# Chapter 2

# Life tables

**LEARNING OUTCOMES:**

1. **To apply life tables**
2. **To understand two assumptions for fractional ages: uniform distribution of death and constant force of mortality**
3. **To understand and model the effect of selection**
4. **To calculate moments for future lifetime random variables**



## 2.1 Life table functions

A life table is a concrete way to look at the survivorship random variable. A life table specifies a certain number of lives at a starting integer age $x_0$. Usually $x_0 = 0$. This number of lives at age $x_0$ is called the radix. Then for each integer $x > x_0$, the expected number of survivors is listed. The notation for the number of lives listed in the table for age $x$ is $\ell_x$.

Assuming for simplicity that $x_0 = 0$, the random variable for the number of lives at each age, $\mathcal{L}(x)$, is a binomial random variable with parameters $\ell_0$ and ${}_xp_0$. so the expected number of lives is

$$\ell_x = \ell_0 \ {}_xp_0. \text{ Similarly, } \ \ell_{x+t} = \ell_x \ {}_tp_x.$$

More importantly, ${}_tp_x$ can be calculated from the table using ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$. A life table also lists the expected number of deaths at each age; $d_x$ is the notation for this concept. Thus

$$d_x = \ell_x - \ell_{x+1} = \ell_x - \ell_x \ p_x = \ell_x \,(1 - p_x) = \ell_x \ q_x.$$

Therefore,

$$q_x = \frac{d_x}{\ell_x}.$$

In other words the difference $\ell_x - \ell_{x+t}$ is the expected number of deaths over the age interval of $[x,\ x+t)$. We denote this by ${}_td_x$. It immediately follows that

$${}_td_x = \ell_x - \ell_{x+t}.$$

We can then calculate ${}_tq_x$ and ${}_{m|n}q_x$, by the following two relations:

$$\begin{aligned}
{}_tq_x &= 1 - \frac{\ell_{x+t}}{\ell_x} = \frac{\ell_x - \ell_{x+t}}{\ell_x} = \frac{{}_td_x}{\ell_x} \\
{}_{m|n}q_x &= \frac{\ell_{x+m} - \ell_{x+m+n}}{\ell_x} = \frac{{}_nd_{x+m}}{\ell_x} = \frac{\sum_{k=0}^{n-1} d_{x+m+k}}{\ell_x} = P((x) \text{ dies between age } x+m \ \text{ and } \ x+m+n).
\end{aligned}$$

The Illustrative Life Table contains a lot of information.

https://www.soa.org/Files/Edu/2018/ltam-standard-ultimate-life-table.pdf

For now, you only need to know and use the first three columns: $x$, $\ell_x$, and $q_x$. For example, to obtain the value $q_{60}$, simply use the column labeled $q_x$. You should obtain $q_{60} = 0.003398$. It is also possible, but more tedious, to calculate $q_{60}$ using the column labeled $\ell_x$ we have

$$q_{60} = 1 - \frac{96,305.8}{96,634.1} = 1 - 0.996603 = 0.003397.$$

To get values of ${}_tp_x$ and ${}_tq_x$ for $t > 1$, you should always use the column labeled $\ell_x$. For example, we have

$${}_5p_{60} = \frac{\ell_{65}}{\ell_{60}} = \frac{94,579.7}{96,634.1} = 0.97874 \quad \text{and} \quad {}_5q_{60} = 1 - {}_5p_{60} = 1 - 0.97874 = 0.02126.$$

**Exercise 2.1.1** *You are given the following excerpt of a life table:*

| $x$ | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|
| $\ell_x$ | $100,000$ | $99,975.0$ | $99,949.7$ | $99,924.0$ | $99,897.8$ | $99,871.1$ |
| $d_x$ | 25 | 25 | 26 | 26 | 27 | 27 |

*Calculate the following: a)* ${}_5p_{20}$*, b)* $q_{24}$*, c)* ${}_{4|}q_{20}$.

**Solution**:

**Example 2.1.1** *You are given* $S_0(t) = 1 - \frac{t}{100}$ *for* $0 \le t \le 100$ *and* $\ell_0 = 100$.
*(a) Find an expression for* $\ell_x$*, for* $0 \le x \le 100$*. (b) Calculate* $q_2$*. (c) Calculate* ${}_3q_2$.

**Solution**:

**Exercise 2.1.2** *Use a radix of* $1,000,000$ *at age* 60 *to complete the following mortality table*

| $x$ | 60 | 61 | 62 | 63 | 64 | 65 |
|---|---|---|---|---|---|---|
| $q_x$ | 0.001 | 0.002 | 0.003 | 0.004 | 0.005 | |
| $\ell_x$ | ? | ? | ? | ? | ? | ? |

**Solution**: We shall recursively calculate $\ell_x$, $x = 61, \ldots, 65$ using $\ell_{x+1} = \ell_x(1 - q_x)$, thus

| $x$ | 60 | 61 | 62 | 63 | 64 | 65 |
|---|---|---|---|---|---|---|
| $q_x$ | 0.001 | 0.002 | 0.003 | 0.004 | 0.005 | **Unknown** |
| $\ell_x$ | $1,000,000$ | $999,000$ | $997,002$ | $994,011$ | $990,035$ | $985,085$ |

**Exercise 2.1.3** *You are given the following life table:*

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $\ell_x$ | | | 890 |
| $d_x$ | 50 | | |
| $p_x$ | | 0.98 | |

*Calculate* ${}_2p_0$.

**Solution**: We know that $\ell_0 = \ell_1 + d_0$ and $\ell_2 = \ell_{1+1} = \ell_1\ p_1$, hence

$$ {}_2p_0 = \frac{\ell_2}{\ell_0} = \frac{\ell_2}{\ell_1 + d_0} = \frac{\ell_2}{\frac{\ell_2}{p_1} + d_0} = \frac{890}{\frac{890}{0.98} + 50} = 0.92886 $$

**Exercise 2.1.4** *You are given:*
*(i) The probability that a person age* 50 *is alive at age* 55 *is* 0.9.
*(ii) The probability that a person age* 55 *is not alive at age* 60 *is* 0.15.
*(iii) The probability that a person age* 50 *is alive at age* 65 *is* 0.54.
*Calculate the probability that a person age* 55 *dies between ages* 60 *and* 65.

**Solution**: The required probability is

$$ P\left(5 \le T_{55} \le 10\right) = {}_{5|5}q_{55} = \begin{cases} {}_5p_{55} - {}_{10}p_{55} \\ {}_{10}q_{55} - {}_5q_{55} \\ {}_5p_{55} \times {}_5q_{60}. \end{cases} $$

From given information we have ${}_5p_{50} = 0.9$, ${}_5q_{55} = 0.15$, ${}_{15}p_{50} = 0.54$. First observe that ${}_{15}p_{50} = {}_5p_{50} \times {}_{10}p_{55}$, hence ${}_{10}p_{55} = \frac{{}_{15}p_{50}}{{}_5p_{50}} = \frac{0.54}{0.9} = 0.6$. Finally ${}_{5|5}q_{55} = 0.85 - 0.6 = 0.25$.

