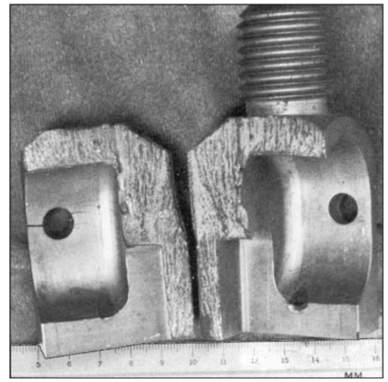
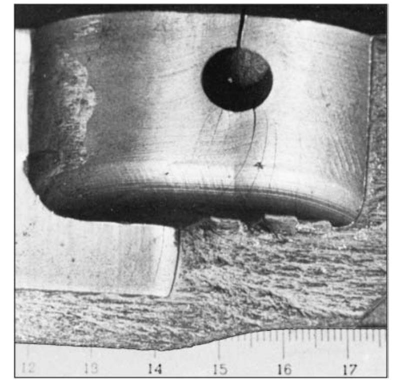
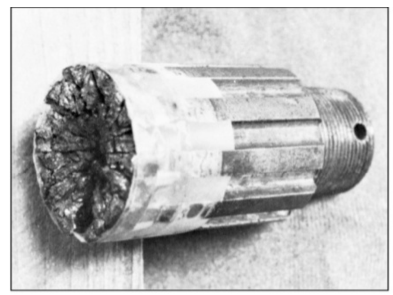
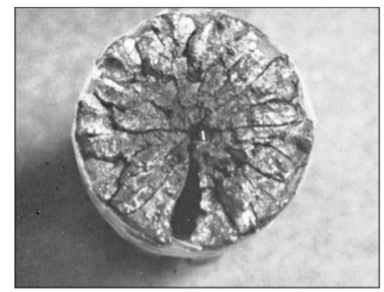
- Strain Failures

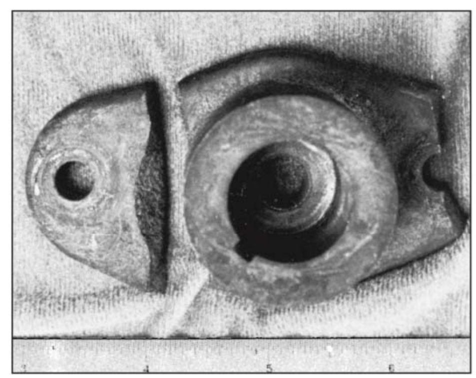
- A *static load* is a stationary force or couple applied to a member.
- Failure can mean a part has separated into two or more pieces; has become permanently distorted, thus ruining its geometry; has had its reliability downgraded; or has had its function compromised, whatever the reason.











- Failure Theories
- There is no universal theory of failure for the general case of material properties and stress state. Instead, over the years several hypotheses have been formulated and tested, leading to today's accepted practices most designers do.
- The generally accepted theories are:

Ductile materials (yield criteria)

–Maximum shear stress (MSS)

–Distortion energy (DE)

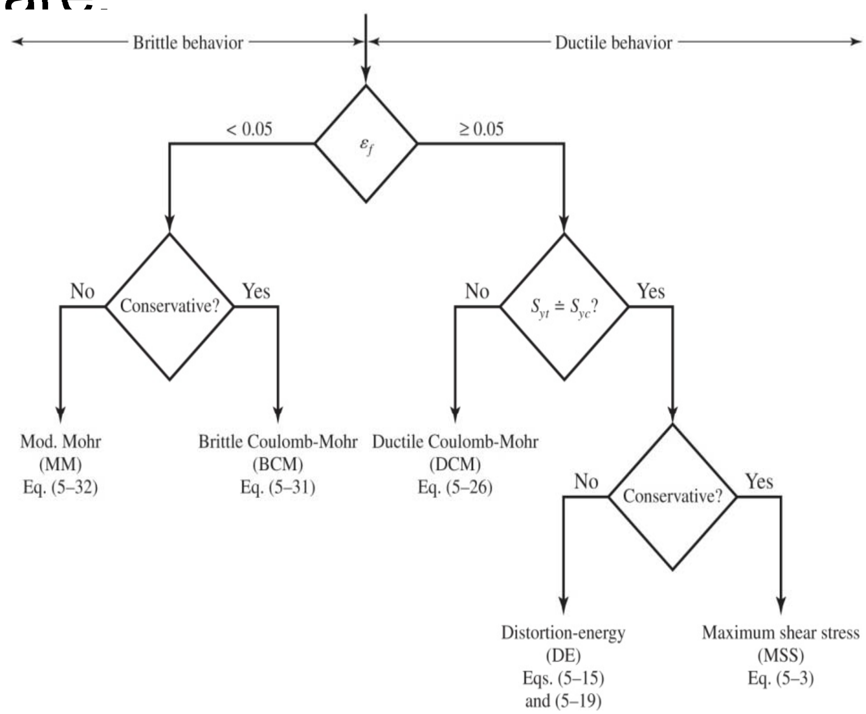
–Ductile Coulomb-Mohr (DCM)

Brittle materials (fracture criteria)

–Maximum normal stress (MNS)

–Brittle Coulomb-Mohr (BCM)

–Modified Mohr (MM)



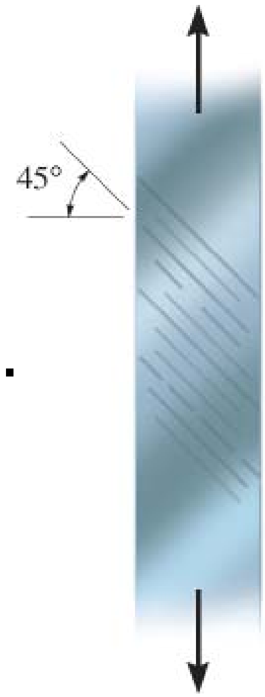
*10.7 THEORIES OF FAILURE

- When engineers design for a material, there is a need to set an upper limit on the state of stress that defines the material's failure.
- For ductile material, failure is initiated by yielding.
- For brittle material, failure is specified by fracture.
- However, criteria for the above failure modes is not easy to define under a biaxial or triaxial stress.
- Thus, theories are introduced to obtain the principal stresses at critical states of stress.

*10.7 THEORIES OF FAILURE

A. Ductile materials

1. Maximum-Shear-Stress Theory



- Most common cause of yielding of ductile material (e.g., steel) is slipping.
- Slipping occurs along the contact planes of randomly-ordered crystals that make up the material.
- Edges of planes of slipping as they appear on the surface of the strip are referred to as Lüder's lines.
- The lines indicate the slip planes in the strip, which occur at approximately 45° with the axis of the strip.

