Chapter 2
Random Vectors and Joint Probability Distributions

Dr. Mohamed Rhaima

Department of Statistics and Operations Research

September, 30, 2019

1. Random Vector.

In many cases it is more natural to describe the outcome of a random experiment by two or more numerical numbers simultaneously. For example, the characterization of both weight and height in a given population, the study of temperature and pressure variations in a physical experiment, and the distribution of monthly temperature readings in a given region over a given year. In these situations, two or more random variables are considered jointly and the description of their joint behavior is our concern. Let us first consider the case of two random variables X and Y. We proceed analogously to the single random variable case in defining their joint probability distributions. We note that random variables X and Y can also be considered as components of a two-dimensional random vector, say Z. Joint probability distributions associated with two random variables are sometimes called bivariate distributions.

1.1 Joint cumulative distribution function

Definition

For a pair of random variables (X, Y), the joint cumulative distribution function (CDF) F_{XY} is given by

$$F_{XY}(x, y) = \mathbb{P}(X \leq x, Y \leq y)$$

For N random variables $X_1, \ldots, X_N$, the joint CDF $F_{X_1,\ldots,X_N}$ is given by

$$\mathbf{F_{X_1,\ldots,X_N}(x_1, \ldots, x_N)} = \mathbb{P}(\mathbf{X_1 \leq x_1, \ldots, X_N \leq x_n})$$

Properties:

Every multivariate CDF is:

1. Monotonically non-decreasing for each of its variables,
2. Right-continuous in each of its variables,
3. $0 \leq F_{X_1 \ldots X_n}(x_1, \ldots, x_n) \leq 1$,
4. $\lim_{x_1, \ldots, x_n \to +\infty} F_{X_1 \ldots X_n}(x_1, \ldots, x_n) = 1$
5. $\lim_{x_1, \ldots, x_n \to -\infty} F_{X_1 \ldots X_n}(x_1, \ldots, x_n) = 0.$

In general, if X and Y are two random variables,the probability distribution that defines their simultaneous behavior is called a **joint probability distribution.**

2. Discrete random vector

2.1 Joint probability mass distribution

Definition

The joint probability distribution of discrete random variables X_1 and X_2 is a function $p_{X_1X_2}(x_1, x_2)$ that possesses the properties:

1. $p_{X_1X_2}(x_1, x_2) \geq 0$
2. $\sum_{x_1}\sum_{x_2} p_{X_1X_2}(x_1, x_2) = 1.$
3. $\mathbb{P}(a_1 \leq X_1 \leq b_1, a_2 \leq X_2 \leq b_2) = \sum_{x_1 \in [a_1, b_1]} \sum_{x_2 \in [a_2, b_2]} p_{X_1X_2}(x_1, x_2).$

The generalization of the preceding two-variable case is the joint probability distribution of n, discrete random variables $X_1, X_2, \ldots, X_n$ which is:

$$p_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n$$

Remark: If X_1 and X_2 are discrete, this distribution can be described with a joint probability mass function.

Example 1: Plastic covers for CDs.
Measurements for the length and width of a rectangular plastic covers for CDs are rounded to the nearest mm (so they are discrete).
Let X denote the length and Y denote the width.
The possible values of X are 129, 130, and 131 mm. The possible values of Y are 15 and 16 mm (Thus, both X and Y are discrete). There are 6 possible pairs (X, Y).
We show the probability for each pair in the following table:

		x=length		
		129	130	131
y=width	15	0.12	0.42	0.06
	16	0.08	0.28	0.04

✓ The sum of all the probabilities is 1.
✓ The joint probability mass function is the function $p_{XY}(x, y) = \mathbb{P}(X = x, Y = y)$. For example, we have $p_{XY}(129, 15) = 0.12$.

