cosine function is $[0, \pi]$.



Graph of $\cos^{-1} x$

Definition (Inverse of Tangent)

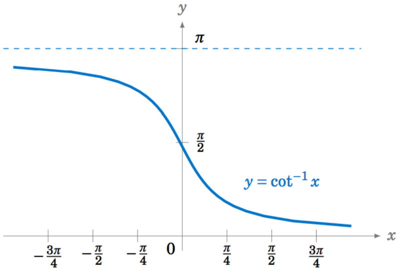
The inverse tangent function is denoted by $\tan^{-1}$ and it is defined as $y = \tan^{-1} x \Leftrightarrow x = \tan y$, where $x \in \mathbb{R}$ and $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$
The domain of the inverse tangent function is $\mathbb{R}$
The range of the inverse tangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$.



Graph of $\tan^{-1} x$

Definition (Inverse of cotangent)

The inverse cotangent function is denoted by $\cot^{-1}$ and it is defined as $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$, where $x \in \mathbb{R}$
The domain of the inverse cotangent function is $\mathbb{R}$
The range of the inverse cotangent function is $(0, \pi)$.



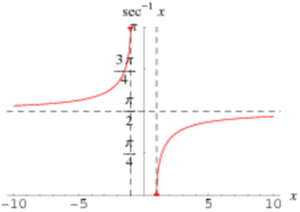
Inverse Trigonometeric

Definition (Inverse secant)

The inverse secant function is denoted by $\sec^{-1}$ and it is defined as $y = \sec^{-1} x \Leftrightarrow x = \sec y$, where $y \in [0, \frac{\pi}{2})$ if $x \geq 1$, and $y \in [\pi, \frac{3\pi}{2})$ if $x \leq -1$

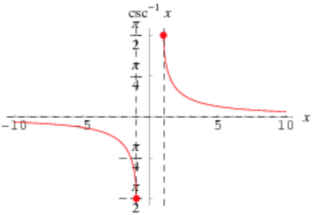
The domain of the inverse secant function is $(-\infty, -1] \cup [1, \infty)$

The range of the inverse secant function is $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.



Definition (Inverse cosecant)

The inverse cosecant function is denoted by $\csc^{-1}$ and it is defined as $\csc^{-1} x = \frac{\pi}{2} - sec^{-1}x$, where $|x| \geq 1$
The domain of the inverse cosecant function is $(-\infty, -1] \cup [1, \infty)$
The range of the inverse cosecant function is $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.



Derivatives of the inverse trigonometric functions

1. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, where $|x| < 1$
2. $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$, where $|x| < 1$
3. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$
4. $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$
5. $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{1-x^2}}$, where $|x| > 1$
6. $\frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$, where $|x| > 1$

Inverse Trigonometeric

Integration of the inverse trigonometric functions

1. $$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}(\frac{x}{a}) + c, \ (|x| < a)$$
   $$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1}(\frac{f(x)}{a}) + c, \ (|f(x)| < a)$$
2. $$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + c$$
   $$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a} \tan^{-1}(\frac{f(x)}{a}) + c$$
3. $$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}(\frac{x}{a}) + c, \ (|x| > a)$$
   $$\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - a^2}} dx = \frac{1}{a} \sec^{-1}(\frac{f(x)}{a}) + c, \ (|f(x)| > a)$$

Inverse Trigonometeric

Examples

1. $$\int \frac{x^2}{5+x^6}dx = \frac{1}{3}\int \frac{3x^2}{(\sqrt{5})^2+(x^3)^2}dx = \frac{1}{3}\frac{1}{\sqrt{5}}\tan^{-1}(\frac{x^3}{\sqrt{5}}) + c.$$
   Here $a=\sqrt{5}$, $f(x)=x^3$ and $f'(x)=3x^2$.
2. $$\int \frac{3x}{\sqrt{9-x^4}}dx = \frac{3}{2}\int \frac{2x}{\sqrt{3^2-(x^2)^2}}dx = \frac{3}{2}\sin^{-1}(\frac{x^2}{3}) + c.$$
3. $$\int \frac{3x}{\sqrt{9-x^2}}dx = \frac{3}{-2}\int (9-x^2)^{-\frac{1}{2}}(-2x)dx = -\frac{3}{2}\frac{(9-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$
4. $$\int \frac{1}{x\sqrt{1-(\ln x)^2}}dx = \int \frac{(\frac{1}{x})}{\sqrt{(1)^2-(\ln x)^2}}dx = \sin^{-1}(\ln x) + c$$
5. $$\int \frac{1}{1+3x^2}dx = \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}}{1^2+(\sqrt{3}x)^2}dx = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + c.$$