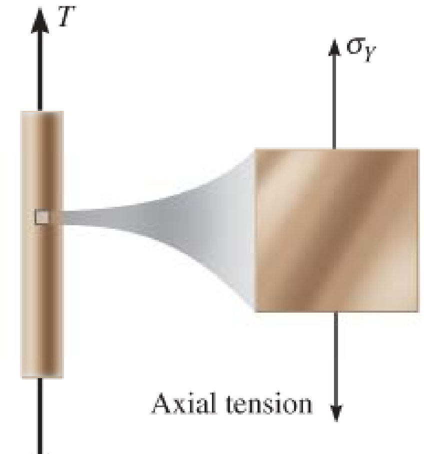
*10.7 THEORIES OF FAILURE

A. Ductile materials

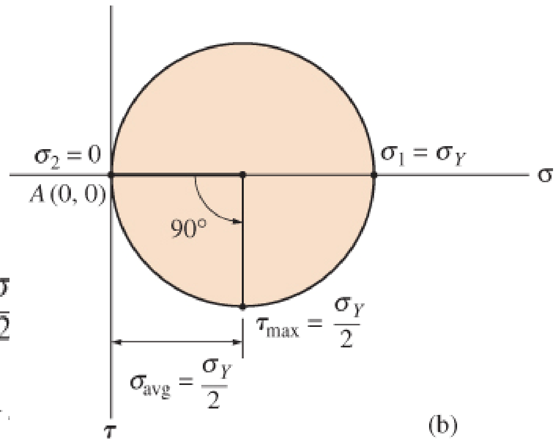
1. *Maximum-Shear-Stress Theory*
2. Consider an element, determine maximum shear stress from Mohr's circle,

$$\tau_{\max} = \frac{\sigma_Y}{2} \qquad (10-26)$$

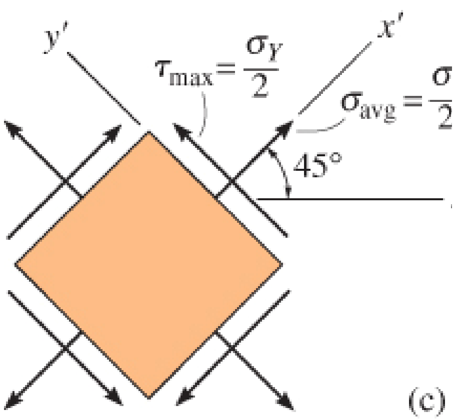
- This implies that the yield strength in shear is **half** the yield strength in tension ($S_{sy}=0.5\ S_y$).
- Thus, in 1868, Henri Tresca proposed the maximum-shear-stress theory or Tresca yield criterion.



(a)



(b)



(c)

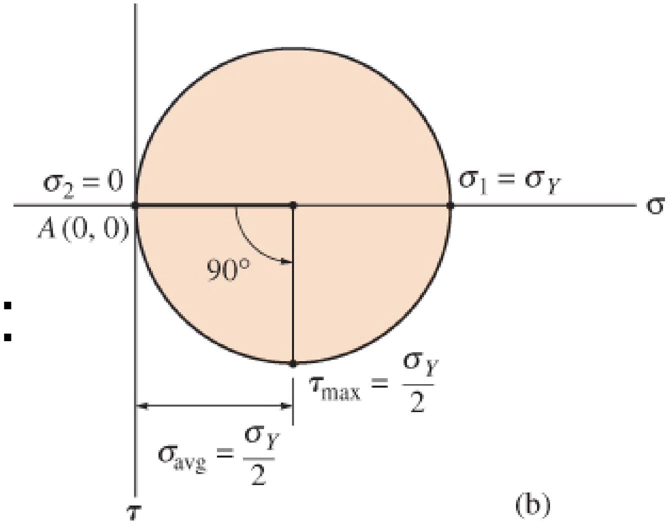
*10.7 THEORIES OF FAILURE

A. Ductile materials

1. Maximum-Shear-Stress Theory

- If the two in-plane principal stresses have the same sign, failure will occur out of the plane:

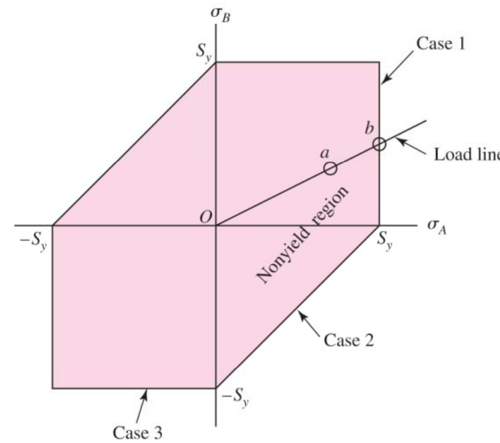
$$\tau_{\substack{\text{abs}\\ \text{max}}} = \frac{\sigma_{\text{max}}}{2} \qquad \tau_{\substack{\text{abs}\\ \text{max}}} = \frac{\sigma_1}{2}$$



- If in-plane principal stresses are of opposite signs, failure occurs in the plane:

$$\tau_{\substack{\text{abs}\\ \text{max}}} = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \qquad \tau_{\substack{\text{abs}\\ \text{max}}} = \frac{\sigma_1 - \sigma_3}{2}$$

- The maximum-shear-stress theory predicts that yielding begins whenever the maximum shear stress in any element equals or exceeds the maximum shear stress in a tension test specimen of the same material when that specimen begins to yield.



$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{S_y}{2} \qquad \text{or} \qquad \sigma_1 - \sigma_3 \geq S_y \tag{5–1}$$

- Assuming a plane stress problem with $\sigma_A \geq \sigma_B$, there are three cases to consider

  ✓ Case 1: $\sigma_A \geq \sigma_B \geq 0$. For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$. Equation (5–1)reduces to a yield condition of

$$\sigma_A \geq S_y$$

  ✓ Case 2: $\sigma_A \geq 0 \geq \sigma_B$. Here, σ1 = σ_A and $\sigma_3 = \sigma_B$, and Eq. (5–1) becomes

$$\sigma_A - \sigma_B \geq S_y$$

  ✓ Case 3: $0 \geq \sigma_A \geq \sigma_B$. For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$, and Eq. (5–1) gives

$$\sigma_B \leq -S_y$$

*10.7 THEORIES OF FAILURE

A. Ductile materials

1. Maximum-Shear-Stress Theory

- Thus, we express the maximum-shear-stress theory for plane stress for any two in-plane principal stresses by the following criteria:

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{S_y}{2} \qquad \text{or} \qquad \sigma_1 - \sigma_3 \geq S_y \qquad (5\text{–}1)$$