**Remark**: we can also write ${}_5q_{60} = 1 - {}_5p_{60} = 1 - {}_5p_{50+10} = 1 - \frac{{}_{15}p_{50}}{{}_5p_{50}}$.

## 2.2 Fractional age assumptions

### 2.2.1 Uniform distribution of death between integral ages

The Uniform Distribution of Death (UDD) assumption is extensively useful to calculate survival and death probabilities for fractional ages. The idea behind this assumption is that we use a bridge, denoted by $U$, to connect the (continuous) future lifetime random variable $T_x$ and the (discrete) curtate future lifetime random variable $K_x$. $T_x = K_x + U$, hence $U$ takes its values in $[0,\ 1]$. It is assumed that $U$ follows a uniform distribution over the interval $[0,\ 1]$, and that $U$ and $K_x$ are independent. Then, for $0 \le r < 1$ and an integral value of $x$, we have

$$ {}_rq_x = P\left(T_x \le r\right) = P\left(K_x + U \le r\right) = P\left(U \le r; K_x = 0\right) = P\left(U \le r\right) P\left(K_x = 0\right) = rq_x. $$

This means that under UDD, we have, for example, ${}_{0.3}q_{35} = 0.3q_{35}$. The value of $q_{35}$ can be obtained straightforwardly from the life table: $q_{35} = 0.000391$. To calculate ${}_rp_x$, for $0 \le r < 1$, we use ${}_rp_x = 1 - {}_rq_x = 1 - rq_x$. For example, we have ${}_{0.7}p_{40} = 1 - 0.7q_{40}$.

**Remark 2.2.1** *Observe that undeer UDD* ${}_rp_x \neq rp_x$. *The equation* ${}_rq_x = rq_x$ *for* $0 \le r < 1$ *and an integral age* $x$ *is equivalent to a linear interpolation between* $\ell_x$ *and* $\ell_{x+1}$*, that is,*

$$ \ell_{x+r} = (1-r)\ell_x + r\ell_{x+1}. $$

*Indeed*

$$ \begin{aligned} \ell_{x+r} &= \ell_x\ {}_rp_x = \ell_x\left(1 - {}_rq_x\right) = \ell_x\left(1 - rq_x\right) \\ &= \ell_x\left(1 - r\left(1 - p_x\right)\right) = \left(1-r\right)\ell_x + r\ell_x p_x = \left(1-r\right)\ell_x + r\ell_{x+1}. \end{aligned} $$

**Exercise 2.2.1** *Calculate under UDD* ${}_{2.7}p_{30}$ *using ILT (see LMS)*

**Solution**: We have under UDD

$$\begin{aligned} {}_{2.7}p_{30} &= {}_{2}p_{30}\ {}_{0.7}p_{32} = {}_{2}p_{30}\left(1 - {}_{0.7}q_{32}\right) = \frac{\ell_{32}}{\ell_{30}}\left(1 - 0.7q_{32}\right) \\ &= \frac{99,663.2}{99,727.3}\left(1 - 0.7 \times 0.000341\right) = 0.99912 \end{aligned}$$

or alternatively

$$\begin{aligned} {}_{2.7}p_{30} &= {}_{2}p_{30}\ {}_{0.7}p_{32} = p_{30}p_{31}\left(1 - {}_{0.7}q_{32}\right) = \left(1 - q_{30}\right)\left(1 - q_{31}\right)\left(1 - 0.7q_{32}\right) \\ &= \left(1 - 0.000315\right)\left(1 - 0.000327\right)\left(1 - 0.7 \times 0.000341\right) = 0.99912 \end{aligned}$$

Under **UDD**, we have the following equation for calculating the density function:

$$f_x(r) = q_x \quad \text{for all } 0 \leq r < 1.$$

Indeed $f_x(r) = \frac{d({}_rq_x)}{dr} = \frac{d(rq_x)}{dr} = q_x$.

We can write also that

$$q_x = f_x(r) = {}_rp_x\mu_{x+r} = \left(1 - {}_rq_x\right)\mu_{x+r} = \left(1 - rq_x\right)\mu_{x+r},$$

therefore the force of mortality under UDD is given by

$$\mu_{x+r} = \frac{q_x}{1 - rq_x} \iff q_x = \frac{\mu_{x+r}}{r\mu_{x+r} + 1} \quad \text{for all } 0 \leq r < 1 \tag{2.2.1}$$

**Question:** What if the subscript on the right–hand–side is not an integer? for example use ILT to calculate ${}_{0.2}p_{25.6}$ and ${}_{0.5}q_{30.7}$ under UDD.

**Solution:**

$${}_{0.2}p_{25.6} = \frac{{}_{0.8}p_{25}}{{}_{0.6}p_{25}} = \frac{1 - {}_{0.8}q_{25}}{1 - {}_{0.6}q_{25}} \overset{\textbf{UDD}}{=} \frac{1 - 0.8q_{25}}{1 - 0.6q_{25}} = \frac{1 - 0.8 \times 0.000273}{1 - 0.6 \times 0.000273} = 0.99995$$

and ${}_{0.5}q_{30.7} = 1 - {}_{0.5}p_{30.7}$. First we have

$$\begin{aligned} {}_{0.5}p_{30.7} &= \frac{{}_{1.2}p_{30}}{{}_{0.7}p_{30}} = \frac{p_{30}\ {}_{0.2}p_{31}}{{}_{0.7}p_{30}} = \left(1 - q_{30}\right)\frac{1 - {}_{0.2}q_{31}}{1 - {}_{0.7}q_{30}} \overset{\textbf{UDD}}{=} \left(1 - q_{30}\right)\frac{1 - 0.2q_{31}}{1 - 0.7q_{30}} \\ &= \left(1 - 0.000315\right)\frac{1 - 0.2 \times 0.000327}{1 - 0.7 \times 0.000315} = 0.99984 \end{aligned}$$

Finally ${}_{0.5}q_{30.7} = 1 - 0.99984 = 0.00016$.

**Exercise 2.2.2** *For a certain mortality table, you are given:*
*(i)* $\mu_{80.5} = 0.020202$, $\mu_{81.5} = 0.040816$ *and* $\mu_{82.5} = 0.061856$
*(ii) Deaths are uniformly distributed between integral ages.*
*Calculate the probability that a person age* 80.5 *will die within two years.*

**Solution:** The required probability is ${}_2q_{80.5} = 1 - {}_2p_{80.5}$. First calculate ${}_2p_{80.5}$. We have

$${}_2p_{80.5} = \frac{{}_{2.5}p_{80}}{{}_{0.5}p_{80}} = \frac{{}_2p_{80}\ {}_{0.5}p_{82}}{1 - {}_{0.5}\,q_{80}} = \frac{p_{80}p_{81}\left(1 - {}_{0.5}q_{82}\right)}{1 - {}_{0.5}\,q_{80}} \overset{\textbf{UDD}}{=} \frac{\left(1 - q_{80}\right)\left(1 - q_{81}\right)\left(1 - 0.5q_{82}\right)}{1 - 0.5q_{80}}.$$

Recall also that from (2.2.1) $q_x = \frac{\mu_{x+r}}{r\mu_{x+r}+1}$ thus

$$\begin{aligned} q_{80} &= \frac{\mu_{80.5}}{0.5 \times \mu_{80.5} + 1} = \frac{0.020202}{0.5 \times 0.020202 + 1} = 0.02 \\ q_{81} &= \frac{\mu_{81.5}}{0.5 \times \mu_{81.5} + 1} = \frac{0.040816}{0.5 \times 0.040816 + 1} = 0.04 \\ q_{82} &= \frac{\mu_{82.5}}{0.5 \times \mu_{82.5} + 1} = \frac{0.061856}{0.5 \times 0.061856 + 1} = 0.06 \end{aligned}$$