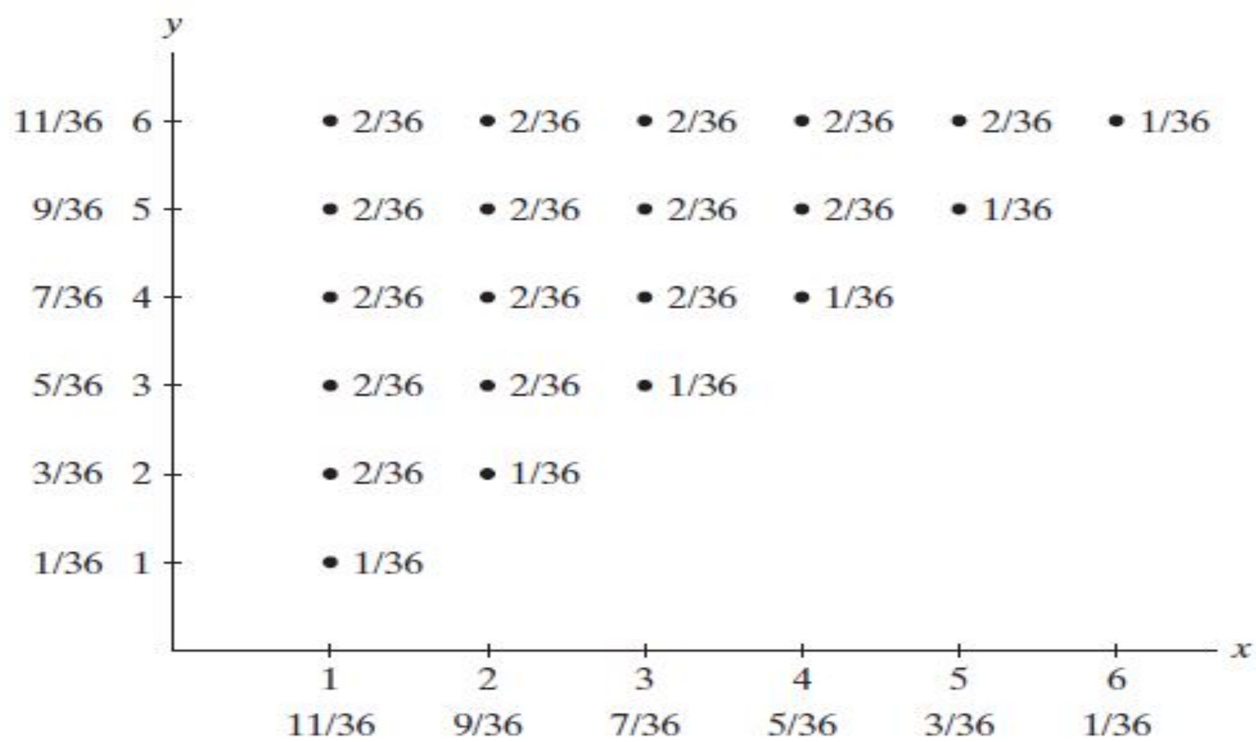
Example 2: Roll a pair of fair dice. For each of the 36 sample points with probability 1/36, let X denote the smaller and Y the larger outcome on the dice. For example, if the outcome is $(3, 2)$, then the observed values are $X = 2, Y = 3$. The event $X = 2, Y = 3$ could occur in one of two ways-$(3, 2)$ or $(2, 3)$-so its probability is

$$\frac{1}{36} + \frac{1}{36} = \frac{2}{36}$$

If the outcome is $(2, 2)$, then the observed values are $X = 2, Y = 2$. Since the event $X = 2, Y = 2$ can occur in only one way, $\mathbb{P}(X = 2, Y = 2) = \frac{1}{36}$. The joint pmf of X and Y is given by the probabilities

$$p_{XY}(x, y) = \begin{cases} \frac{1}{36}, & \text{if } 1 \le x = y \le 6 \\ \frac{2}{36}, & \text{if } 1 \le x < y \le 6 \end{cases}$$

when x and y are integers.This figure depicts the probabilities of the various points of the space Ω.



1.2 Marginal probability mass function

Definition

Let X and Y have the joint probability mass function $p_{XY}(x,y)$ with space Ω.

1. The **marginal probability mass function p_X** of X, is defined by

$$p_X(x) = \sum_y p_{XY}(x,y) = \mathbb{P}(X = x), \quad x \in S_X.$$

2. The **marginal probability mass function p_Y** of Y is defined by

$$p_Y(y) = \sum_x p_{XY}(x,y) = \mathbb{P}(Y = y), \quad y \in S_Y.$$

Theorem

The random variables X and Y are independent if and only if, for every $x \in S_X$ and every $y \in S_Y$,

$$\mathbb{P}(X = x, Y = y) = P(X = x)P(Y = y)$$

or, equivalently,

$$p_{XY}(x, y) = p_X(x)p_Y(y),$$

otherwise, X and Y are said to be dependent.

We note in **Example 2** that X and Y are dependent because there are many x and y values for which $p_{XY}(x, y) \neq p_X(x)p_Y(y)$. For instance,

$$p_X(1)p_Y(1) = (\frac{11}{36}(\frac{1}{36}) \neq \frac{1}{36} = p_{XY}(1, 1).$$

Example 3: Let X and Y two random variables with joint probability mass function given by

$$p_{XY}(x,y) = \frac{xy^2}{30}, \qquad x = 1,2,3, \qquad y = 1,2.$$

1. Find the marginal probability mass function p_X of X.
2. Find the marginal probability mass function p_Y of Y.
3. Prove that X and Y are independents.