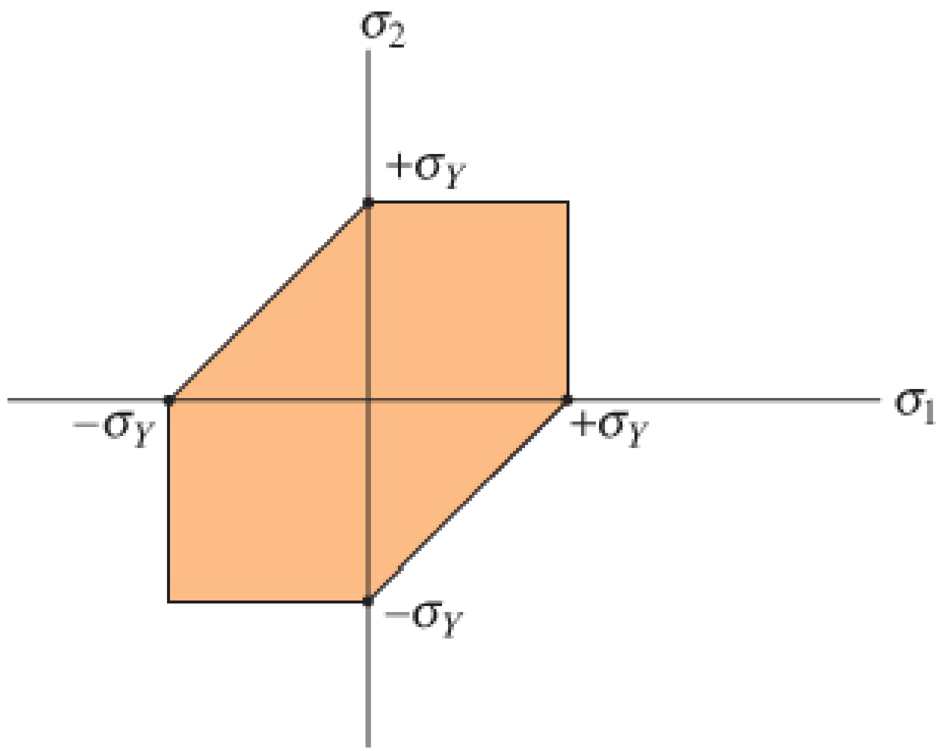
$$|\sigma_1| = \sigma_Y \qquad \} \ \sigma_1,\ \sigma_2 \ \text{have} + VE \ \text{signs}, \sigma_3 = 0.$$

$$|\sigma_1 - \sigma_3| = \sigma_Y \quad \} \ \sigma_1,\ \sigma_3 \ \text{have opposite signs}, \sigma_2 = 0.$$

*10.7 THEORIES OF FAILURE

A. Ductile materials

1. Maximum-Shear-Stress Theory



Maximum-shear-stress theory

*10.7 THEORIES OF FAILURE

A. Ductile materials

2. Maximum-Distortion-Energy Theory

- Energy per unit volume of material is called the strain-energy density.
- Material subjected to a uniaxial stress , the strain-energy density is written as

$$u = \frac{1}{2}\sigma\varepsilon \qquad (10\text{-}28)$$

Linear elastic behavior (Hooke's law)

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

$$u = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$

Triaxial stress

*10.7 THEORIES OF FAILURE

A. Ductile materials

2. Maximum-Distortion-Energy Theory

- For linear-elastic behavior, applying Hooke's law into above eqn (Energy needed to cause a volume change as well as needed to distort the element):

$$u = \frac{1}{2E}\left[\begin{array}{l}\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 \\ \qquad -2\upsilon\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_3\sigma_2\right)\end{array}\right] \qquad (10\text{-}29)$$

- Part responsible for volume change is σ_{avg}, part responsible for distortion is ($\sigma_{1,2,3} - \sigma_{avg}$).
- Maximum-distortion-energy theory is defined as the yielding of a ductile material occurs when the distortion energy per unit volume of the material equals or exceeds the distortion energy per unit volume of the same material when subjected to yielding in a simple tension test.

*10.7 THEORIES OF FAILURE

A. Ductile materials

2. Maximum-Distortion-Energy Theory

- To obtain distortion energy per unit volume,

$$u_d = \frac{1+\upsilon}{6E}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$$

- In the case of plane stress,

$$u_d = \frac{1+\upsilon}{3E}\left(\sigma_1{}^2-\sigma_1\sigma_2+\sigma_2{}^2\right)$$

- For uniaxial tension test, $\sigma_1 = \sigma_Y$, $\sigma_2 = \sigma_3 = 0$

$$(u_d)_Y = \frac{1+\nu}{3E}\sigma_Y{}^2$$

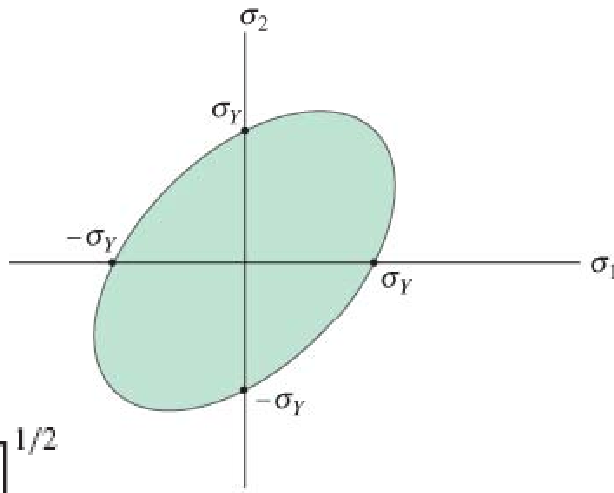
*10.7 THEORIES OF FAILURE

A. Ductile materials

2. Maximum-Distortion-Energy Theory

- Since maximum-distortion energy theory requires $u_d = (u_d)_Y$, then for the case of plane or biaxial stress, we have

$$\sigma_1^{\;2} - \sigma_1\sigma_2 + \sigma_2^{\;2} = \sigma_Y^{\;2} \qquad (10\text{-}30)$$



If a point in a material is stressed such that (σ1, σ2) is plotted on the boundary or outside the shaded area, the material is said to fail.

$$\sigma' = \sigma_y$$

$$\sigma' = \left[\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}\right]^{1/2}$$

$$\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_x-\sigma_y)^2 + (\sigma_y-\sigma_z)^2 + (\sigma_z-\sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)\right]^{1/2}$$

Distortion-Energy Theory for Ductile Materials

- The distortion-energy theory predicts that yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion strain energy per unit volume for yield in simple tension or compression of the same material.
- For unit volume subjected to any three-dimensional stress state designated by the stresses σ_1, σ_2, and σ_3, effective stress is usually called the von Mises stress, σ' as

$$\sigma' = \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{1/2}$$

- Using *xyz* components of three-dimensional stress, the von Mises stress can be written as

$$\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)\right]^{1/2}$$

and for plane stress,

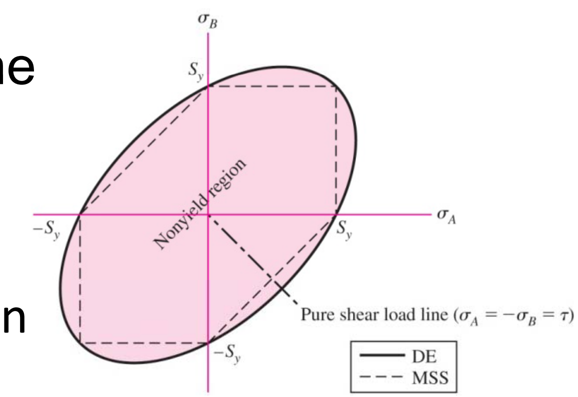
$$\sigma' = \left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2\right)^{1/2}$$