Therefore

$$ {}_2p_{80.5} = \frac{(1-q_{80})(1-q_{81})(1-0.5q_{82})}{1-0.5q_{80}} = \frac{(1-0.02)(1-0.04)(1-0.5\times 0.06)}{1-0.5\times 0.02} = 0.92179. $$

Finally ${}_2q_{80.5} = 1 - {}_2p_{80.5} = 1 - 0.92179 = 0.07821.$

### 2.2.2 Constant force of mortality or Exponential distribution between integral ages

The idea behind this assumption is that for every age $x$ and any $0 \le r < 1$ we assume that $\mu_{x+r} = \mu$. This implies that

$$ p_x = e^{-\int_0^1 \mu_{x+u} du} = e^{-\int_0^1 \mu\ du} = e^{-\mu}. $$

For any integral age $x$, we can write

$$ {}_rp_x = e^{-\int_0^r \mu_{x+u} du} = e^{-\int_0^r \mu\ du} = e^{-\mu r} = \left(e^{-\mu}\right)^r = p_x^r \quad \forall\ \ 0 \le r < 1 $$

Moreover, for any integral age $x$ and any $r$, $u$ such that $0 \le r < 1$ and $r + u \le 1$ we have

$$ {}_rp_{x+u} = e^{-\int_0^r \mu_{x+u+s}\ ds} = e^{-\int_0^r \mu\ du} = e^{-\mu r} = \left(e^{-\mu}\right)^r = p_x^r. $$

**Exercise 2.2.3** *Calculate ${}_{0.2}p_{25.6}$ and ${}_{0.5}q_{30.7}$ under CFM using ILT (see LMS).*

**Solution:**

$$ {}_{0.2}p_{25.6} \overset{\textbf{CFM}}{=} p_{25}^{0.2} = (1-q_{25})^{0.2}(1-0.000273)^{0.2} = 0.99995 $$

and ${}_{0.5}q_{30.7} = 1 - {}_{0.5}p_{30.7}$. First we have

$$ \begin{aligned} {}_{0.5}p_{30.7} &= \frac{{}_{1.2}p_{30}}{{}_{0.7}p_{30}} = \frac{p_{30}\ {}_{0.2}p_{31}}{{}_{0.7}p_{30}} \overset{\textbf{CFM}}{=} p_{30}\frac{p_{31}^{0.2}}{p_{30}^{0.7}} = (1-q_{30})^{0.3}(1-q_{31})^{0.2} \\ &= (1-0.000315)^{0.3}(1-0.000327)^{0.2} = 0.99984. \end{aligned} $$

Finally ${}_{0.5}q_{30.7} = 1 - 0.99984 = 0.00016.$

**Exercise 2.2.4** *For a certain mortality table, you are given:*
*(i)* $\mu_{80.5} = 0.020202$, $\mu_{81.5} = 0.040816$ *and* $\mu_{82.5} = 0.061856$
*(ii) Assume constant force of mortality between integral ages.*
*Calculate the probability that a person age* 80.5 *will die within two years.*

**Solution:** The required probability is ${}_2q_{80.5} = 1 - {}_2p_{80.5}$. First calculate ${}_2p_{80.5}$. We have

$$ {}_2p_{80.5} = \frac{{}_{2.5}p_{80}}{{}_{0.5}p_{80}} = \frac{{}_2p_{80}\ {}_{0.5}p_{82}}{{}_{0.5}p_{80}} \overset{\textbf{CFM}}{=} \frac{p_{80}\ p_{81}\ p_{82}^{0.5}}{p_{80}^{0.5}} = p_{80}^{0.5}\ p_{81}\ \left(p_{82}^{0.5}\right) = p_{81}\left(p_{80}\ p_{82}\right)^{0.5} $$

Recall that

$$ \begin{aligned} p_{80} &= e^{-\int_0^1 \mu_{80+u} du} = e^{-\mu_{80.5}} = e^{-0.020202} = 0.98, \\ p_{81} &= e^{-\int_0^1 \mu_{81+u} du} = e^{-\mu_{81.5}} = e^{-0.040816} = 0.96, \\ p_{82} &= e^{-\int_0^1 \mu_{82+u} du} = e^{-\mu_{82.5}} = e^{-0.061856} = 0.94. \end{aligned} $$

Therefore

$$ {}_2p_{80.5} = (0.96)(0.98 \times 0.94)^{0.5} = 0.92140. $$

Finally ${}_2q_{80.5} = 1 - {}_2p_{80.5} = 1 - 0.92140 = 0.0786.$

## 2.3 Select–and–ultimate tables

Suppose you selected two 40–year–old men from the population. The first one was selected randomly, whereas the second one had recently purchased a life insurance policy. Would the mortality rate for both of these, $q_{40}$ be the same? No. The second person was unde1written for a life insurance policy, which means his medical situation was reviewed. He had to satisfy certain guidelines regarding weight, blood pressure, blood lipids, family history, existing medical conditions, and possibly even driving record and credit history. The fact he was approved for an insurance policy implies that his mortality rate is lower than that of a randomly selected 40–year–old male.

Not only would $q_{40}$ be different. If both men survived 5 years, $q_{45}$ would be different as well. A man whose health was established 5 years ago will have better mortality than a randomly selected man.

A mortality table for the insured population must consider both the age of issue and the duration since issue. Mortality rates would require two arguments and need a notation like $q(x,t)$ where $x$ is the issue age and $t$ the duration since issue.

International Actuarial Notation provides two-parameter notation for all actuarial functions. The parameters are written as subscripts with a bracket around the first parameter and a plus sign between the parameters. In other words, the subscript is of the form $[x]+t$. When $t=0$, it is omitted. Thus the mortality rate for a 40–year–old who just purchased a life insurance policy would be written as $q_{[40]}$. The mortality rate for a 45–year–old who purchased a policy at age 40 would be written $q_{[40]+5}$

**Exercise 2.3.1** *You are given:* $\ell_{[45]}=1000$, ${}_5q_{[45]}=0.04$, ${}_5q_{[45]+5}=0.05$. *Calculate* $\ell_{[45]+10}$.

**Solution**: We know ${}_5q_{[45]+5}=1-\ {}_5p_{[45]+5}=1-\frac{\ell_{[45]+10}}{\ell_{[45]+5}}$

Due to the effect of underwriting, a select death probability $q_{[x]+t}$ must be no greater than the corresponding ordinary death probability $q_{x+t}$. However, the effect of underwriting will not last forever. The period after which the effect of underwriting is completely gone is called the select period. Suppose that the **select period** is $n$ years, we have

$$q_{[x]+t}<q_{x+t} \text{ for all } \ t<n \ \text{ and } \ q_{[x]+t}=q_{x+t} \text{ for all } \ t\geq n$$

The ordinary death probability $q_{x+t}$ is called the **ultimate death probability** and $q_{[x]+t}$ is called the **select death probability**. A life table that contains both select probabilities and ultimate probabilities is called a select–and–ultimate life table. The following is an excerpt of a (hypothetical) select–and–ultimate table with a select period of **two** years:

| $x$ | $q_{[x]}$ | $q_{[x]+1}$ | $q_{x+2}$ | $x+2$ |
|---|---|---|---|---|
| 40 | 0.04 | 0.06 | 0.08 | 42 |
| 41 | 0.05 | 0.07 | 0.09 | 43 |
| 42 | 0.06 | 0.08 | 0.10 | 44 |
| 43 | 0.07 | 0.09 | 0.11 | 45 |