Solution :

1. The marginal probability mass function of X is

$$p_X(x) = \sum_{y=1}^{2} \frac{xy^2}{30} = \frac{x}{6}, \qquad x = 1, 2, 3.$$

2. The marginal probability mass function of Y is

$$p_Y(y) = \sum_{x=1}^{3} \frac{xy^2}{30} = \frac{y^2}{5}, \qquad y = 1, 2.$$

3. Since

$$p_{XY}(x, y)) = \frac{xy^2}{30} = \frac{x}{6} \times \frac{y^2}{5} = p_X(x)p_Y(y) \text{ for } x = 1, 2, 3 \text{ and } y = 1, 2;$$

then, X and Y are independent.

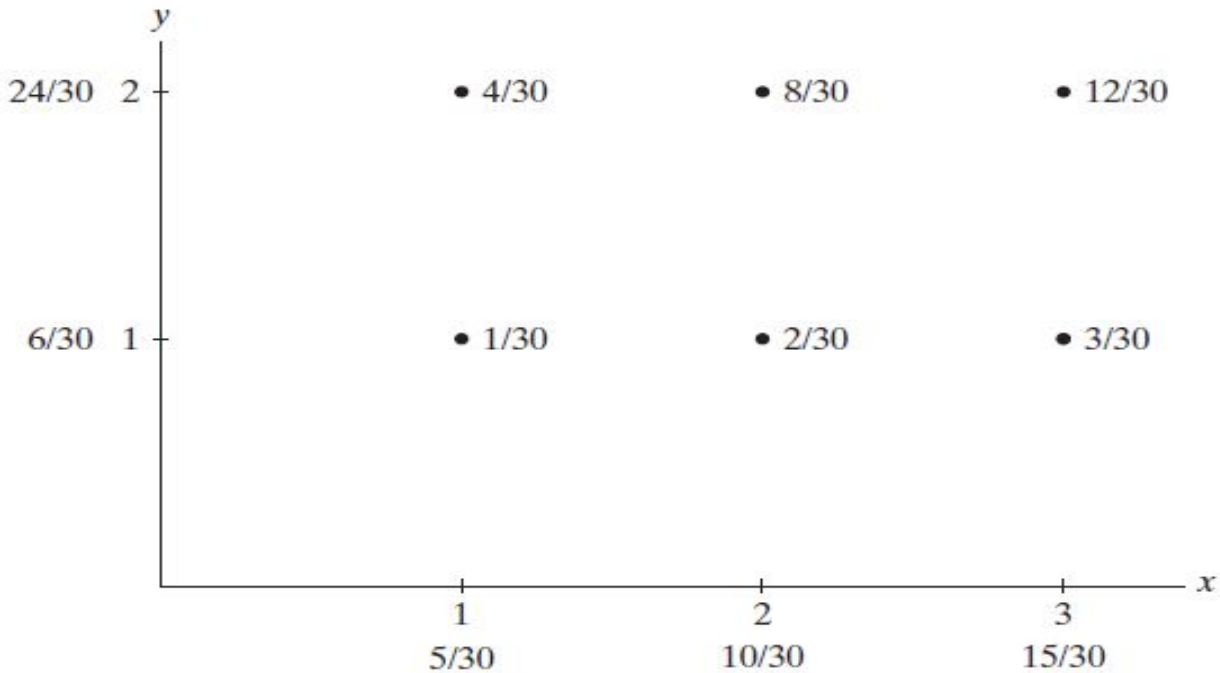


Figure : Joint pmf $p_{XY}(x,y) = \frac{xy^2}{30}$, $x = 1, 2, 3,\ y = 1, 2.$

Example 4: Let X and Y two random variables with joint probability mass function given by

$$p_{XY}(x,y) = \frac{xy^2}{13}, \qquad (x,y) = (1,1), (1,2), (2,2).$$

Prouve that X and Y are dependent.

Solution: The pmf of X is

$$p_X(x) = \begin{cases} \frac{5}{13}, & x = 1 \\ \frac{8}{13}, & x = 2 \end{cases}$$

and that of Y is

$$p_Y(y) = \begin{cases} \frac{1}{13}, & y = 1 \\ \frac{12}{13}, & y = 2 \end{cases}$$

X and Y are dependent because $p_X(2)p_Y(1) = \frac{8}{13} \times \frac{1}{13} \neq 0 = p_{XY}(2,1)$.

2.3 Joint Cumulative Distribution Function

Theorem

Let X and Y be two discrete random variables with joint probability mass function p_{XY} then

1. *The cumulative distribution function F_{XY} of X and Y is given by*

$$F_{XY}(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \sum_{X \leq x} \sum_{Y \leq y} p_{XY}(x,y)$$

2. *The marginal cumulative distribution function of X is given by*

$$F_X(x) = F_{XY}(x,\infty) = \mathbb{P}(X \leq x, Y < \infty)$$

3. *The marginal cumulative distribution function of Y is given by*

$$F_Y(y) = F_{XY}(\infty,y) = \mathbb{P}(X < \infty, Y \leq y)$$

Example 5:

Let $X \sim Bernoulli(p)$ and $Y \sim Bernoulli(q)$ be independent, where $0 < p, q < 1$. Find the joint probability mass distribution and joint cumulative distribution function for X and Y.