- Consider a case of pure shear τ_{xy} ,where for plane stress $\sigma_x = \sigma_y = 0$. For yield

$$\left(3\tau_{xy}^2\right)^{1/2} = S_y \qquad \text{or} \qquad \tau_{xy} = \frac{S_y}{\sqrt{3}} = 0.577S_y$$

Thus, the shear yield strength predicted by the distortion energy theory is

$$S_{sy} = 0.577S_y$$



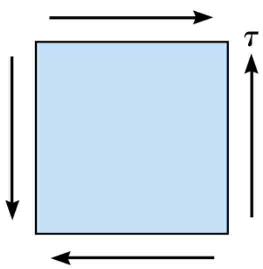
*10.7 THEORIES OF FAILURE

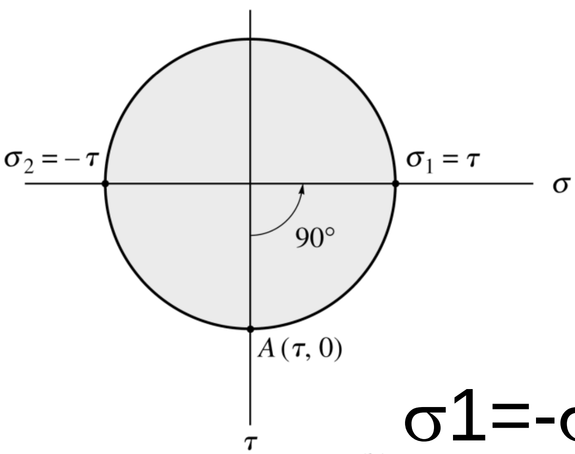
A. Ductile materials

2. Maximum-Distortion-Energy Theory

- Comparing both theories, we get the following graph.

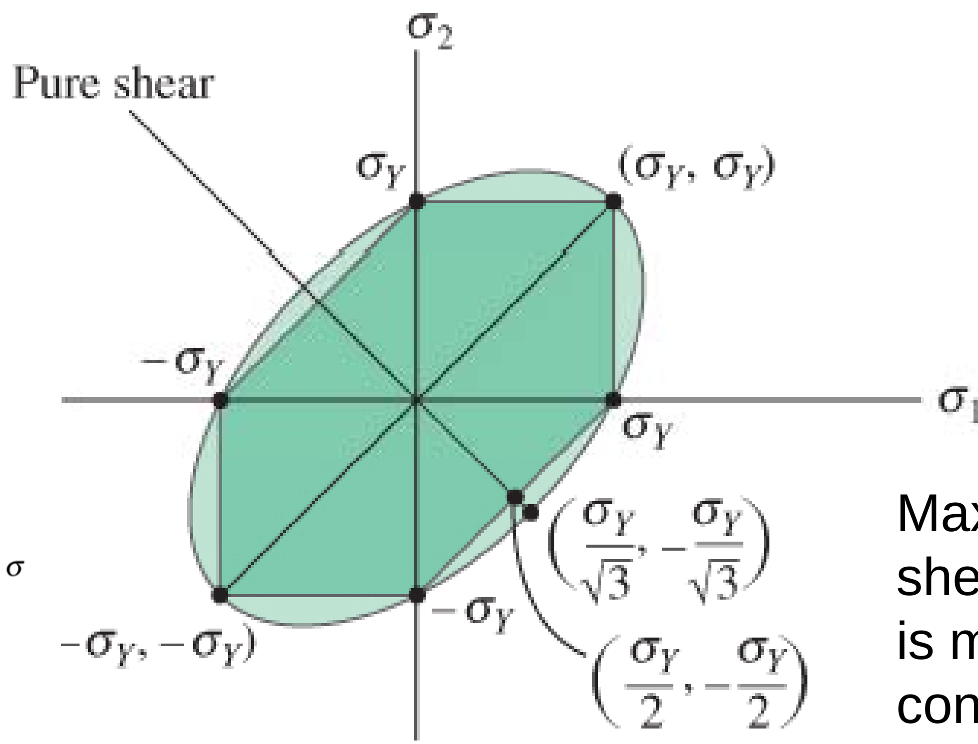
Pure Shear





(b)

$\sigma1 = -\sigma2 = \tau$



Maximum shear stress is more conservative by 15%

EXAMPLE 5–1 A hot-rolled steel has a yield strength of $S_{yt} = S_{yc} = 700$ MPa and a true strain at fracture of $\varepsilon_f = 0.55$. Estimate the factor of safety for the following principal stress states:

(*a*) 490, 490, 0 MPa.
(*b*) 210, 490, 0 MPa.
(*c*) 0, 490, −210 MPa.
(*d*) 0, −210, −490 MPa.
(*e*) 210, 210, 210 MPa.

Solution Since $\varepsilon_f > 0.05$ and S_{yc} and S_{yt} are equal, the material is ductile and the distortion-energy (DE) theory applies. The maximum-shear-stress (MSS) theory will also be applied and compared to the DE results. Note that cases *a* to *d* are plane stress states.

(*a*) The ordered principal stresses are $\sigma_A = \sigma_1 = 490$, $\sigma_B = \sigma_2 = 490$, $\sigma_3 = 0$ MPa.

DE From Eq. (5–13),

$$\sigma' = [490^2 - 490(490) + 490^2]^{1/2} = 490 \text{ MPa}$$

Answer

$$n = \frac{S_y}{\sigma'} = \frac{700}{490} = 1.43$$

MSS Case 1, using Eq. (5–4) with a factor of safety,

Answer

$$n = \frac{S_y}{\sigma_A} = \frac{700}{490} = 1.43$$

(*b*) The ordered principal stresses are $\sigma_A = \sigma_1 = 490$, $\sigma_B = \sigma_2 = 210$, $\sigma_3 = 0$ MPa.

DE

$$\sigma' = [490^2 - 490(210) + 210^2]^{1/2} = 426 \text{ MPa}$$

Answer

$$n = \frac{S_y}{\sigma'} = \frac{700}{426} = 1.64$$

MSS Case 1, using Eq. (5–4),

Answer $$n = \frac{S_y}{\sigma_A} = \frac{700}{490} = 1.43$$

(c) The ordered principal stresses are $\sigma_A = \sigma_1 = 490$, $\sigma_2 = 0$, $\sigma_B = \sigma_3 = -210$ MPa.

DE $$\sigma' = [490^2 - 490(-210) + (-210)^2]^{1/2} = 622 \text{ MPa}$$

Answer $$n = \frac{S_y}{\sigma'} = \frac{700}{622} = 1.13$$

MSS Case 2, using Eq. (5–5),

Answer $$n = \frac{S_y}{\sigma_A - \sigma_B} = \frac{700}{490 - (-210)} = 1.00$$

(d) The ordered principal stresses are $\sigma_1 = 0$, $\sigma_A = \sigma_2 = -210$, $\sigma_B = \sigma_3 = -490$ MPa.