Inverse Trigonometeric

Examples

6. $$\int \frac{e^{2x}}{e^{4x}+16}dx = \frac{1}{2}\int \frac{2e^{2x}}{4^2+(2^{2x})^2}dx = \frac{1}{2*4}\tan^{-1}\left(\frac{e^{2x}}{4}\right)+c.$$
7. $$\int \frac{1}{\sqrt{e^{2x}-36}}dx = \int \frac{e^x}{e^x\sqrt{(e^x)^2-(6)^2}}dx = \frac{1}{6}\sec^{-1}\left(\frac{e^x}{6}\right)+c.$$

Exercises

Solve the following integrals :

1. $$\int \frac{x+\sin^{-1}x}{\sqrt{1-x^2}}dx$$
2. $$\int \frac{x+1}{x^2+1}dx$$

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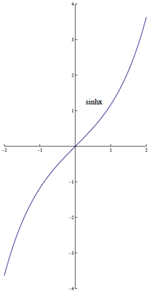
Hyperbolic Functions

Definition (The hyperbolic sine function)

It is denoted by $\sinh x$ and it is defined as $\sinh x = \frac{e^x - e^{-x}}{2}$

Notes

1. The domain of $\sinh x$ is $\mathbb{R}$ and the range of $\sinh x$ is $\mathbb{R}$.
2. It is an odd function and $\sinh(0) = 0$



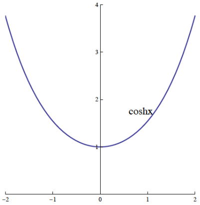
Hyperbolic Functions

Definition (The hyperbolic cosine function)

It is denoted by $\cosh x$ and it is defined as $\cosh x = \frac{e^x + e^{-x}}{2}$

Notes

1. The domain of $\cosh x$ is $\mathbb{R}$ and the range of $\cosh x$ is $[1, \infty]$.
2. It is an even function and $\cosh(0) = 1$



Hyperbolic Functions

Definitions

1. The hyperbolic tangent function is denoted by $\tanh x$ and it is defined as $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ for every $x \in \mathbb{R}$
2. The hyperbolic cotangent function is denoted by $\coth x$ and it is defined as $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^x}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$
3. The hyperbolic secant function is denoted by $sech\ x$ and it is defined as $sech\ x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x - e^{-x}}$ for every $x \in \mathbb{R}$
4. The hyperbolic cosecant function is denoted by $csch\ x$ and it is defined as $csch\ x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$

Hyperbolic Functions

Notes

1. $cosh^2\ x - sinh^2\ x = 1$ for every $x \in \mathbb{R}$
2. $1 - tanh^2\ x = sech^2\ x$ for every $x \in \mathbb{R}$
3. $coth^2\ x - 1 = csch^2\ x$ for every $x \in \mathbb{R} - \{0\}$

Derivatives of the hyperbolic functions

1. $\dfrac{d}{dx}\sinh x = \cosh x,$
2. $\dfrac{d}{dx}\cosh x = \sinh x,$
3. $\dfrac{d}{dx}tanh\ x = sech^2\ x$
4. $\dfrac{d}{dx}coth\ x = -csch^2\ x$
5. $\dfrac{d}{dx}sech\ x = -sech\ x\ tanh\ x$
6. $\dfrac{d}{dx}csch\ x = -csch\ x\ coth\ x$

Hyperbolic Functions

Derivatives of the hyperbolic functions

1. $$\frac{d}{dx}\sinh(f(x)) = \cosh(f(x))f'(x)$$
2. $$\frac{d}{dx}\cosh(f(x)) = \sinh(f(x))f'(x)$$
3. $$\frac{d}{dx}tanh\ (f(x)) = sech^2\ (f(x))f'(x)$$
4. $$\frac{d}{dx}coth\ (f(x)) = -csch^2\ (f(x))f'(x)$$
5. $$\frac{d}{dx}sech\ (f(x)) = -sech\ (f(x))\ tanh\ (f(x))f'(x)$$
6. $$\frac{d}{dx}csch\ (f(x)) = -csch\ (f(x))\ coth\ (f(x))f'(x)$$

Examples

1. Find the value of $f(0)$ if $f(x) = \ln[cosh(3x)]$.
   $f(0) = \ln[cosh(0)] = \ln(1) = 0.$
2. Find the value of $f'(0)$ if $f(x) = \ln|1 + sinh(x)|$.
   $$f'(x) = \frac{cosh(x)}{1 + sinh(x)} \to f'(0) = \frac{cosh(0)}{1 + sinh(0)} = \frac{1}{1+0} = 1$$
3. Find $f'(x)$ if $f(x) = e^{sinh(x)}$.
   $f'(x) = e^{sinh(x)} cosh(x).$
4. Find $f'(x)$ if $f(x) = \tan^{-1}(sinh(x))$.
   $$f'(x) = \frac{cosh(x)}{1 + (sinh(x))^2} = \frac{cosh(x)}{(cosh(x))^2} = \frac{1}{cosh(x)} = sech(x).$$