Solution: First note that the joint range of X and Y is given by

$$\Omega_{XY} = (0,0), (0,1), (1,0), (1,1).$$

Since X and Y are independent, we have

$$p_{XY}(i,j) = p_X(i)p_Y(j), \quad \text{for } i,j = 0,1.$$

Thus, we conclude

$$\begin{aligned}
p_{XY}(0,0) &= p_X(0)p_Y(0) = (1-p)(1-q), \\
p_{XY}(0,1) &= p_X(0)p_Y(1) = (1-p)q, \\
p_{XY}(1,0) &= p_X(1)p_Y(0) = p(1-q), \\
p_{XY}(1,1) &= p_X(1)p_Y(1) = pq.
\end{aligned}$$

Now that we have the joint PMF, we can find the joint CDF

$$F_{XY}(x,y) = \mathbb{P}(X \le x, Y \le y).$$

Specifically, since $0 \le X,Y \le 1$, we conclude

$$\begin{aligned} F_{XY}(x,y) &= 0, \quad \text{if } x < 0, \\ F_{XY}(x,y) &= 0, \quad \text{if } y < 0, \\ F_{XY}(x,y) &= 1, \quad \text{if } x \ge 1 \text{ and } y \ge 1. \end{aligned}$$

Now, for $0 \le x < 1$ and $y \ge 1$, we have

$$F_{XY}(x,y) = \mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X = 0, y \le 1) = \mathbb{P}(X = 0) = 1 - p.$$

Similarly, for $0 \le y < 1$ and $x \ge 1$, we have

$$\begin{aligned} F_{XY}(x,y) &= \mathbb{P}(X \le x, Y \le y) \\ &= \mathbb{P}(X \le 1, y = 0) \\ &= \mathbb{P}(Y = 0) = 1 - q. \end{aligned}$$

Finally, for $0 \leq x < 1$ and $0 \leq y < 1$, we have

$$
\begin{aligned}
F_{XY}(x,y) &= \mathbb{P}(X \leq x, Y \leq y) \\
&= \mathbb{P}(X = 0, y = 0) \\
&= \mathbb{P}(X = 0)\mathbb{P}(Y = 0) = (1-p)(1-q).
\end{aligned}
$$