It is important to know how to apply such a table. Let us consider a person who is currently age 41 and is just selected. To further illustrate, let us consider a person who is currently age 41 and was selected at age 40. The death probabilities for this person are as follows:

| | | |
|---|---|---|
| Age | 41 | $q_{[41]}=0.05$ |
| Age | 42 | $q_{[41]+1}=0.07$ |
| Age | 43 | $q_{[41]+2}=q_{43}=0.09$ |
| Age | 44 | $q_{[41]+3}=q_{44}=0.10$ |
| Age | 45 | $q_{[41]+4}=q_{45}=0.11$ |

and

| | | |
|---|---|---|
| Age | 41 | $q_{[40]+1}=0.06$ |
| Age | 42 | $q_{[40]+2}=0.08$ |
| Age | 43 | $q_{[40]+3}=q_{43}=0.09$ |
| Age | 44 | $q_{[40]+4}=q_{44}=0.10$ |
| Age | 45 | $q_{[40]+5}=q_{45}=0.11$ |

We may measure the effect of underwriting by the index of selection, which is defined as follows:

$$I(x,k)=1-\frac{q_{[x]+k}}{q_{x+k}}.$$

One can interpret this formula as follows: If the effect of underwriting is strong, then $q_{[x]+k}$ would be small compared to $q_{x+k}$, and therefore $I(x,k)$ would be close to one. By contrast, if the effect of underwriting is weak, then $q_{[x]+k}$ would be close to $q_{x+k}$, and therefore $I(x,k)$ would be close to zero.

For example, on the basis of the preceding table,

$$I(41,1) = 1 - \frac{q_{[41]+1}}{q_{42}} = 1 - \frac{0.07}{0.08} = 1 - \frac{7}{8} = \frac{1}{8} = 0.125.$$

**Example 2.3.1** *Aicha was a newly selected life on* 01/01/2000 *and her age on* 01/01/2001 *is* 21. *Let* $p$ *denotes the probability on* 01/01/2001 *that Aicha will be alive on* 01/01/2006. *For a select–and–ultimate mortality table with a* 3*–year select period:*

| $x$ | $q_{[x]}$ | $q_{[x]+1}$ | $q_{[x]+2}$ | $q_{x+3}$ | $x+3$ |
|---|---|---|---|---|---|
| 20 | 0.09 | 0.11 | 0.13 | 0.15 | 23 |
| 21 | 0.10 | 0.12 | 0.14 | 0.16 | 24 |
| 22 | 0.11 | 0.13 | 0.15 | 0.17 | 25 |
| 23 | 0.12 | 0.14 | 0.16 | 0.18 | 26 |
| 24 | 0.13 | 0.15 | 0.17 | 0.19 | 27 |

*Calculate* $p$.

**Solution**: Aicha is now age 21 and was selected at age 20. So the probability that White will be alive 5 years from now can be expressed as $p = {}_5p_{[20]+1}$. We have

$$\begin{aligned}
{}_5p_{[20]+1} &= p_{[20]+1}\ {}_4p_{[20]+2} = p_{[20]+1}\ p_{[20]+2}\ {}_3p_{[20]+3} = p_{[20]+1}\ p_{[20]+2}\ p_{[20]+3}\ {}_2p_{[20]+4} \\
&= p_{[20]+1}\ p_{[20]+2}\ p_{[20]+3}\ p_{[20]+4}\ p_{[20]+5} = \prod_{k=1}^{5} p_{[20]+k} = \prod_{k=1}^{5} \left(1 - q_{[20]+k}\right) \\
&= \left(1 - q_{[20]+1}\right)\left(1 - q_{[20]+2}\right)\left(1 - q_{23}\right)\left(1 - q_{24}\right)\left(1 - q_{25}\right) \\
&= (1 - 0.11)(1 - 0.13)(1 - 0.15)(1 - 0.16)(1 - 0.17) = 0.45887
\end{aligned}$$

Sometimes, you may be given a select–and–ultimate table that contains the life table function $\ell_{[x]}$. In this case, you can calculate survival and death probabilities by using the following equations:

$${}_tp_{[x]+u} = \frac{\ell_{[x]+u+t}}{\ell_{[x]+u}} \quad \text{and} \quad {}_tq_{[x]+u} = 1 - \frac{\ell_{[x]+u+t}}{\ell_{[x]+u}} = \frac{\ell_{[x]+u} - \ell_{[x]+u+t}}{\ell_{[x]+u}}$$

**Exercise 2.3.2** *A select–and–ultimate table with a select period of* 2*–years is given as follows:*

| $x$ | $\ell_{[x]}$ | $\ell_{[x]+1}$ | $\ell_{x+2}$ | $x+2$ |
|---|---|---|---|---|
| 70 | 22507 | 22200 | 21722 | 72 |
| 71 | 21500 | 21188 | 20696 | 73 |
| 72 | 20443 | 20126 | 19624 | 74 |
| 73 | 19339 | 19019 | 18508 | 75 |
| 74 | 18192 | 17871 | 17355 | 76 |

1. *Compute* ${}_3q_{73}$.

2. *Compute the probability that a life age* 71 *dies between ages* 75 *and* 76, *given that the life was selected at age* 70.

3. *Assuming UDD between integral ages, calculate* ${}_{0.5}q_{[70]+0.7}$.

4. *Assuming CFM between integral ages, calculate* ${}_{0.5}q_{[70]+0.7}$.

**Solution**:

1. We have
$$ {}_{3}q_{73} = 1 - \frac{\ell_{73+3}}{\ell_{73}} = 1 - \frac{17355}{20696} = 0.161 $$

2. The probability that a life age 71 dies between ages 75 and 76, given that the life was selected at age 70 is given by $P\left(4 \leq T_{[70]+1} < 5\right) = {}_{4|}q_{[70]+1}$, hence
$$ \begin{aligned} {}_{4|}q_{[70]+1} &= {}_{5}q_{[70]+1} - {}_{4}q_{[70]+1} = {}_{4}p_{[70]+1} - {}_{5}p_{[70]+1} \\ &= \frac{\ell_{[70]+5} - \ell_{[70]+6}}{\ell_{[70]+1}} = \frac{\ell_{75} - \ell_{76}}{\ell_{[70]+1}} = \frac{18508 - 17355}{22200} = 0.0519 \end{aligned} $$

3. Assume UDD, we have
$$ \begin{aligned} {}_{0.5}q_{[70]+0.7} &= 1 - {}_{0.5}p_{[70]+0.7} = 1 - \frac{{}_{1.2}p_{[70]}}{{}_{0.7}p_{[70]}} \\ &= 1 - \frac{p_{[70]} \times_{0.2} p_{[70]+1}}{{}_{0.7}p_{[70]}} = 1 - \frac{p_{[70]}\left(1 -_{0.2} q_{[70]+1}\right)}{1 -_{0.7} q_{[70]}} \\ &= 1 - \frac{p_{[70]}\left(1 - 0.2q_{[70]+1}\right)}{1 - 0.7q_{[70]}} \quad \text{(UDD)} \\ &= 1 - \frac{\frac{\ell_{[70]+1}}{\ell_{[70]}}\left(1 - 0.2\left(1 - \frac{\ell_{[70]+2}}{\ell_{[70]+1}}\right)\right)}{1 - 0.7\left(1 - \frac{\ell_{[70]+1}}{\ell_{[70]}}\right)} \\ &= 1 - \frac{\frac{22200}{22507}\left(1 - 0.2\left(1 - \frac{21722}{22200}\right)\right)}{1 - 0.7\left(1 - \frac{22200}{22507}\right)} \simeq 0.008 \end{aligned} $$