DE $$\sigma' = [(-490)^2 - (-490)(-210) + (-210)^2]^{1/2} = 426 \text{ MPa}$$

Answer $$n = \frac{S_y}{\sigma'} = \frac{700}{426} = 1.64$$

MSS Case 3, using Eq. (5–6),

Answer $$n = -\frac{S_y}{\sigma_B} = -\frac{700}{-490} = 1.43$$

(e) The ordered principal stresses are $\sigma_1 = 210$, $\sigma_2 = 210$, $\sigma_3 = 210$ MPa

DE From Eq. (5–12),

$$\sigma' = \left[\frac{(210-210)^2 + (210-210)^2 + (210-210)^2}{2}\right]^{1/2} = 0 \text{ MPa}$$

Answer $$n = \frac{S_y}{\sigma'} = \frac{700}{0} \to \infty$$

MSS From Eq. (5–3),

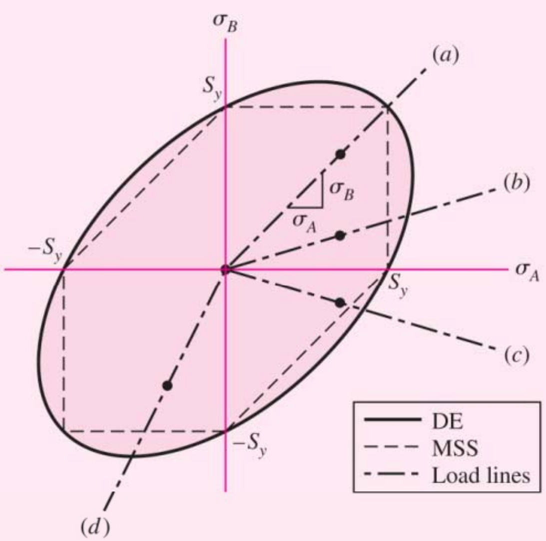
Answer

$$n = \frac{S_y}{\sigma_1 - \sigma_3} = \frac{700}{210 - 210} \rightarrow \infty$$

A tabular summary of the factors of safety is included for comparisons.

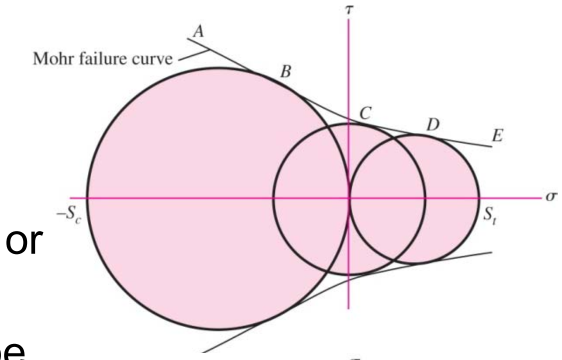
	(a)	(b)	(c)	(d)	(e)
DE	1.43	1.64	1.13	1.64	∞
MSS	1.43	1.43	1.00	1.43	∞

ple 5–1.

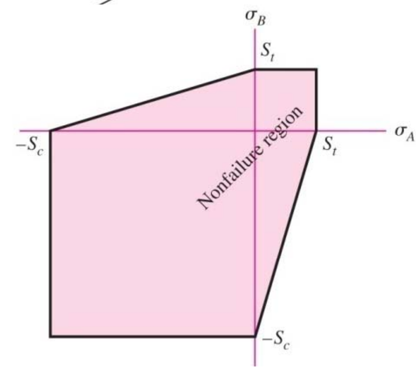


Since the MSS theory is on or within the boundary of the DE theory, it will always predict a factor of safety equal to or less than the DE theory, as can be seen in the table. For each case, except case (*e*), the coordinates and load lines in the σ_A, σ_B plane are shown in Fig. 5–11. Case (*e*) is not plane stress. Note that the load line for case (*a*) is the only plane stress case given in which the two theories agree, thus giving the same factor of safety.

Coulomb-Mohr Theory for Ductile Materials



- Not all materials have compressive strengths equal to their corresponding tensile values.
- The idea of Mohr is based on three "simple" tests: tension, compression, and shear, to yielding if the material can yield, or to rupture.
- The practical difficulties lies in the form of the failure envelope.
- A variation of Mohr's theory, called the **Coulomb-Mohr theory** or the **internal-friction theory**, assumes that the boundary is straight.
- For plane stress, when the two nonzero principal stresses are $\sigma_A \geq \sigma_B$, we have a situation similar to the three cases given for the MSS theory



- ✓ Case 1: $\sigma_A \geq \sigma_B \geq 0$. For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$. Equation (5–22) reduces to a failure condition of

$$\sigma_A \geq S_t$$

- ✓ Case 2: $\sigma_A \geq 0 \geq \sigma_B$. Here, σ1 = σ_A and $\sigma_3 = \sigma_B$, and Eq. (5–22) becomes

$$\frac{\sigma_A}{S_t} - \frac{\sigma_B}{S_c} \geq 1$$

- ✓ Case 3: $0 \geq \sigma_A \geq \sigma_B$. For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$, and Eq. (5–22) gives

$$\sigma_B \leq -S_c$$

*10.7 THEORIES OF FAILURE

- Carry out a uniaxial tensile test to determine the ultimate tensile stress $(\sigma_y)_t$
- Carry out a uniaxial compressive test to determine the ultimate compressive stress $(\sigma_y)_c$
- Carry out a torsion test to determine the ultimate shear stress τ_y.
- Results are plotted in Mohr circles.

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = \frac{1}{n} \qquad (5\text{–}26)$$

$$S_{sy} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}} \qquad (5\text{–}27)$$

EXAMPLE 5–2 A 25-mm-diameter shaft is statically torqued to 230 N · m. It is made of cast 195-T6 aluminum, with a yield strength in tension of 160 MPa and a yield strength in compression of 170 MPa. It is machined to final diameter. Estimate the factor of safety of the shaft.

Solution The maximum shear stress is given by

$$\tau = \frac{16T}{\pi d^3} = \frac{16(230)}{\pi\left[25\left(10^{-3}\right)\right]^3} = 75\left(10^6\right)\mathrm{N/m^2} = 75\,\mathrm{MPa}$$

The two nonzero principal stresses are 75 and −75 MPa, making the ordered principal stresses $\sigma_1 = 75$, $\sigma_2 = 0$, and $\sigma_3 = -75$ MPa. From Eq. (5–26), for yield,

Answer

$$n = \frac{1}{\sigma_1/S_{yt} - \sigma_3/S_{yc}} = \frac{1}{75/160 - (-75)/170} = 1.10$$

Alternatively, from Eq. (5–27),

$$S_{sy} = \frac{S_{yt}S_{yc}}{S_{yt} + S_{yc}} = \frac{160(170)}{160 + 170} = 82.4\,\mathrm{MPa}$$