Hyperbolic Functions

Integration of the hyperbolic functions

1. $$\int \sinh x \; dx = \cosh x + c,$$
2. $$\int \sinh(f(x))f'(x)dx = \cosh(f(x)) + c$$
3. $$\int \cosh x \; dx = \sinh x + c,$$
4. $$\int \cosh(f(x))f'(x)dx = \sinh(f(x)) + c$$
5. $$\int sech^2 x \; dx = \tanh x + c$$
6. $$\int sech^2(f(x))f'(x)dx = \tanh(f(x)) + c$$

Integration of the hyperbolic functions

7. $$\int csch^2 x \; dx = -\coth x + c$$
8. $$\int csch^2(f(x))f'(x)dx = -\coth(f(x)) + c$$

Examples

1. $$\int x^2 \cosh x^3 \ dx = \frac{1}{3}\int \cosh x^3 \ (3x^2)dx = \frac{1}{3}\sinh x^3 + c$$
2. $$\int (e^x - e^{-x}) sech^2(e^x + e^{-x})dx = \tanh(e^x + e^{-x}) + c$$
3. $$\int \frac{\sinh x}{1+\sinh^2 x}dx = \int \frac{\sinh x}{\cosh^2 x}dx = \int \frac{1}{\cosh x}\frac{\sinh x}{\cosh x}dx$$
$$= \int sech \ x \tanh x \ dx = -sech \ x + c$$
4. $$\int \frac{1}{sech \ x\sqrt{4-\sinh^2 x}}dx = \int \frac{\cosh x}{\sqrt{(2)^2 - (\sinh x)^2}}dx$$
$$= \sin^{-1}(\frac{\sinh x}{2}) + c$$

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Definition

The inverse hyperbolic sine function is denoted by $\sinh^{-1}$ and it is defined as $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$

Definition

The inverse hyperbolic cosine function is denoted by $\cosh^{-1}$ and it is defined as $y = \cosh^{-1} x \Leftrightarrow x = \cosh y$, where $x \in [1, \infty)$ and $y \in [0, \infty)$

Definition

The inverse hyperbolic tangent function is denoted by $\tanh^{-1}$ and it is defined as $y = \tanh^{-1} x \Leftrightarrow x = \tanh y$, where $x \in [-1, 1]$ and $y \in \mathbb{R}$

Definition

The inverse hyperbolic cotangent function is denoted by $\coth^{-1}$ and it is defined as $y = \coth^{-1} x \Leftrightarrow x = \coth y$, where $|x| > 1$ and $y \in \mathbb{R}$.

Definition

The inverse hyperbolic secant function is denoted by $sech^{-1}$ and it is defined as $y = sech^{-1}x \Leftrightarrow x = sech\ y$, where $x \in [0, 1]$ and $y \in [0, \infty)$

Definition

The inverse hyperbolic cosecant function is denoted by $csch^{-1}$ and it is defined as $y = csch^{-1}x \Leftrightarrow x = csch\ y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R} - \{0\}$

The Inverse Hyperbolic Functions

Derivatives of the inverse hyperbolic functions

- $\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$,
- $\dfrac{d}{dx}\sinh^{-1} f(x) = \dfrac{f'(x)}{\sqrt{1+f((x))^2}}$.
- $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}$, where $x > 1$
- $\dfrac{d}{dx}\cosh^{-1} f(x) = \dfrac{f'(x)}{\sqrt{(f(x))^2-1}}$, where $|f(x)| > 1$
- $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}$, where $|x| > 1$
- $\dfrac{d}{dx}\tanh^{-1} f(x) = \dfrac{f'(x)}{1-(f(x))^2}$, where$|f(x)| > 1$

The Inverse Hyperbolic Functions

Derivatives of the inverse hyperbolic functions

1. $\dfrac{d}{dx}coth^{-1}x = \dfrac{-1}{1-x^2}$ where $|x| > 1$
2. $\dfrac{d}{dx}coth^{-1}f(x) = \dfrac{-f'(x)}{1-(f(x))^2}$ where $|f(x)| > 1$
3. $\dfrac{d}{dx}sech^{-1}x = \dfrac{-1}{x\sqrt{1-x^2}}$ where $0 < x < 1$
4. $\dfrac{d}{dx}sech^{-1}f(x) = \dfrac{-f'(x)}{f(x)\sqrt{1-(f(x))^2}}$ where $0 < f(x) < 1$
5. $\dfrac{d}{dx}csch^{-1}x = \dfrac{-1}{|x|\sqrt{1+x^2}}$, where $x \neq 0$
6. $\dfrac{d}{dx}csch^{-1}f(x) = \dfrac{-f'(x)}{|f(x)|\sqrt{1+(f(x))^2}}$, where $f(x) \neq 0$