4. Assume CFM, we have from 3.
$$ \begin{aligned} {}_{0.5}q_{[70]+0.7} &= 1 - \frac{p_{[70]}\ {}_{0.2}p_{[70]+1}}{{}_{0.7}p_{[70]}} = 1 - \frac{p_{[70]}\left(p_{[70]+1}\right)^{0.2}}{\left(p_{[70]}\right)^{0.7}} \\ &= 1 - \left(p_{[70]}\right)^{0.3}\left(p_{[70]+1}\right)^{0.2} \\ &= 1 - \left(\frac{\ell_{[70]+1}}{\ell_{[70]}}\right)^{0.3}\left(\frac{\ell_{[70]+2}}{\ell_{[70]+1}}\right)^{0.2} \\ &= 1 - \left(\frac{22200}{22507}\right)^{0.3}\left(\frac{21722}{22200}\right)^{0.2} \simeq 0.008. \end{aligned} $$

**Exercise 2.3.3** *A select–and–ultimate table with a select period of 4–years is given as follows:*

| $x$ | $q_{[x]}$ | $q_{[x]+1}$ | $q_{[x]+2}$ | $q_{[x]+3}$ | $q_{x+4}$ | $x+4$ |
|---|---|---|---|---|---|---|
| 40 | 0.00101 | 0.00175 | 0.00205 | 0.00233 | 0.00257 | 44 |
| 41 | 0.00113 | 0.00188 | 0.00220 | 0.00252 | 0.00293 | 45 |
| 42 | 0.00127 | 0.00204 | 0.00240 | 0.00280 | 0.00337 | 46 |
| 43 | 0.00142 | 0.00220 | 0.00262 | 0.00316 | 0.00384 | 47 |
| 44 | 0.00157 | 0.00240 | 0.00301 | 0.00367 | 0.00445 | 48 |

1. *The index of selection $x$ is defined by $I(x,k) = 1 - \frac{q_{[x]+k}}{q_{x+k}}$. Calculate the index at age 44 for $k = 0$, 1, 2, 3.*

2. *Construct the select life table function of table $\ell_{[x]+k}$ for $x = 40,\ 41,\ 42$ and $k = 0,\ 1,\ 2,\ 3,4$ given that $\ell_{[40]} = 10,000$*

3. *Calculate i) ${}_3q_{[41]+1}$, ii) ${}_{3|2}q_{[41]}$.*

**Solution:**

1. We have

$$I(44,0) = 1 - \frac{q_{[44]}}{q_{44}} = 1 - \frac{0.00157}{0.00257} = 0.38911, \quad I(44,1) = 1 - \frac{q_{[44]+1}}{q_{45}} = 1 - \frac{0.00240}{0.00293} = 0.18089,$$

and

$$I(44,2) = 1 - \frac{q_{[44]+2}}{q_{46}} = 1 - \frac{0.00301}{0.00337} = 0.10682, \; I(44,3) = 1 - \frac{q_{[44]+3}}{q_{47}} = 1 - \frac{0.00367}{0.00384} = 0.044271$$

2. The select life table function of table $\ell_{[x]+k}$ for $x = 40,\ 41,\ 42$ and $k = 0,\ 1,\ 2,\ 3,4$ is given by the following table using the formulas

$$\ell_{[40]+k+1} = \left(1 - q_{[40]+k}\right)\ell_{[40]+k} \;\text{ and }\; \ell_{[41]+4} = \ell_{45} = (1 - q_{44})\,\ell_{44},$$

moreover

$$\ell_{[41]+k} = \frac{\ell_{[41]+k+1}}{1 - q_{[41]+k}} \text{ for } k = 0,\ 1,\ 2,\ 3 \text{ and } \ell_{[42]+4} = \ell_{46} = (1 - q_{45})\,\ell_{45},$$

and then use again $\ell_{[42]+k} = \frac{\ell_{[42]+k+1}}{1-q_{[42]+k}}$ for $k = 0,\ 1,\ 2,\ 3.$

| $x$ | $\ell_{[x]}$ | $\ell_{[x]+1}$ | $\ell_{[x]+2}$ | $\ell_{[x]+3}$ | $\ell_{x+4}$ | | $x+4$ |
|---|---|---|---|---|---|---|---|
| | | $\longrightarrow$ | $\longrightarrow$ | $\longrightarrow$ | $\longrightarrow$ | | |
| 40 | 10000.00 | 9989.90 | 9972.42 | 9951.97 | 9928.79 | $\downarrow$ | 44 |
| | | $\longleftarrow$ | $\longleftarrow$ | $\longleftarrow$ | $\longleftarrow$ | | |
| 41 | 9980.20 | 9968.92 | 9950.18 | 9928.29 | 9903.27 | $\downarrow$ | 45 |
| | | $\longleftarrow$ | $\longleftarrow$ | $\longleftarrow$ | $\longleftarrow$ | | |
| 42 | 9958.74 | 9946.09 | 9925.80 | 9901.98 | 9874.25 | | 46 |

3. By the definition in terms of the select life table function we have

$${}_3q_{[41]+1} = 1 - \frac{\ell_{[41]+4}}{\ell_{[41]+1}} = 1 - \frac{9903.27}{9968.92} = 0.006585$$

and

$${}_{3|2}q_{[41]} = {}_3p_{[41]} - {}_5p_{[41]} = \frac{\ell_{[41]+3} - \ell_{[41]+5}}{\ell_{[41]}} = \frac{9928.29 - 9874.25}{9980.20} = 0.005415.$$

**Exercise 2.3.4** *You are given the following select–and–ultimate life table with a* ***two*** *year select period.*

| $x$ | $\ell_{[x]}$ | $\ell_{[x]+1}$ | $\ell_{x+2}$ |
|---|---|---|---|
| 90 | | | |
| 91 | 1250 | | 920 |
| 92 | 1000 | 900 | |

*and* $q_{[x]+t} = \frac{t+1}{3} q_{x+t}$

*Calculate* $\ell_{[90]+1}$

1. By definition
$$\ell_{[90]+1}=\frac{\ell_{[90]+2}}{1-q_{[90]+1}}=\frac{\ell_{92}}{1-q_{[90]+1}}=\frac{\ell_{92}}{1-\frac{2}{3}q_{91}}.$$
So we need to find $q_{91}$. For $t=0$, $q_{[91]}=\frac{1}{3}q_{91}$ and
$$q_{[91]}=1-\frac{\ell_{[91]+1}}{\ell_{[91]}}=1-\frac{1}{\ell_{[91]}}\frac{\ell_{[91]+2}}{1-q_{[91]+1}}=1-\frac{1}{1250}\frac{920}{1-\frac{2}{3}q_{92}}.$$
Moreover
$$q_{92}=3q_{[92]}=3\left(1-\frac{\ell_{[92]+1}}{\ell_{[92]}}\right)=3\left(1-\frac{900}{1000}\right)=0.3.$$
Thus
$$q_{91}=3q_{[91]}=3\left(1-\frac{1}{1250}\frac{920}{1-\frac{0.6}{3}}\right)=0.24.$$
Finally
$$\ell_{[90]+1}=\frac{\ell_{[90]+2}}{1-q_{[90]+1}}=\frac{\ell_{92}}{1-q_{[90]+1}}=\frac{1}{1-\frac{2}{3}q_{91}}\frac{\ell_{93}}{1-q_{92}}=\frac{1}{1-\frac{2}{3}\times 0.24}\times\frac{920}{1-0.3}\simeq 1565$$

**Exercise 2.3.5** *Select mortality rates for* [45] *are half of the Illustrative Life Table's mortality rates for a selection period of* 3 *years. Calculate* ${}_{2|3}q_{[45]}$.