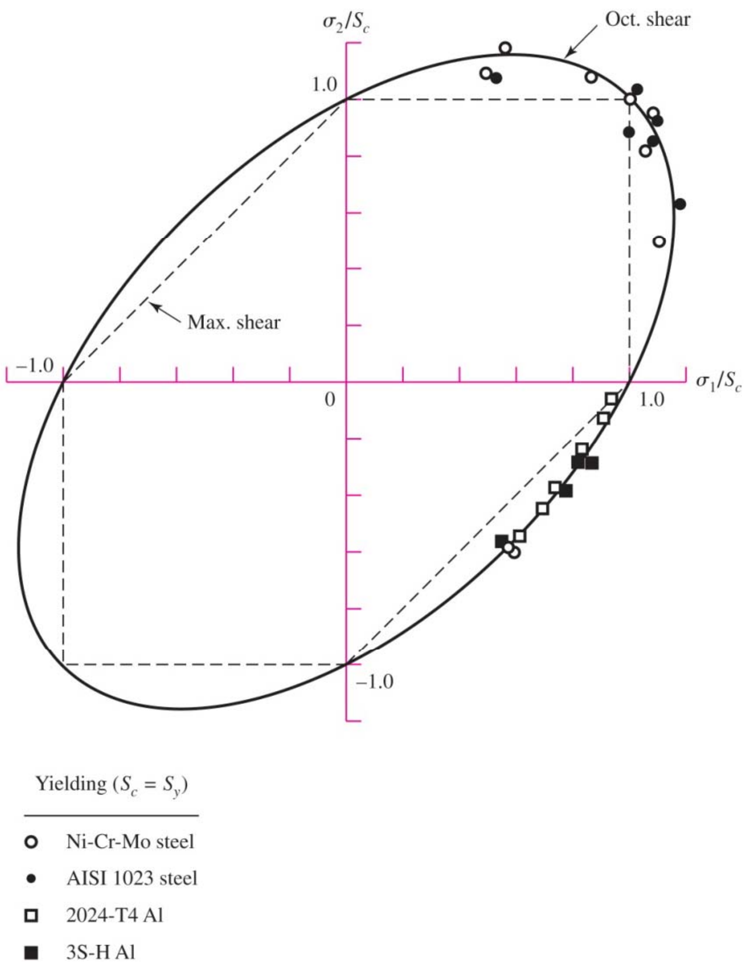
and $\tau_{max} = 75$ MPa. Thus,

Answer

$$n = \frac{S_{sy}}{\tau_{max}} = \frac{82.4}{75} = 1.10$$

Failure of Ductile Materials Summary

- Either the maximum-shear-stress theory or the distortion-energy theory is acceptable for design and analysis of materials that would fail in a ductile manner.
- For design purposes the maximum-shear-stress theory is easy, quick to use, and conservative.
- If the problem is to learn why a part failed, then the distortion-energy theory may be the best to use.
- For ductile materials with unequal yield strengths, S_{yt} in tension and S_{yc} in compression, the Mohr theory is the best available.

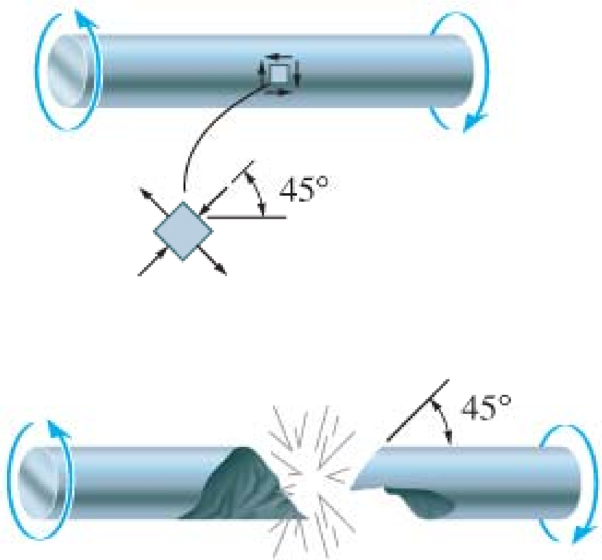


*10.7 THEORIES OF FAILURE

A. Brittle materials

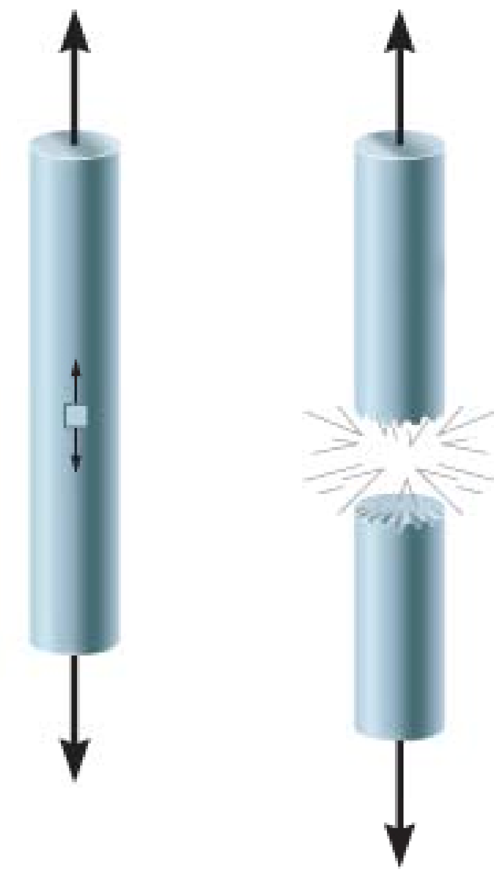
3. Maximum-Normal-Stress Theory

- Figure shows how brittle materials fail.



Failure of a brittle material in torsion

(b)



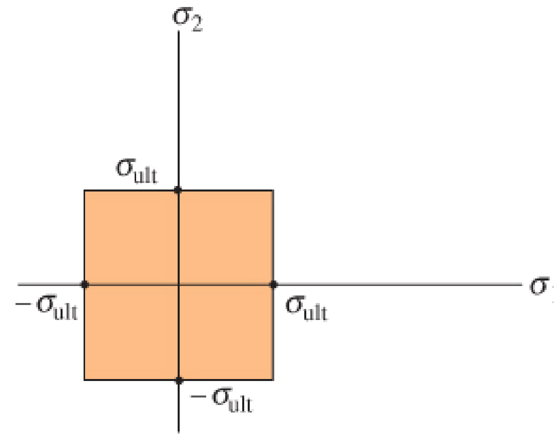
Failure of a brittle material in tension

(a)

*10.7 THEORIES OF FAILURE

A. Brittle materials

3. Maximum-Normal-Stress Theory



Maximum-normal-stress theory

- The maximum-normal-stress theory states that a brittle material will fail when the maximum principal stress σ_1 in the material reaches a limiting value that is equal to the ultimate normal stress the material can sustain when subjected to simple tension.
- For the material subjected to plane stress

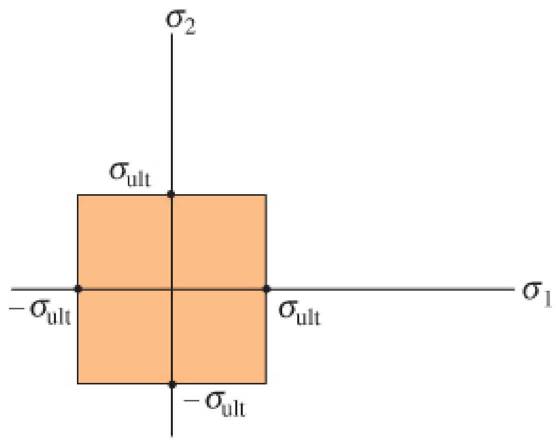
$$|\sigma_1| = \sigma_{\text{ult}}$$
$$|\sigma_3| = \sigma_{\text{ult}} \qquad (10\text{-}31)$$

*10.7 THEORIES OF FAILURE

A. Brittle materials

3. Maximum-Normal-Stress Theory

- Experimentally, it was found to be in close agreement with the behavior of brittle materials that have stress-strain diagrams similar in both tension and compression.