Examples

1. Find $f'(x)$ if $f(x) = \tanh^{-1} 3x$?
$$f'(x) = \frac{3}{1-(3x)^2} = \frac{3}{1-9x^2}$$
2. Find $f'(x)$ if $f(x) = \sinh^{-1} \sqrt{x}$?
$$f'(x) = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1+(\sqrt{x})^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}}$$
3. Find $f'(x)$ if $f(x) = sech^{-1}(\cos 2x)$?
$$f'(x) = \frac{-(-2\sin 2x)}{\cos 2x\sqrt{1-(\cos 2x)^2}} = \frac{2\sin 2x}{\cos 2x\sqrt{1-\cos^2 2x}}$$

The Inverse Hyperbolic Functions

Integration of the inverse hyperbolic functions

- $\int \frac{1}{\sqrt{a^2+x^2}}dx = \sinh^{-1}(\frac{x}{a}) + c$
- $\int \frac{f'(x)}{\sqrt{a^2+(f(x))^2}}dx = \sinh^{-1}(\frac{f(x)}{a}) + c$
- $\int \frac{1}{\sqrt{x^2-a^2}}dx = cosh^{-1}(\frac{x}{a}) + c, (x > a)$
- $\int \frac{f'(x)}{\sqrt{(f(x))^2-a^2}}dx = cosh^{-1}(\frac{f(x)}{a}) + c, \ (f(x) > a)$
- $\int \frac{1}{a^2-x^2}dx = \frac{1}{a}tanh^{-1}(\frac{x}{a}) + c(|x| < a)$
- $\int \frac{f'(x)}{a^2-(f(x))^2}dx = \frac{1}{a}tanh^{-1}(\frac{f(x)}{a}) + c, \ (|f(x)| < a)$

The Inverse Hyperbolic Functions

Integration of the inverse hyperbolic functions

- $\displaystyle\int \frac{1}{x\sqrt{a^2-x^2}}dx = -\frac{1}{a}sech^{-1}(\frac{x}{a}) + c,\ \ (0<x<a)$
- $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{a^2-(f(x))^2}}dx = -\frac{1}{a}sech^{-1}(\frac{f(x)}{a}) + c,$ $(0<f(x)<a)$
- $\displaystyle\int \frac{1}{x\sqrt{x^2+a^2}}dx = -\frac{1}{a}csch^{-1}(\frac{x}{a}) + c,\ \ (x\neq 0)$
- $\displaystyle\int \frac{f'(x)}{x\sqrt{(f(x))^2+a^2}}dx = -\frac{1}{a}csch^{-1}(\frac{f(x)}{a}) + c,\ \ (f(x)\neq 0)$

Examples

1. $$\int \frac{e^x}{1-e^{2x}}dx = \int \frac{e^x}{(1)^2-(e^x)^2}dx = \tanh^{-1}(e^x)+c$$

2. $$\int \frac{1}{\sqrt{x}\sqrt{4+x}}dx = 2\int \frac{\frac{1}{2\sqrt{x}}}{\sqrt{(2)^2+(\sqrt{x})^2}}dx$$
$$= 2\sinh^{-1}(\frac{\sqrt{x}}{2})+c$$

3. $$\int \frac{1}{\sqrt{1+e^{2x}}}dx = \int \frac{e^x}{e^x\sqrt{1+e^{2x}}}dx = -csch^{-1}(e^x)+c$$

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Method of Partial fractions

Definition: linear factor

A linear factor is a polynomial of degree 1. It has the form $ax + b$ where $a, b \in \mathbb{R}$ and $a \neq 0$.
Such $x, 3x$, and $2x - 7$

Definition: irreducible quadratic

An irreducible quadratic is a polynomial of degree 2. It has the form $ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $b^2 - 4ac < 0$.
Such $x^2 + 9$ and $x^2 + x + 1$.

Method of Partial fractions

What is the Partial Fraction?

It is re-expressing a rational function (a ratio of polynomial function) as a sum of simpler fraction.

Let $h(x) = \frac{P(x)}{Q(x)} = \dfrac{a_0x^n + a_1x^{n-1} + \cdots + a - n - 1x + a_n}{b_o x^m + b_1 x^{m-1} + \cdots + b_{m-1}x + b_m}$ be a rational function, where $P(x), Q(x)$ are polynomials, we have two cases:

1. degree $P(x) <$ degree $Q(x)$ use method of partial fractions.
2. degree $P(x) \geq$ degree $Q(x)$ use long division of polynomials, then use method of partial fractions.

How do we create partial functions?