**Solution**: We know that
$${}_{2|3}q_{[45]}={}_{2}p_{[45]}\ {}_{3}q_{[45]+2}={}_{2}p_{[45]}\left(1-{}_{3}p_{[45]+2}\right),$$
so we need ${}_{2}p_{[45]}$ and ${}_{3}p_{[45]+2}$. So
$${}_{2}p_{[45]}=p_{[45]}\ p_{[45]+1}=\left(1-q_{[45]}\right)\left(1-q_{[45]+1}\right)$$
and
$${}_{3}p_{[45]+2}=p_{[45]+2}\ p_{[45]+3}\ p_{[45]+4}=\left(1-q_{[45]+2}\right)\left(1-q_{[45]+3}\right)\left(1-q_{[45]+4}\right).$$
For $q_{[45]}$ and $q_{[45]+1}$ we use half of the ILT rates. Then
$${}_{2}p_{[45]}=(1-0.5(0.004))(1-0.5(0.00431)=0.995849,$$
moreove $q_{[45]+2}=0.5q_{47}$ but $q_{[45]+3}=q_{48}$ and $q_{[45]+4}=q_{49}$ since the selection period ends after 3 years. Mortality for duration 3 and on is no different from standard mortality.
$${}_{3}p_{[45]+2}=(1-0.5(0.00466))(1-0.00504)(1-?????)=$$
The answer is
$${}_{2|3}q_{[45]}={}_{2}p_{[45]}\ {}_{3}q_{[45]+2}={}_{2}p_{[45]}\left(1-{}_{3}p_{[45]+2}\right)=0.995849(1-??????)=0????????.$$

## 2.4 Moments of future lifetime random variables

First, let us focus on the moments of the future lifetime random variable $T_x$. We call $E\left[T_x\right]$ the complete expectation of life at age $x$, and denote it by $\mathring{e}_x$. We have
$$E\left[T_x\right]=:\mathring{e}_x=\int_0^\infty tf_x(t)dt=-\int_0^\infty tS_x'(t)dt.$$

Using integration by parts and the fact that $\lim_{t\longrightarrow\infty} tS_x(t) = 0$, we can show that $\mathring{e}_x = \int_0^\infty {}_tp_x dt$.

Note that if there is a limiting age, we replace $\infty$ with $\omega - x$: that

$$\mathring{e}_x = \int_0^{\omega-x} {}_tp_x dt.$$

The second moment of $T_x$ can be expressed as

$$E\left[T_x^2\right] = \int_0^\infty t^2 f_x(t)dt = -\int_0^\infty t^2 S_x'(t)dt$$

Using integration by parts and the fact that $\lim_{t\longrightarrow\infty} t^2 S_x(t) = 0$, we can show that the above formula can be rewritten as

$$E\left[T_x^2\right] = \int_0^\infty 2t\ {}_tp_x dt$$

which is generally easier to apply. Again, if there is a limiting age, we replace $\infty$ with $\omega - x$: that is

$$E\left[T_x^2\right] = \int_0^{\omega-x} 2t\ {}_tp_x dt$$

In the exam, you may also be asked to calculate $E\left[T_x \wedge n\right] = E[\min(T_x; n)]$. This expectation is known as the $n$–year temporary complete expectation of life at age $x$, and is denoted by $\mathring{e}_x n$. It can be shown that

$$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x dt.$$

**Exercise 2.4.1** *You are given $\mu_x = 0.01$ for all $x \geq 0$. Calculate the following: a. $\mathring{e}_{x:\overline{n}|}$, b. $\operatorname{Var}(T_x)$.*

Second remember that if $K_x$ stands for the **Curtate** function we have:

$$e_x := E\left[K_x\right] = \sum_{k=1}^{\infty} {}_kp_x \quad \text{and} \quad E\left[K_x^2\right] = \sum_{k=1}^{\infty} (2k-1)\ {}_kp_x$$

If there is a limiting age, we replace $\infty$ with $\omega - x$ that is

$$e_x := E\left[K_x\right] = \sum_{k=1}^{\omega-x} {}_kp_x \quad \text{and} \quad E\left[K_x^2\right] = \sum_{k=1}^{\omega-x} (2k-1)\ {}_kp_x.$$

and

$$E\left[\min(K_x, n)\right] = \sum_{k=1}^{n} {}_kp_x \quad \text{and} \quad E\left[(\min(K_x, n))^2\right] = \sum_{k=1}^{n} (2k-1)\ {}_kp_x.$$

This is called the $n$–year temporary curtate expectation of life at age $x$, and is denoted by $e_{x:\overline{n}|}$. It can be shown that

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x.$$

that is, instead of summing to infinity, we just sum to $n$.

There are two other equations that you need to know. First, you need to know that $e_x$ and $e_{x+1}$ are related to each other as follows:

$$e_x = p_x(1 + e_{x+1}).$$

Formulas of this form are called recursion formulas.

Assume that **UDD** holds, we have $T_x = K_x + U$, where $U$ follows a uniform distribution over the interval $[0,\ 1]$. Taking expectation on both sides, we have the following relation:

$$\mathring{e}_x = e_x + \frac{1}{2}.$$

### 2.4.1 Uniform distribution of death for all ages

Recall that the uniform distribution on $[0,\ \kappa]$ has a mean of $\frac{\kappa}{2}$, which is its median and midrange. Its variance is $\frac{\kappa^2}{12}$. It is traditional to use the letter $\omega$ to indicate the upper limit of a mortality table. For a uniform model, age at death $T_0$ is uniformly distributed on $(0,\ \omega)$.

While the uniform distribution has memory–after all, you can not live beyond $\omega$, so the older one is, the less time until certain death–It has the following property that simplifies calculation: if age at death is uniform on $(0, \omega]$, then survival time for $(x)$ is uniform on $(0, \omega - x)$.

Here are the probability functions for $T_x$ if $T_0$ is uniform on $(0, \omega)$: After working out several problems, you should recognize the uniform distribution on sight.

For the uniform distribution of death (**UDD**) or **De Moivre's law** we have:

$$\begin{aligned} \mu_x &= \frac{1}{\omega - x}, \quad S_x(t) = {}_t p_x = \frac{\omega - x - t}{\omega - x} = 1 - \frac{t}{\omega - x}, \quad F_x(t) = {}_t q_x = \frac{t}{\omega - x}, \\ \ell_x &= \omega - x, \quad f_x(t) = \frac{1}{\omega - x}, \quad E\left[T_x\right] = \frac{\omega - x}{2} \ \text{ and } \ \mathrm{Var}(T_x) = \frac{(\omega - x)^2}{12} \end{aligned}$$

### 2.4.2 Constant force of mortality or Exponential distribution for all ages

If the force of mortality is the constant $\mu$, the distribution of survival time is exponential. Survival probabilities are then independent of age; ${}_t p_x$ does not depend on $x$. The constant force of mortality $\mu$ is the reciprocal of mean survival time; in other words, a life with constant force of mortality $\mu$ has expected future lifetime $\frac{1}{\mu}$, regardless of the life's current age.