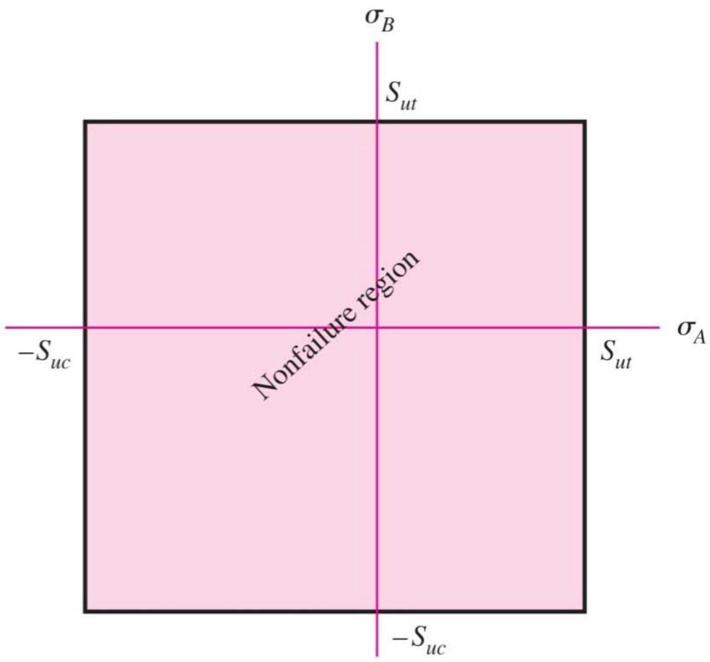
Maximum-normal-stress theory

Maximum-Normal-Stress Theory for Brittle Materials

- The maximum-normal-stress (MNS) theory states that failure occurs whenever one of the three principal stresses equals or exceeds the strength.
- For a general stress state in the ordered form $\sigma_1 \geq \sigma_2 \geq \sigma_3$. This theory then predicts that failure occurs whenever

$$\sigma_1 \geq S_{ut} \qquad \text{or} \qquad \sigma_3 \leq -S_{uc}$$

where S_{ut} and S_{uc} are the ultimate tensile and compressive strengths, respectively, given as positive quantities.



*10.7 THEORIES OF FAILURE

A. Brittle materials

4. Mohr's Failure Criterion

- Use for brittle materials where the tension and compression properties are different.
- Three tests need to be performed on material to determine the criterion.

*10.7 THEORIES OF FAILURE

A. Brittle materials

4. Mohr's Failure Criterion

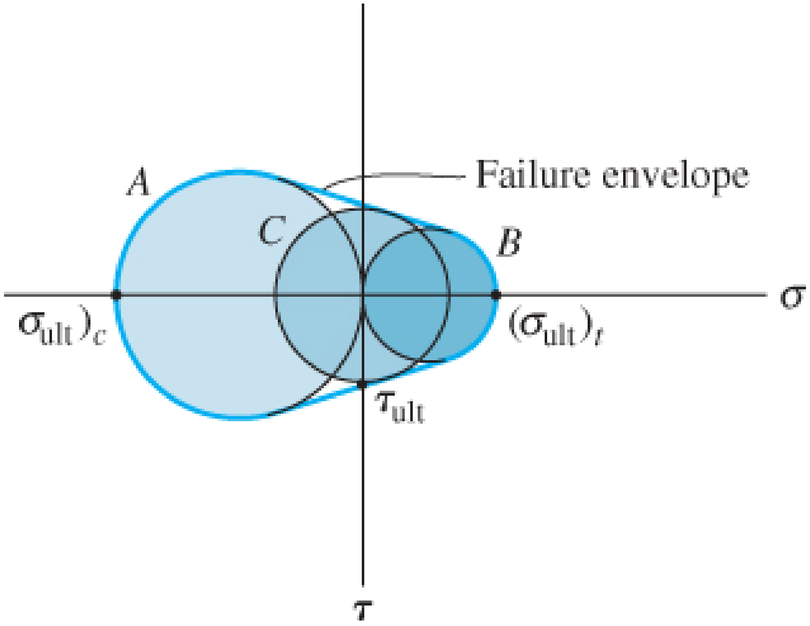
- Carry out a uniaxial tensile test to determine the ultimate tensile stress $(\sigma_{\text{ult}})_t$
- Carry out a uniaxial compressive test to determine the ultimate compressive stress $(\sigma_{\text{ult}})_c$
- Carry out a torsion test to determine the ultimate shear stress τ_{ult}.
- Results are plotted in Mohr circles.

*10.7 THEORIES OF FAILURE

A. Brittle materials

4. Mohr's Failure Criterion

- Circle A represents the stress condition $\sigma_1 = \sigma_2 = 0$, $\sigma_3 = -(\sigma_{ult})_c$
- Circle B represents the stress condition $\sigma_1 = (\sigma_{ult})_t$, $\sigma_2 = \sigma_3 = 0$
- Circle C represents the pure-shear-stress condition caused by τ_{ult}.



*10.7 THEORIES OF FAILURE

A. Brittle materials

4. Mohr's Failure Criterion

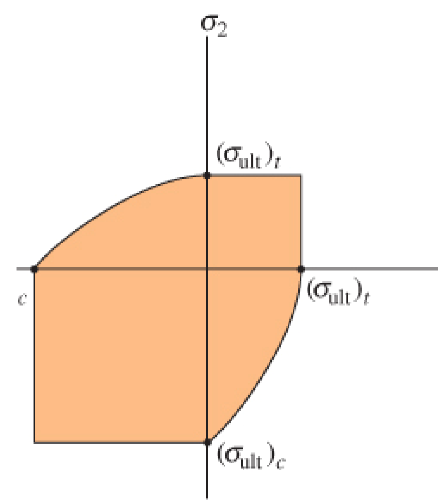
- The Criterion can also be represented on a graph of principal stresses σ_1 and σ_2 (σ_3 = 0).