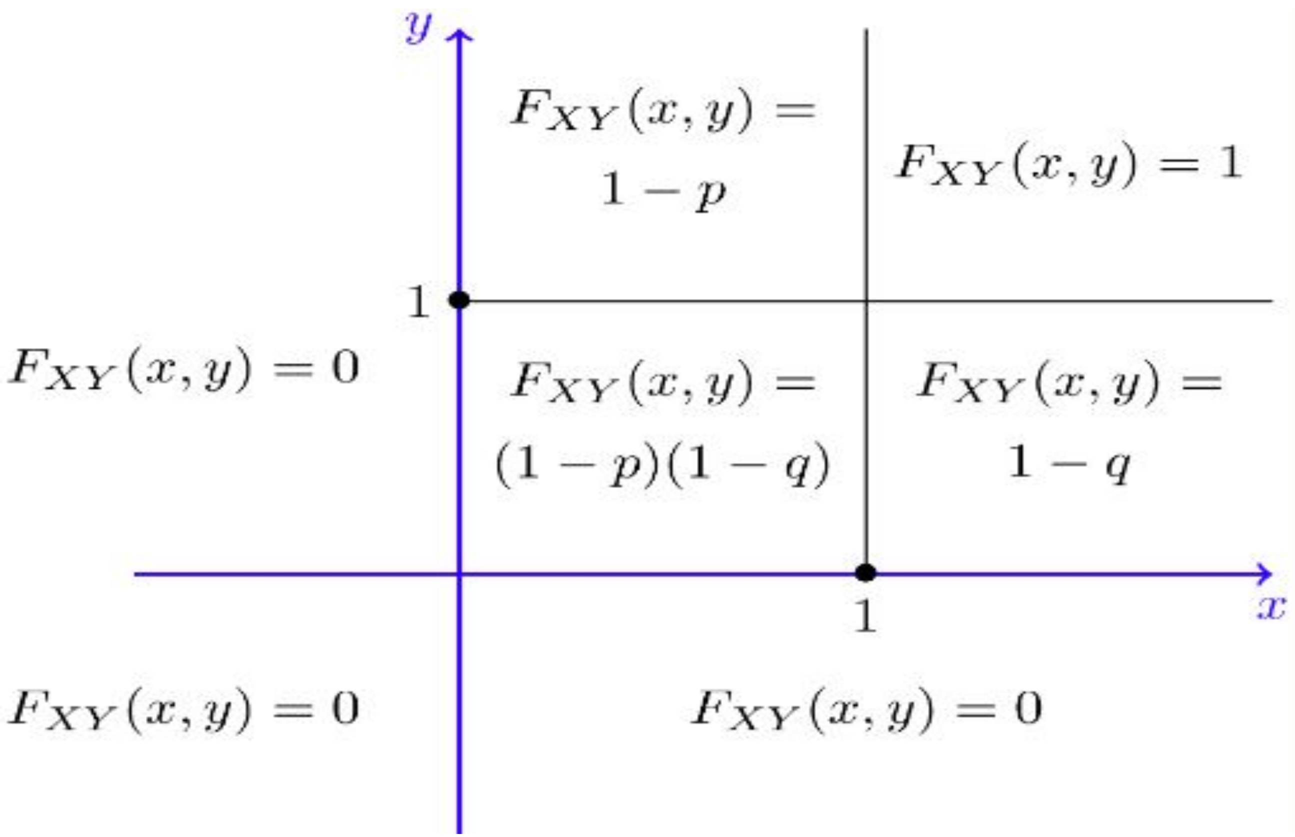


Figure : Joint CDF for X and Y

2.4 Expected value of two discrete random variables

Definition

Let X and Y be random variables of the discrete type with the joint pmf $p_{XY}(x, y)$ on the space Ω. If $u(X, Y)$ is a function of these two random variables, then

$$\mathbb{E}(u(X, Y)) = \sum_{(x,y)\in\Omega} u(x, y)p_{XY}(x, y)$$

if it exists, is called the **mathematical expectation** (or **expected value**) **of** $u(X, Y)$.

Example 5: There are eight similar chips in a bowl: three marked $(0, 0)$, two marked $(1, 0)$, two marked $(0, 1)$, and one marked $(1, 1)$. A player selects a chip at random and is given the sum of the two coordinates in dollars. If X and Y represent those two coordinates, respectively, their joint pmf is

$$p_{XY}(x, y) = \frac{3 - x - y}{8}, \quad x = 0, 1 \quad \text{and} \quad y = 0, 1.$$

Find the expected value of $X + Y$.

Solution:

We define $u(X, Y) = X + Y$, therefore

$$\begin{aligned}
\mathbb{E}(u(X,Y)) = \mathbb{E}(X+Y) &= \sum_{y=0}^{1}\sum_{x=0}^{1}(x+y)p_{XY}(x,y) \\
&= \sum_{y=0}^{1}\sum_{x=0}^{1}(x+y)\frac{3-x-y}{8} \\
&= (0)\frac{3}{8} + (1)\frac{2}{8} + (1)\frac{2}{8} + (2)\frac{1}{8} = \frac{3}{4}
\end{aligned}$$

That is, the expected payoff is $75d$.

3. Continuous random vector

3.1 Joint probability density function

Definition

The joint probability density function (joint pdf) of two continuous-type random variables X and Y is an integrable function $f_{XY}(x,y)$ with the following properties:

1. $f_{XY}(x,y) \geq 0$,
2. $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)dxdy = 1$,
3. $\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = \int_c^d \int_a^b f_{XY}(x,y)dxdy$

Example 6: Let f be a function given by

$$f(x,y) = \begin{cases} \frac{1}{3600}, & \text{if } 0 \leq x \leq 60,\ 0 \leq y \leq 60 \\ 0, & \text{elsewhere.} \end{cases}$$

Prove that f is a joint probability density function for two random variables.

Solution:

1. For every x and y, $f(x,y) \geq 0$
2.

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy &= \int_0^{60}\left(\int_0^{60} f(x,y)dx\right)dy \\ &= \int_0^{60}\left(\int_0^{60} \frac{1}{3600}dx\right)dy \\ &= \frac{60}{3600}\times\frac{60}{3600} = \frac{3600}{3600} = 1\end{aligned}$$

Then f is probability density function.

3.2 Marginal probability density function

Definition

Let X and Y be two continuous random variables with joint probability density function $f_{XY}(x, y)$ then

1. The marginal pdf of continuous-type random variables X is given by

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y)dy, \qquad x \in S_X$$

2. The marginal pdf of continuous-type random variables Y is given by

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y)dx, \qquad y \in S_Y$$

Theorem

The random variables X and Y are independent if and only if the joint pdf factors into the product of their marginal pdfs; namely,

$$f_{XY}(x,y) = f_X(x) f_Y(y) \quad x \in S_X \;\; y \in S_Y,$$

Example 7: Let X and Y have the joint probability density function

$$f_{XY}(x,y) = cx^2 y \quad 0 \le y \le 1 \;\; -y \le x \le 1.$$

1. Find the value of the constant c.
2. Find the marginal pdf of X.
3. Find the marginal pdf of Y.
4. Calculate two probabilities, $\mathbb{P}(X \le 0)$ and $\mathbb{P}(0 \le Y \le X \le 1)$.