Here are the probability functions for $T_x$ if $T_x$ has constant force of mortality $\mu$. Notice that none of the functions vary with age $x$

$$\forall\ x,\ \forall\ t,\ \mu_{x+t} = \mu, \quad S_x(t) = {}_t p_x = e^{-\mu t}, \quad F_x(t) = {}_t q_x = 1 - e^{-\mu t}, \quad f_x(t) = \mu e^{-\mu t},$$

consequently

$$E\left[T_x\right] = \frac{1}{\mu} \ \text{ and } \ \mathrm{Var}(T_x) = \frac{1}{\mu^2}.$$

**Practice**

**Exercise 2.4.2** *You are given: (i) Deaths are uniformly distributed over each year of age.*
*(ii)* $\mathring{e}_{55.2:\overline{0.4}|} = 0.396$. *Calculate* $\mu_{55.2}$.

**Solution**: We have $\mu_{55.2} = \frac{q_{55}}{1 - 0.2 q_{55}}$ and

$$\begin{aligned} \mathring{e}_{55.2:\overline{0.4}|} &= \int_0^{0.4} {}_t p_{55.2} dt = \int_0^{0.4} \frac{{}_{0.2+t} p_{55}}{{}_{0.2} p_{55}} dt = \frac{1}{{}_{0.2} p_{55}} \int_0^{0.4} \left(1 - {}_{0.2+t} q_{55}\right) dt = \frac{1}{{}_{0.2} p_{55}} \int_0^{0.4} \left(1 - (0.2 + t)\, q_{55}\right) dt \\ &= \frac{1}{1 - 0.2 q_{55}} \left(0.4 - q_{55} \int_0^{0.4} (0.2 + t)\, dt\right) = \frac{0.4 - q_{55} \times 0.16}{1 - 0.2 q_{55}} = 0.396. \end{aligned}$$

Therefore $q_{55} = 0.04950$. Thus $\mu_{55.2} = \frac{0.04950}{1 - 0.2 \times 0.04950} = 0.049995$.

**Exercise 2.4.3** *For a select–and–ultimate mortality table with a one–year select period, you are given::*

| $x$ | $\ell_{[x]}$ | $d_{[x]} = q_{[x]}\ \ell_{[x]}$ | $\mathring{e}_{[x]}$ |
|---|---|---|---|
| 85 | 1000 | 100 | 5.225 |
| 86 | 850 | 100 | |

*Assume deaths are uniformly distributed over each year of age.*

1. *Calculate* $p_{[85]}$, $p_{[86]}$ *and* $p_{86}$.
2. *Use the decomposition* ${}_tp_{[85]} = {}_{1+t-1}p_{[85]}$ *to calculate* $\int_1^2 {}_tp_{[85]}dt$
3. *Calculate* $\mathring{e}_{[85]} - \int_2^\infty {}_tp_{[85]}dt$ *and deduce the value of* $\int_2^\infty {}_tp_{[85]}dt$.
4. *Calculate* $\mathring{e}_{[86]}$

**Solution**:

1. We have

$$p_{[85]} = 1 - \frac{d_{[85]}}{\ell_{[85]}} = 1 - \frac{100}{1000} = 0.9, \quad p_{[86]} = 1 - \frac{d_{[86]}}{\ell_{[86]}} = 1 - \frac{100}{850} = \frac{15}{17} = 0.88235,$$

and

$$p_{86} = \frac{\ell_{87}}{\ell_{86}} = \frac{\ell_{[86]+1}}{\ell_{[85]+1}} = \frac{\ell_{[86]} - d_{[86]}}{\ell_{[85]} - d_{[85]}} = \frac{850-100}{1000-100} = \frac{750}{900} = \frac{5}{6} = 0.83333.$$

2. We want to calculate $\int_1^2 {}_tp_{[85]}dt$. Use the change of variable $t = r+1$, so when $t=1$, $r=0$ and when $t=2$, $r=1$, and $dt = dr$, thus

$$\begin{aligned}
\int_1^2 {}_tp_{[85]}dt &= \int_0^1 {}_{r+1}p_{[85]}dr = \int_0^1 p_{[85]}\ {}_rp_{[85]+1}dr = p_{[85]}\int_0^1 {}_rp_{[85]+1}dr = p_{[85]}\int_0^1 \left(1 - {}_rq_{[85]+1}\right)dr \\
&\overset{\textbf{UDD}}{=} p_{[85]}\int_0^1 \left(1 - rq_{[85]+1}\right)dr = p_{[85]}\left(1 - q_{[85]+1}\int_0^1 rdr\right) = p_{[85]}\left(1 - \frac{1}{2}q_{[85]+1}\right) \\
&= p_{[85]}\left(1 - \frac{1}{2}q_{86}\right) = p_{[85]}\left(1 - \frac{1}{2}\left(1 - p_{86}\right)\right) = \frac{1}{2}p_{[85]}\left(1 + p_{86}\right) \\
&= \frac{1}{2}0.9\left(1 + \frac{5}{6}\right) = 0.825
\end{aligned}$$

3. We know that $\mathring{e}_{[85]} - \int_2^\infty {}_tp_{[85]}dt = \int_0^1 {}_tp_{[85]}dt + \int_1^2 {}_tp_{[85]}dt$. And

$$\begin{aligned}
\int_0^1 {}_tp_{[85]}dt &= \int_0^1 \left(1 - {}_tq_{[85]}\right)dt = \int_0^1 \left(1 - tq_{[85]}\right)dt = \int_0^1 \left(1 - t\left(1 - p_{[85]}\right)\right)dt \\
&= \int_0^1 \left(1 - t\left(1 - 0.9\right)\right)dt = 0.95
\end{aligned}$$

Therefore $\mathring{e}_{[85]} - \int_2^\infty {}_tp_{[85]}dt = 0.95 + 0.825 = 1.775$. We deduce that

$$\int_2^\infty {}_tp_{[85]}dt = \mathring{e}_{[85]} - 1.775 = 5.225 - 1.775 = 3.45$$

4. We have

$$\begin{aligned}
\mathring{e}_{[86]} &= \int_0^1 {}_tp_{[86]}dt + \int_1^\infty {}_tp_{[86]}dt = \int_0^1 \left(1 - tq_{[86]}\right)dt + p_{[86]}\int_1^\infty {}_{t-1}p_{[86]+1}dt \\
&= \int_0^1 \left(1 - tq_{[86]}\right)dt + p_{[86]}\int_1^\infty {}_{t-1}p_{[86]+1}dt = \int_0^1 \left(1 - t\left(1 - p_{[86]}\right)\right)dt + p_{[86]}\int_0^\infty {}_tp_{87}dt \\
&= \int_0^1 \left(1 - t\left(1 - \frac{15}{17}\right)\right)dt + p_{[86]}\int_0^\infty {}_tp_{87}dt = \frac{16}{17} + \frac{15}{17}\frac{1}{{}_2p_{[85]}}\int_0^\infty {}_{t+2}p_{[85]}dt \\
&= \frac{16}{17} + \frac{15}{17}\frac{1}{p_{[85]}p_{86}}\int_2^\infty {}_tp_{[85]}dt = \frac{16}{17} + \frac{15}{17}\frac{3.45}{\frac{9}{10}\frac{5}{6}} = 5.
\end{aligned}$$

**Exercise 2.4.4**

1. *A survival function is given by*

$$S_0(t) = \frac{(10-t)^2}{100} \quad \textit{for} \quad 0 \le t < 10,$$

*Calculate the difference between the force of mortality at age* 1*, and the probability that a life age* (1) *dies before age* 2*.*