1. If we can write $Q(x)$ as a linear factors
   $b_ox^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m = (x-a)^m, \ a \in \mathbb{R}, \ m \in \mathbb{N}$
   Then: $h(x) = \dfrac{A_0}{(x-a)^m} + \dfrac{A_1}{(x-a)^{m-1}} + \cdots + \dfrac{A_{m-1}}{x-a}, m \in \mathbb{N}$
2. If we can write $Q(x)$ as a irreducible quadratic factors
   $b_ox^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m = (ax^2+bx+c)^n, \ a, b, c \in \mathbb{N}$
   and $b^2 - 4ac < 0$
   Then:
   $$h(x) = \frac{B_0x + C_0}{(ax^2+bx+c)^n} + \frac{B_1x + C_1}{(ax^2+bx+c)^{n-1}} + \cdots + \frac{B_{n-1}x + C_{n-1}}{ax^2+bx+c}.$$

Some time we can write $Q(x)$ as a product of linear factors and irreducible quadratics.
Then

$$h(x) = \frac{A_0}{(x-a)^m} + \frac{A_1}{(x-a)^{m-1}} + \cdots + \frac{A_{m-1}}{x-a}$$

$$+\frac{B_0 x + C_0}{(ax^2+bx+c)^n} + \frac{B_1 x + C_1}{(ax^2+bx+c)^{n-1}} + \cdots + \frac{B_{n-1}x + C_{n-1}}{ax^2+bx+c}.$$

Method of Partial fractions (Examples)

Determine the partial fraction for: $\dfrac{x-3}{x^2-4}$

$\dfrac{x-3}{(x-2)(x+2)} = \dfrac{A}{x-2} + \dfrac{B}{x+2} \Rightarrow x-3 = A(x+2) + B(x-2)$

$x = -2 \Rightarrow -5 = -4B \Rightarrow B = \frac{5}{4}$

$x = 2 \Rightarrow -1 = 4A \Rightarrow A = \frac{-1}{4}$

So: $\dfrac{x-3}{(x-2)(x+2)} = \dfrac{-1}{4(x-2)} + \dfrac{5}{4(x+2)}$

Now Integrate:

Determine $\displaystyle\int \frac{x-3}{x^2-4}dx$

$$\int \frac{x-3}{x^2-4}dx = \int \left[\frac{-1}{4(x-2)} + \frac{5}{4(x+2)}\right]dx$$

$$= -\frac{-1}{4}\ln\left|x-2\right| + \frac{5}{4}\ln\left|x+2\right| + c$$

Determine $\int \frac{x-3}{x^2+4x}dx$

Note that degree $P(x) <$ degree $Q(x)$

$$\frac{x-3}{x^2+4x} = \frac{x-3}{x(x+4)} = \frac{A}{x} + \frac{B}{x+4} \Rightarrow x-3 = A(x+4) + Bx$$

$x = -4 \Rightarrow -7 = -4B \Rightarrow B = \frac{7}{4}$

$x = 0 \Rightarrow -3 = 4A \Rightarrow A = \frac{-3}{4}$

Method of Partial fractions (Examples)

Now we can write:

$$\frac{x-3}{x^2-4x} = \frac{-3}{4x} + \frac{7}{4(x+4)}.$$

$$\int \frac{x-3}{x^2-4x}dx = \int \frac{-3}{4x}dx + \int \frac{7}{4(x+4)}dx$$

$$= -\frac{3}{4}\ln|x| + \frac{7}{4}\ln|x+4| + C$$

$$= \frac{\ln|x+4|^{\frac{7}{4}}}{\ln|x|^{\frac{3}{4}}} + C$$

Method of Partial fractions (Examples)

Determine $\int \frac{x^2-2}{x^3+3x^2+2x}dx$

Note that degree $P(x)$ < degree $Q(x)$

$$\frac{x^2-2}{x^3+3x^2+2x} = \frac{x^2-2}{x(x+2)(x+1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1}$$

$\Rightarrow x^2-2 = A(x+2)(x+1) + Bx(x+1) + Cx(x+2)$

$x = 0 \Rightarrow -2 = 2A \Rightarrow A = -1$

$x = -2 \Rightarrow 2 = 2B \Rightarrow b = 1$

$x = -1 \Rightarrow -1 = -C \Rightarrow c = 1$

Method of Partial fractions (Examples)

Now we can write:

$$\frac{x^2-2}{x^3+3x^2+2x} = \frac{-1}{x} + \frac{1}{x+2} + \frac{1}{x+1}.$$

$$\int \frac{x^2-2}{x^3+3x^2+2x}dx = \int \frac{-1}{x}dx + \int \frac{1}{x+2}dx + \int \frac{1}{x+1}$$

$$= -\ln|x| + \ln|x+2| + \ln|x+1| + c$$

Method of Partial fractions (Examples)

Determine $\int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} dx$

Note that degree $P(x) <$ degree $Q(x)$.