Solution:

1. To determine the value of the constant c, we evaluate

$$\int_0^1 \int_{-y}^1 cx^2 ydxdy = \int_0^1 \frac{c}{3}(y + y^4)dy$$
$$+ \frac{c}{3}(\frac{1}{2} + \frac{1}{2}) = \frac{7c}{30}$$

$$\text{so} \quad \frac{7c}{30} = 1 \quad \text{and thus} \quad c = \frac{30}{7}$$

2. The marginal pdfs are

$$f_X(x) = \begin{cases} \frac{15}{7}x^2(1 - x^2), & \text{if } -1 \leq x \leq 0 \\ \frac{15}{7}x^2, & \text{if } 0 < x < 1. \end{cases}$$

and

$$f_Y(y) = \frac{10}{7}(y + y^4), \quad 0 \leq y \leq 1$$

3. For illustration, we calculate two probabilities, in the first of which we use the marginal pdf $f_X(x)$.We have

$$\mathbb{P}(X \leq 0) = \int_{-1}^{0} \frac{15}{7} x^2(1 - x^2)dx = \frac{15}{7}(\frac{1}{3} - \frac{1}{5}) = \frac{2}{7}$$

4.

$$\begin{aligned} \mathbb{P}(0 \leq Y \leq X \leq 1) &= \int_0^1 \int_0^x \frac{30}{7} x^2 y dy dx \\ &= \int_0^1 \frac{15}{7} x^4 dx \\ &= \frac{3}{7} \end{aligned}$$

Exercice: In Example 7, the reader is asked to calculate the means and the variances of X and Y.

Example 8:

Let X and Y have the joint pdf

$$f_{XY}(x,y) = 1, \quad x \leq y \leq x+1, \quad 0 \leq x \leq 1.$$

Find the expected value of X and Y.

Solution: The marginal pdf of X is equal to

$$f_x(x) = \int_x^{x+1} 1dy = 1 \quad 0 \leq x \leq 1$$

and the marginal pdf of Y is given by

$$f_Y(y) = \begin{cases} \int_0^y 1dx = y, & \text{if} \quad 0 \leq y \leq 1 \\ \int_{y-1}^1 1dx = 2-y, & \text{if} \quad 1 \leq y \leq 2 \end{cases}$$

Also,

$$\begin{aligned}
\mathbb{E}(X) = \mu_X &= \int_0^1 x.1dx = \frac{1}{2} \\
\mathbb{E}(Y) = \mu_Y &= \int_0^1 y.ydy + \int_1^2 y(2-y)dy = \frac{1}{3} + \frac{2}{3} = 1 \\
\mathbb{E}(X^2) &= \int_0^1 x^2.1dx = \frac{1}{3} \\
\mathbb{E}(Y^2) &= \int_0^1 y^2.ydy + \int_1^2 y^2(2-y)dy = \frac{7}{6}.
\end{aligned}$$

Thus,

$$\sigma_X^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}, \qquad \sigma_Y^2 = \frac{7}{6} - 1^2 = \frac{1}{6}.$$

3.3 Joint Cumulative Distribution Function

Definition

The joint cumulative distribution function F_{XY} of the continuous random variables X and Y with joint probability distribution function f_{XY} is defined by

$$F_{XY}(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u,v)dvdu$$

Theorem

Let F_{XY} be the joint cumulative distribution function of the continuous random variables X and Y. Then, the joint probability density function (jpdf) of X and Y , is given by the partial derivative

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

Theorem

The random variables X and Y are independent if and only if the joint cumulative probability function factors into the product of their marginal cumulative probability function; namely,

$$F_{XY}(x,y) = F_X(x)F_Y(y) \quad x \in S_X \;\; y \in S_Y,$$

Theorem

Let X and Y be two continuous random variables with joint probability distribution function f_{XY} then

1. *The marginal cumulative distribution function F_X of X is given by*

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f_{XY}(u, v) dv du$$

2. *The marginal cumulative distribution function F_Y of Y is given by*

$$F_Y(y) = \mathbb{P}(Y \leq y) = \int_{-\infty}^{y} \int_{-\infty}^{+\infty} f_{XY}(u, v) du dv$$

Example 9:

Let f be a function defined by with joint (pdf)

$$f_{XY}(x,y) = \begin{cases} 6xy^2, & \text{if} \quad 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

1. Prove that f is a joint (pdf) for two continuous random variables X and Y.
2. Find the cumulative distribution function of X and Y.
3. Find the marginal cumulative distribution of X and Y.
4. Are X and Y independent?
5. Compute $\mathbb{P}(X < 0.6, Y < 0.6)$ and $\mathbb{P}(X + Y > 1.)$.