2. *For life aged* 60*, the probability of survival is* ${}_tp_{60} = (1 - \frac{t}{60})^{0.4}$*. Calculate* $\mu_{80}$*.*

3. *Assume that the future lifetime* $T_{20}$ *is subject to force of mortality*

$$\mu_x = \frac{1}{100-x} \quad \textit{for} \;\; 0 \le x < 100.$$

*a. Calculate* $e_{20}$*,* *b. Calculate* $e_{20:\overline{50}|}$*,* *c. Calculate* $\mathrm{Var}(K_{20})$*.*

**Hint**

$$\mathring{e}_{x:\overline{n}|} = e_{x:\overline{n}|} + \frac{1}{2}\,{}_nq_x, \quad \mathring{e}_x = e_x + \frac{1}{2} \text{ and } \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

**Solution:**

1. Recall that $\mu_{x+t} = \frac{-S_x'(t)}{S_x(t)}$ where $S_x(t) = \frac{S_0(x+t)}{S_0(x)}$ then

$$\mu_{x+t} = \frac{-S_0'(x+t)}{S_0(x+t)} = \frac{2\,(10-(x+t))}{(10-(x+t))^2} = \frac{2}{10-x-t}$$

therefore

$$\mu_1 = \frac{-S_0'(1)}{S_0(1)} = \frac{2}{10-1} = \frac{2}{9} = 0.22222$$

and the probability that a life age (1) dies before age 2 is given by

$$P\,(0 \le T_1 \le 1) = q_1 = 1 - p_1 = 1 - \frac{S_0(2)}{S_0(1)} = 1 - \frac{8^2}{9^2}$$

So $\mu_1 - q_1 = 0.22222 - 0.20988 = 0.01234$.

2. Given $S_{60}(t) = \left(1 - \frac{t}{60}\right)^{0.4}$, we have

$$\mu_{60+t} = \frac{-S_{60}'(t)}{S_{60}(t)} = \frac{0.4}{60}\frac{60}{60-t} = \frac{0.4}{60-t},$$

then

$$\mu_{80} = \mu_{60+20} = \frac{0.4}{60-20} = \frac{0.4}{60-20} = 0.01.$$

3. a. We know $e_{20} = \mathring{e}_x - 0.5$,

$$\mathring{e}_{20} = \int_0^{100-20} {}_tp_{20}dt \;\text{ and }\; {}_tp_{20} = e^{\int_{20}^{20+t} \frac{-du}{100-u}} = e^{[\ln(80-t) - \ln(80)]} = 1 - \frac{t}{80}$$

therefore

$$\mathring{e}_{20} = \int_0^{80} \left(1 - \frac{t}{80}\right) dt = 40, \text{ thus } \; e_{20} = 40 - 0.5 = 39.5.$$

b. and

$$\mathring{e}_{20:\overline{50}|} = \int_0^{50} {}_tp_{20} dt = \int_0^{50} \left(1 - \frac{t}{80}\right) dt = \int_0^{50} \left(1 - \frac{t}{80}\right) dt = \frac{275}{8} = 34.375$$

Therefore using the formula

$$\mathring{e}_{20:\overline{50}|} = e_{20:\overline{50}|} + \frac{1}{2}\ {}_{50}q_{20} = e_{20:\overline{50}|} + \frac{1}{2}\left(1 - {}_{50}p_{20}\right)$$

we get

$$e_{20:\overline{50}|} = \mathring{e}_{20:\overline{50}|} - \frac{1}{2}\left(1 - {}_{50}p_{20}\right) = 34.375 - \frac{1}{2}\frac{50}{80} = 34.063$$

c. By definition $\text{Var}(K_{20}) = E\left[K_{20}^2\right] - \left(E\left[K_{20}\right]\right)^2$ and

$$E\left[K_{20}\right] = \sum_{k=1}^{80} {}_kp_{20} = \sum_{k=1}^{80}\left(1 - \frac{k}{80}\right) = 80 - \frac{1}{80}\sum_{k=1}^{80} k = 80 - \frac{1}{80}\left(\frac{1+80}{2}80\right) = \frac{79}{2} = 39.5$$

and $\frac{79}{2} = 39.5$

$$\begin{aligned} E\left[K_{20}^2\right] &= \sum_{k=1}^{80}(2k-1)\ {}_kp_{20} = \sum_{k=1}^{80}(2k-1)\left(1 - \frac{k}{80}\right) = \sum_{k=1}^{80}(2k-1) - \frac{1}{80}\sum_{k=1}^{80}\left(2k^2 - k\right) \\ &= 2\left(\frac{1+80}{2}80\right) - 80 - \frac{2}{80}\frac{80(80+1)(160+1)}{6} + \frac{1}{80}\left(\frac{1+80}{2}80\right) = \frac{4187}{2} = 2093.5 \end{aligned}$$

Finally $\text{Var}(K_{20}) = 2093.5 - (39.5)^2 = 533.25$.

**Exercise 2.4.5**

1. *The force of mortality of a life is* $\mu_x = \frac{x}{1000}$. *Calculate* $e_{35.5:\overline{3}|}$
2. *The force of mortality for life* $(x)$ *is the constant* $\mu_x = 0.01$. *Calculate the* ***curtate life expectancy*** *of* $(x)$.

**Recursive formulas**

| |
|---|
| $\mathring{e}_x = \mathring{e}_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n}, \quad \mathring{e}_{x:\overline{n}|} = \mathring{e}_{x:\overline{m}|} + {}_mp_x\mathring{e}_{x+m:\overline{n-m}|}$, for $m < n$ |
| $e_x = e_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n} = e_{x:\overline{n-1}|} + {}_np_x\left(1 + \mathring{e}_{x+n}\right)$ |
| $e_x = p_x + p_x\mathring{e}_{x+1} = p_x\left(1 + \mathring{e}_{x+1}\right)$ |
| $e_{x:\overline{n}|} = e_{x:\overline{m}|} + {}_mp_xe_{x+m:\overline{n-m}|} = e_{x:\overline{m-1}|} + {}_mp_x(1 + e_{x+m:\overline{n-m}|})$, for $m < n$ |
| $e_{x:\overline{n}|} = p_x + p_xe_{x-1:\overline{n-1}|} = p_x\left(1 + e_{x-1:\overline{n-1}|}\right)$. |

| **For any integral age** $x$, $n \geq 1$, $0 \leq r < 1$ and $0 \leq r + t < 1$ | | |
|---|---|---|
| **Function** | **Uniform distribution of deaths** | **Constant force of mortality** |
| $\ell_{x+r}$ | $\ell_x - rd_x$ | $\ell_xp_x^r$ |
| ${}_rq_x$ | $rq_x$ | $1 - p_x^r$ |
| ${}_rp_x$ | $1 - rq_x$ | $p_x^r$ |
| ${}_rq_{x+t}$ | $\frac{rq_x}{1-tq_x}$ | $1 - p_x^r$ |
| $\mu_{x+r}$ | $\frac{q_x}{1-rq_x}$ | $-\ln\left(p_x\right)$ |
| ${}_rp_x\mu_{x+r}$ | $q_x$ | $-\ {}_rp_x\ln\left(p_x\right)$ |
| $\mathring{e}_{x:\overline{n}|}$ | $e_{x:\overline{n}|} + 0.5\ {}_nq_x$ | |
| $\mathring{e}_{x:\overline{1}|}$ | $p_x + 0.5q_x$ | |
| $\mathring{e}_x$ | $e_x + 0.5$ | |