We can write: $x^4 + 5x^2 + 4 = (x^2 + 4)(x^2 + 1)$

$$\frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} = \frac{Ax + b}{x^2 + 4} + \frac{Cx + D}{x^2 + 1}$$

$$\Rightarrow x^3 - 2x^2 + x + 1 = (Ax + B)(x^2 + 1) + (Cx + D)(x^2 + 4)$$
$$= (A + C)x^3 + (B + D)x^2 + (A + 4C)x + (B + 4D)$$

$A = 1, \quad B = -3, \quad C = 0, \quad D = 1$

Method of Partial fractions (Examples)

Now we can write:

$$\frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} = \frac{x - 3}{x^2 + 4} + \frac{1}{x^2 + 1}$$

$$\int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} dx = \int [\frac{x - 3}{x^2 + 4} + \frac{1}{x^2 + 1}] dx$$

$$= \frac{1}{2} \ln |x^2 + 4| - \frac{3}{2} \tan^{-1} \frac{x}{2} + \tan^{-1} x + c$$

Determine $\int \frac{x^2+3}{x^2-x-2}dx$

Note that degree $P(x) \geq$ degree $Q(x)$

Here Divide First

$\frac{x^2+3}{x^2-x-2} = 1 + \frac{x+5}{x^2-x-2}$

$\frac{x+5}{x^2-x-2} = \frac{x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \Rightarrow$

$x+5 = A(x+1) + B(x-2)x = 2 \Rightarrow 7 = 3A \Rightarrow A = \frac{7}{3}$

$x = -1 \Rightarrow 4 = -3B \Rightarrow B = \frac{-4}{3}$

Method of Partial fractions (Examples)

Now we can write:

$$\frac{x^2+3}{x^2-x-2} = 1 + \frac{7}{3(x-2)} - \frac{4}{3(x+1)}$$

$$\int \frac{x^2+3}{x^2-x-2}dx = \int \left[1 + \frac{7}{3(x-2)} - \frac{4}{3(x+1)}\right]dx$$

$$= x + \frac{7}{3}\ln\left|x-2\right| - \frac{4}{3}\ln\left|x+1\right| + c$$

Determine $I = \int \frac{x^4+1}{(x+1)(x^2+x+1)}dx$

Note that degree $P(x) \geq$ degree $Q(x)$

$$\frac{x^4+1}{(x+1)(x^2+x+1)} = (x-2) + \frac{2x^2+3x+3}{(x+1)(x^2+x+1)}$$

$$I = \underbrace{\int (x-2)\ dx}_{I_1} + \underbrace{\int \frac{2x^2+3x+3}{(x+1)(x^2+x+1)}dx}_{I_2}$$

Method of Partial fractions (Examples)

Now for I_2 we have de to applied method of partial fraction

$$\frac{2x^2+3x+3}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1},$$

$$\begin{aligned}\Rightarrow 2x^2+3x+3 &= A(x^2+x+1)+(Bx+C)(x+1)\\ &= Ax^2+Ax+A+Bx^2+Bx+Cx+C\\ &= (A+B)x^2+(A+B+C)x+(A+C)\end{aligned}$$

$A+B=2$

$A+B+C=3$

$A+C=3$

So: $A=2,\ B=0,$ and $C=1.$

Method of Partial fractions (Examples)

Now we can write:

$$\frac{2x^2+3x+3}{(x+1)(x^2+x+1)} = \frac{2}{x+1} + \frac{1}{x^2+x+1}$$

$$I = I_1 + I_2 = \int (x-2)\ dx + \int \frac{2x^2+3x+3}{(x+1)(x^2+x+1)} dx$$

$$= \underbrace{\int (x-2)\ dx}_{I_1} + \underbrace{\underbrace{\int \frac{2}{x+1} dx}_{J_1} + \underbrace{\int \frac{1}{x^2+x+1} dx}_{J_2}}_{I_2}$$

Method of Partial fractions (Examples)

$$J_2 = \int \frac{1}{x^2+x+1}dx = \int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$$
$$= \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx = \frac{2}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}})+c$$

So,

$$\int \frac{x^4+1}{(x+1)(x^2+x+1)}dx = I_1 + \underbrace{J_1+J_2}_{I_2}$$
$$= \underbrace{x^2-2x}_{I_1} + \underbrace{2\ln|x+1|}_{J_1} + \underbrace{\frac{2}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}})}_{J_2} + c$$

Method of Partial fractions (Examples)

Exercises

1. $\displaystyle\int \frac{6x+7}{(x+2)^2}dx$
2. $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx$
3. $\displaystyle\int \frac{x}{x^2+2x-3}dx$
4. $\displaystyle\int \frac{x^2}{(x-1)^2(x+1)}dx$
5. $\displaystyle\int \frac{x^3-5x+7}{x^2+x-6}dx$
6. $\displaystyle\int \frac{x^3-11x-26}{x^2-2x-8}dx$
7. $\displaystyle\int \frac{1}{x(x^2+1)^2}$

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Half-Angle Substitution

It is used to solve integrals of rational functions involving $\sin x$ or $\cos x$

Example

$\displaystyle\int \frac{1}{2+\cos x}dx$, and $\displaystyle\int \frac{1}{1-\sin x}dx$

Question

How to solve an integral using half angle trigonometric substitution?