3.4 Expected value of two continuous random variables

Definition

Let X and Y be random variables of the continuous type with the joint pmf $f_{XY}(x, y)$ on the space Ω. If $u(X, Y)$ is a function of these two random variables, then

$$\mathbb{E}(u(X,Y)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(x,y) f_{XY}(x,y)dxdy$$

if it exists, is called the **mathematical expectation** (or **expected value) of** $u(X, Y)$**.**

Example 11:

Random variables X and Y have joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{5x^2}{2}, & \text{if} \quad -1 \leq x \leq 1,\ 0 \leq y \leq x^2 \\ 0, & \text{elsewhere.} \end{cases}$$

1. What are $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.
2. Find $\mathbb{E}(X+Y)$

4. Conditional distribution and expectation

The conditional cumulative distribution function of a random variable X, given that another random variable Y has taken a value y, is defined by

$$F_{X\backslash Y}(x\backslash y) = \mathbb{P}(X \leq x\backslash Y = y) \quad (1)$$

4.1 Discrete case

Let X and Y have a joint discrete distribution with pmf $p_{XY}(x, y)$ on space S. Say the marginal probability mass functions are $p_X(x)$ and $p_Y(y)$ with spaces S_X and S_Y, respectively. Let event $A = \{X = x\}$ and event $B = \{Y = y\}$, $(x, y) \in S$. Thus, $A \cap B = \{X = x, Y = y\}$. Because

$$\mathbb{P}(A \cap B) = \mathbb{P}(X = x, Y = y) = p_{XY}(x, y)$$

and

$$\mathbb{P}(B) = \mathbb{P}(Y = y) = p_Y(y) > 0 \qquad (\text{since} \quad y \in S_Y)$$

the conditional probability of event A given event B is

$$\mathbb{P}(A \backslash B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{p_{XY}(x, y)}{p_Y(y)}$$

This formula leads to the following definition.

Definition

1. The conditional probability mass function of X, given that $Y = y$, is defined by

$$p_{X\backslash Y}(x\backslash y) = \mathbb{P}(X = x\backslash Y = y) = \frac{p_{XY}(x, y)}{p_Y(y)} \quad \text{provided that } p_Y(y) > 0.$$

2. The conditional probability mass function of Y, given that $X = x$, is defined by

$$p_{X\backslash Y}(y\backslash x) = \mathbb{P}(Y = y\backslash X = x) = \frac{p_{XY}(x, y)}{p_X(x)} \quad \text{provided that } p_X(x) > 0.$$

Remark:

For discrete random variables one has

1. $p_{Y\backslash X}(y\backslash x) \geq 0$
2. $\sum_{y} p_{Y\backslash X}(y\backslash x) = 1$
3. $p_{Y\backslash X}(y\backslash x) = \mathbb{P}(Y = y\backslash X = x)$

Theorem

If X and Y two independent random variables then,

$$p_{X\backslash Y}(x\backslash y) = p_X(x)$$

and

$$p_{Y\backslash X}(y\backslash x) = p_Y(y)$$

Notice that, if the random variables X and Y are discrete then

$$F_{X\backslash Y}(x\backslash y) = \sum_{x_i \leq x} p_{X\backslash Y}(x_i\backslash y)$$

Example 12: Let X and Y have the joint pmf

$$p_{XY}(x,y) = \frac{x+y}{21} \qquad x = 1,2,3 \quad y = 1,2.$$

1. Are X and Y independent?
2. Find $p_{X\backslash Y}(x\backslash y)$ and compute $\mathbb{P}(X = 2\backslash Y = 2) = p_{X\backslash Y}(2\backslash 2)$.
3. Find $p_{Y\backslash X}(y\backslash x)$ and compute $\mathbb{P}(Y = 1\backslash X = 2) = p_{X\backslash Y}(1\backslash 2)$.