To solve this type of integral we have to concentrate on:

1. $u = \tan\frac{x}{2}$
2. $\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2u}{1+u^2}$
3. $\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-u^2}{1+u^2}$
4. $\frac{du}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}(\tan^2\frac{x}{2}+1) = \frac{1}{2}(u^2+1) \Rightarrow$

$$dx = \frac{2}{(u^2+1)}du$$

Examples

Evaluate $\int \frac{1}{2+\cos x}dx$ and $\int \frac{1}{1-\sin x}dx$

to solve $\int \frac{1}{2+\cos x}dx$

we put $u = \tan\frac{x}{2}$ So $\cos x = \frac{1-u^2}{1+u^2}$, and $dx = \frac{2}{(u^2+1)}du$

$$\int \frac{1}{2+\cos x}dx = \int \frac{2}{(2+\frac{1-u^2}{1+u^2})(u^2+1)}du = \int \frac{2}{u^2+3}du$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + c$$

Half-Angle Substitution

To solve $\displaystyle\int \frac{1}{1-\sin x}dx$

We put $u = \tan\dfrac{x}{2}, \sin x = \dfrac{2u}{1+u^2}$ and $dx = \dfrac{2}{(u^2+1)}du$

$$\int \frac{1}{1-\sin x}dx = \int \frac{2}{1-\frac{2u}{1+u^2}(1+u^2)}du$$

$$= -\frac{2}{(u-1)} + c = \frac{-2}{tan\frac{x}{2}-1} + c$$

Example

How we can integrate using tangent half angle substitution.

$$\int \frac{1}{3 - 5\sin x} dx$$

We put $u = \tan \frac{x}{2}, \sin x = \frac{2u}{u^2 + 1}$ and $dx = \frac{2}{u^2 + 1}$

Hence, the given integral becomes:

$$\int \frac{1}{3 - 5\sin x} dx = \int \frac{\frac{2}{u^2+1}}{3 - 5(\frac{2u}{u^2+1})} du = \int \frac{2}{3u^2 - 10u + 3} du$$

Now, we need to do partial fraction decomposition.

Half-Angle Substitution

$$\frac{2}{3u^2-10u+3}=\frac{2}{(u-3)(3u-1)}=\frac{A}{u-3}+\frac{B}{3u-1}$$

$$2=A(3u-1)+B(u-3)\Rightarrow 2=(3A+B)u-A-3B$$

$3A+B=0$
$-A-3B=2$
$A=\frac{1}{4}$, and $B=-\frac{3}{4}$

Half-Angle Substitution

$$\int \frac{1}{3-5\sin x}dx = \int \frac{2}{3u^2-10u+3}du$$

$$= \int \frac{1}{4(u-3)}\ du - \int \frac{3}{4(3u-1)}\ du$$

$$= \frac{1}{4}\ln|u-3| - \frac{3}{4}\frac{1}{3}\ln|3u-1| + c$$

$$\int \frac{1}{3-5\sin x}dx = \frac{1}{4}\ln|\tan^{-1}\frac{x}{2}-3| - \frac{1}{4}\ln|3\tan^{-1}\frac{x}{2}-1| + c$$

Half-Angle Substitution

$$\int \frac{\sin x}{\sqrt{5 - 2\cos x + \cos^2 x}}\, dx$$

Put $u = \cos x \Rightarrow -du = \sin x$

$$\int \frac{\sin x}{\sqrt{5 - 2\cos x + \cos^2 x}}\, dx = \int \frac{-1}{\sqrt{5 - 2u + u^2}}\, du$$

$$= -\int \frac{1}{\sqrt{(u^2 - 2u + 1) + 4}}\, du = -\int \frac{1}{\sqrt{(u-1)^2 + (2)^2}}\, du$$

$$= -\sinh^{-1}\left(\frac{u-1}{2}\right) + c$$

$$\int \frac{\sin x}{\sqrt{5 - 2\cos x + \cos^2 x}}\, dx = -\sinh^{-1}\left(\frac{\cos x - 1}{2}\right) + c$$

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Improper Integrals

Definition

1. If f is continuous on $[a, b)$ and $\left|f(x)\right| \to \infty$ as $x \to b^-$ then
$$\int_a^b f(x)\ dx = \lim_{t \to b^-} \int_a^t f(x)\ dx$$
2. If f is continuous on $(a, b]$ and $|f(x)| \to \infty$ as $x \to a^+$ then
$$\int_a^b f(x)\ dx = \lim_{t \to a^+} \int_t^b f(x)\ dx$$

If the limit exists (and equals a value L) then the improper integral **converges** (to L).
If the limit does not exist then the improper integral **diverges**.

Rmark

If f is continuous on $[a, b]$ except at a point $c \in (a, b)$ and $|f(x)| \to \infty$ as $x \to c$ then

$$\int_a^b f(x)\ dx = \lim_{t \to c^-} \int_a^t f(x)\ dx + \lim_{t \to c^+} \int_t^b f(x)\ dx$$

If both limits exist (and equals L_1 and L_2 respectively) then the improper integral **converges** (to $L_1 + L_2$).

If at least one of the limits does not exist then the improper integral **diverges**.

Improper Integrals

Definition

1. If f is continuous on $[a, \infty)$ then
$$\int_a^\infty f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx$$
2. If f is continuous on $(-\infty, a$ then
$$\int_{-\infty}^a f(x)\ dx = \lim_{t\to-\infty} \int_t^a f(x)\ dx$$

if the limit exists (and equals a value L) then the improper integral **converges** (to L).
If the limit does not exist then the improper integral **diverges**.

Rmark

If f is continuous on $(-\infty, \infty)$ then for any constant a

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{t \to -\infty} \int_{t}^{a} f(x)\ dx + \lim_{t \to \infty} \int_{a}^{t} f(x)\ dx$$

If both limits exist (and equals L_1 and L_2 respectively) then the improper integral **converges** (to $L_1 + L_2$).
If at least one of the limits does not exist then the improper integral **diverges**.

Example

1. $\int_1^\infty x^{-2}\,dx$ is an improper integral.
   Some such integrals can sometimes be computed by replacing infinite limits with finite values
   $$\int_1^\infty x^{-2}\,dx = \lim_{y\to\infty}\int_1^y x^{-2}\,dx = \lim_{y\to\infty}\left[-\frac{1}{x}\right]_1^y = \lim_{y\to\infty}\left(-\frac{1}{y}+1\right)$$
   $$= 1$$
2. $$\int_1^\infty \frac{1}{x}\,dx = \lim_{c\to\infty}\int_1^c \frac{1}{x}\,dx = \lim_{c\to\infty}\Big[\ln|x|\Big]_1^c$$
   $$= \lim_{c\to\infty}\Big(\ln|c| - \ln|1|\Big) = \infty$$

Improper Integrals: Examples

Example

Evaluate : $\int_{-\infty}^{\infty} \frac{1}{1+X^2}\, dx$

First, we split the integral in two.

$$\int_{-\infty}^{\infty} \frac{1}{1+X^2}\, dx = \int_{-\infty}^{0} \frac{1}{1+X^2}\, dx + \int_{0}^{\infty} \frac{1}{1+X^2}\, dx$$

Second, we turn each part into a limit.

$$\int_{-\infty}^{\infty} \frac{1}{1+X^2}\,dx = \lim_{c\to-\infty}\int_{c}^{0} \frac{1}{1+X^2}\,dx + \lim_{c\to\infty}\int_{0}^{c} \frac{1}{1+X^2}\,dx$$

Finally, we evaluate each part and add up the results.

$$\int_{-\infty}^{\infty} \frac{1}{1+X^2}\,dx = \lim_{c\to-\infty}\Big[\tan^{-1}x\Big]_c^0 + \lim_{c\to\infty}\Big[\tan^{-1}x\Big]_0^c$$

$$= \lim_{c\to-\infty}\Big[\tan^{-1}0 - \tan^{-1}c\Big] + \lim_{c\to\infty}\Big[\tan^{-1}c - \tan^{-1}0\Big]$$

$$= \Big(0 - (-\tfrac{\pi}{2})\Big) + \Big(\tfrac{\pi}{2} - 0\Big) = \pi$$

Example

Evaluate $\int_1^{\infty} \frac{1}{x^p}\, dx$, where p is a real number.

We have to consider every possible value of p.

First, for $p = 1$

$$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\, dx = \lim_{t\to\infty} \Big[\ln x\Big]_1^t = \lim_{b\to\infty} \ln\left|t\right| = \infty$$

so the integral diverges when p = 1.

Now, for $p \neq 1$, the power rule applies:

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^t$$

$$= \lim_{t\to\infty}\left(\frac{1}{1-p}t^{1-p} - \frac{1}{1-p}\right) = \frac{1}{1-p}$$

This means that for $p \leq 1$, the integral diverges, and for $p > 1$, it converges and equals $\frac{-1}{1-p}$

Example

Evaluate $\int\limits_{-\infty}^{\infty} \frac{dx}{x^2+16} dx$

$$\int\limits_{-\infty}^{\infty} \frac{dx}{x^2+16} dx = \lim_{a \to -\infty} \int\limits_{a}^{0} \frac{dx}{x^2+16} dx + \lim_{b \to \infty} \int\limits_{0}^{b} \frac{dx}{x^2+16} dx$$

$$= \lim_{a \to -\infty} \left(0 - \frac{1}{4} tan^{-1} \frac{a}{4} \right) + \lim_{b \to \infty} \left(\frac{1}{4} tan^{-1} \frac{b}{4} - 0 \right)$$

$$= \frac{1}{4} \frac{\pi}{2} + \frac{1}{4} \frac{\pi}{2} = \frac{\pi}{4}$$