Definition

Let X and Y two discrete random variables then

1. The conditional mean of Y given $X = x$, denoted as $\mathbb{E}(Y \backslash X = x)$ or $\mu_{Y \backslash X}$ is

$$\mathbb{E}(Y \backslash X = x) = \sum_y y p_{Y \backslash X}(y \backslash x)$$

2. The conditional variance of Y given $X = x$, denoted as $V(Y \backslash X = x)$ or $\sigma^2_{Y \backslash X}$ is

$$\begin{aligned} V(Y \backslash X = x) &= \sum_y (y - \mu_{Y \backslash X})^2 p_{Y \backslash X}(x \backslash y) \\ &= \sum_y y^2 p_{Y \backslash X}(x \backslash y) - \mu^2_{Y \backslash X} \\ &= \mathbb{E}(Y^2 \backslash X = x) - \mathbb{E}(Y \backslash X = x)^2 \\ &= \sigma^2_{Y \backslash X} \end{aligned}$$

Remark:

$p_{X\backslash Y}$ satisfies the conditions of a probability mass function, and we can compute conditional probabilities such as

$$\mathbb{P}(a < Y < b\backslash X = x) = \sum_{a<y<b} p_{Y\backslash X}(y\backslash x)$$

and conditional expectations such as

$$\mathbb{E}(u(Y)\backslash X = x) = \sum_{y} u(y)p_{Y\backslash X}(y\backslash x)$$

Example 13: We use the background of **Example 12** and compute $\mu_{Y\backslash X}$ and $\sigma_{Y\backslash X}$ when $x = 3$.

4.2 Continuous case

Let X be a continuous random variable. A consistent definition of the conditional density function of X given $Y = y$, $f_{X\backslash Y}(x\backslash y)$ is the derivative of its corresponding conditional cumulative distribution function. Hence,

$$f_{X\backslash Y}(x\backslash y) = \frac{dF_{X\backslash Y}(x\backslash y)}{dx}$$

where $F_{X\backslash Y}(x\backslash y)$ is defined in (1).

Definition

1. The conditional probability density function of X, given that $Y = y$, is defined by

$$f_{X\backslash Y}(x\backslash y) = \frac{f_{XY}(x, y)}{f_Y(y)} \quad f_Y(y) > 0$$

2. The conditional probability density function of Y, given that X = x, is defined by

$$f_{Y\backslash X}(y\backslash x) = \frac{f_{XY}(x, y)}{f_X(x)} \quad f_X(x) > 0$$

Remark:

For continuous random variables one has

1. $f_{Y\backslash X}(y\backslash x) \geq 0$
2. $\int_{-\infty}^{+\infty} f_{Y\backslash X}(y\backslash x)dy = 1$
3. $\mathbb{P}(a < Y < b\backslash X = x) = \int_{a}^{b} f_{Y\backslash X}(y\backslash x)dy$

Theorem

If X and Y two independent random variables then,

$$f_{X\backslash Y}(x\backslash y) = f_X(x)$$

and

$$f_{Y\backslash X}(y\backslash x) = f_Y(y)$$

Notice that, if the random variables X and Y are continuous then

$$F_{X\backslash Y}(x\backslash y) = \int_{-\infty}^{x} f_{X\backslash Y}(u\backslash y)du$$

Example 14: Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6}, & \text{if} \quad 0 \le x \le 1, \quad 0 \le y \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

For $0 \le y \le 2$, find

1. the conditional PDF of X given $Y = y$;
2. $\mathbb{P}(X < \frac{1}{2}\backslash Y = y)$
3. $\mathbb{P}(X < \frac{1}{2}\backslash Y = 1)$

Definition

Let X and Y two continuous random variables then

1. The conditional mean of Y given $X = x$, denoted as $\mathbb{E}(Y \backslash X = x)$ or $\mu_{Y \backslash X}$ is

$$\mathbb{E}(Y \backslash X = x) = \int_y y f_{Y \backslash X}(y \backslash x) dy$$

2. The conditional variance of Y given $X = x$, denoted as $V(Y \backslash X = x)$ or $\sigma^2_{Y \backslash X}$ is

$$\begin{aligned} V(Y \backslash X = x) &= \int_y (y - \mu_{Y \backslash X})^2 f_{Y \backslash X}(y \backslash x) dy \\ &= \int_y y^2 f_{Y \backslash X}(y \backslash x) dy - \mu^2_{Y \backslash X} \end{aligned}$$

Remark:

$f_{X\backslash Y}$ satisfies the conditions of a probability density function, and we can compute conditional probabilities such as

$$\mathbb{P}(a < Y < b \backslash X = x) = \int_a^b f_{Y\backslash X}(y\backslash x)dy$$

and conditional expectations such as

$$\mathbb{E}(u(Y)\backslash X = x) = \int_y u(y) f_{Y\backslash X}(y\backslash x)dy$$

Example 15:

Let X and Y have the pdf

$$f_{XY}(x,y) = \begin{cases} axy, & \text{if} \quad 0 < x < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

1. Prove that $a = 8$
2. Compute $\mathbb{E}(X^2 Y)$
3. Find $f_{X \backslash Y}$
4. Compute $\mathbb{E}(X^2 \backslash Y = 0.5)$