Brittle-Coulomb-Mohr

$$\sigma_A = \frac{S_{ut}}{n} \qquad \sigma_A \geq \sigma_B \geq 0$$

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n} \qquad \sigma_A \geq 0 \geq \sigma_B$$

$$\sigma_B = -\frac{S_{uc}}{n} \qquad 0 \geq \sigma_A \geq \sigma_B$$



Mohr's failure criteria

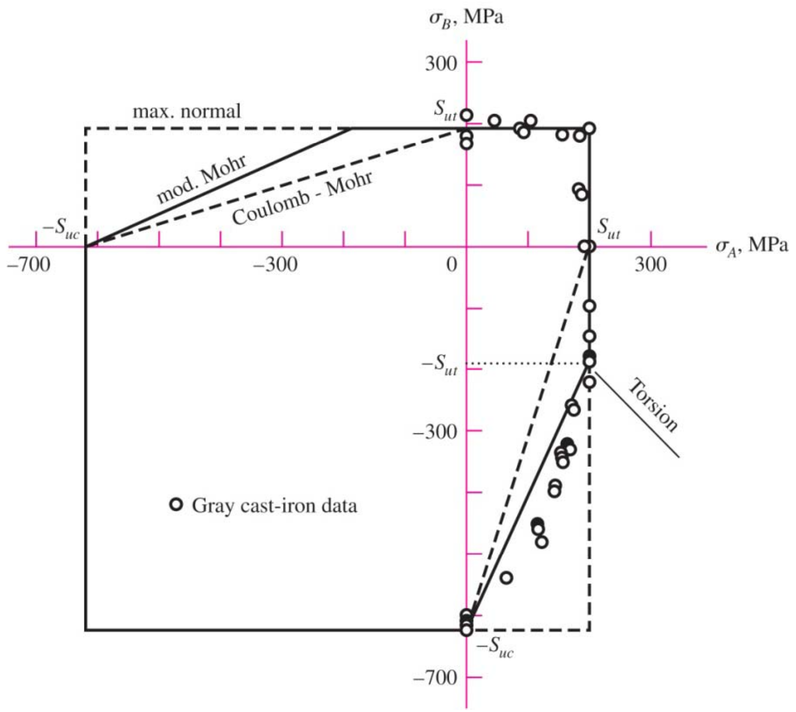
Modifications of the Mohr Theory for Brittle Materials

Brittle-Coulomb-Mohr

$$\sigma_A = \frac{S_{ut}}{n} \qquad \sigma_A \geq \sigma_B \geq 0$$

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n} \qquad \sigma_A \geq 0 \geq \sigma_B$$

$$\sigma_B = -\frac{S_{uc}}{n} \qquad 0 \geq \sigma_A \geq \sigma_B$$



Modified Mohr

$$\sigma_A = \frac{S_{ut}}{n} \qquad \sigma_A \geq \sigma_B \geq 0$$

$$\sigma_A \geq 0 \geq \sigma_B \quad \text{and} \quad \left|\frac{\sigma_B}{\sigma_A}\right| \leq 1$$

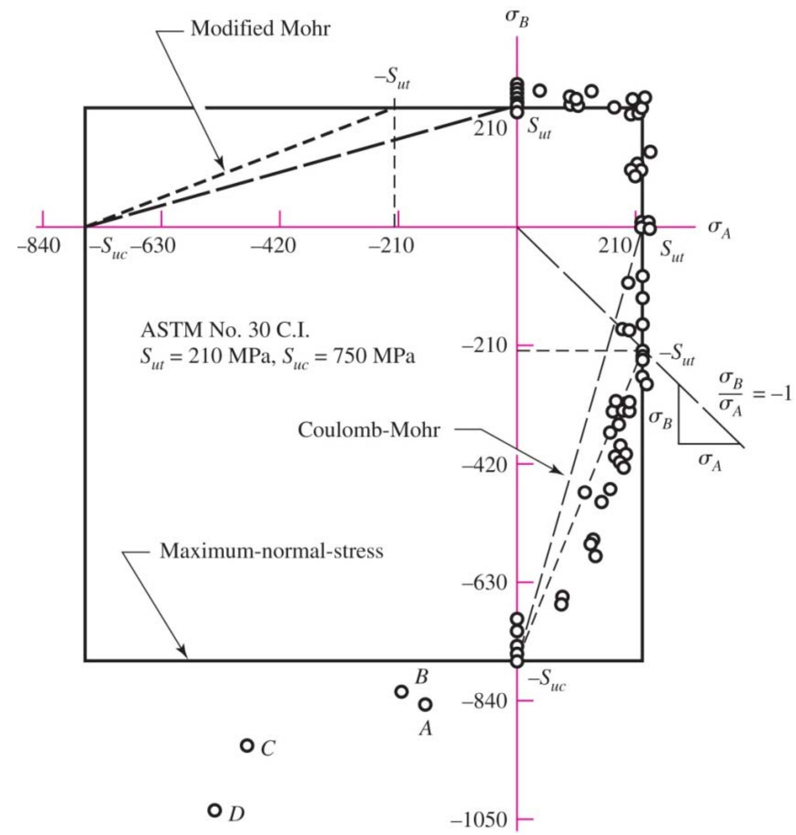
$$\frac{(S_{uc} - S_{ut})\,\sigma_A}{S_{uc} S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n} \qquad \sigma_A \geq 0 \geq \sigma_B \quad \text{and} \quad \left|\frac{\sigma_B}{\sigma_A}\right| > 1$$

$$\sigma_B = -\frac{S_{uc}}{n} \qquad 0 \geq \sigma_A \geq \sigma_B$$

Failure of Brittle Materials Summary

Brittle materials have true strain at fracture is 0.05 or less.

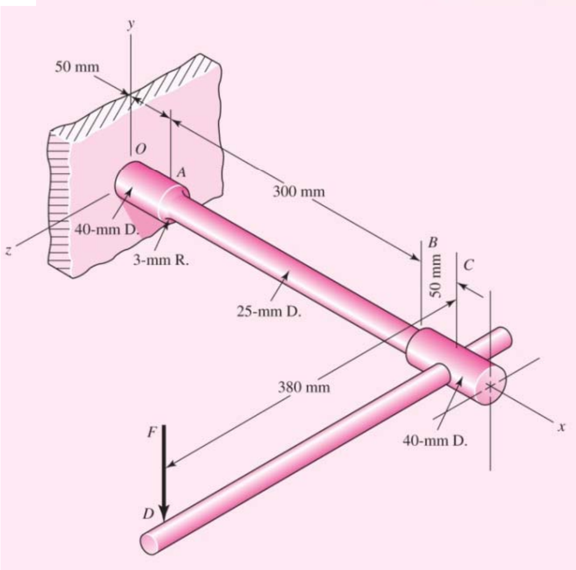
- In the first quadrant the data appear on both sides and along the failure curves of maximum-normal-stress, Coulomb-Mohr, and modified Mohr. All failure curves are the same, and data fit well.
- In the fourth quadrant the modified Mohr theory represents the data best.
- In the third quadrant the points *A*, *B*, *C*, and *D* are too few to make any suggestion concerning a fracture locus.



EXAMPLE 5–3 This example illustrates the use of a failure theory to determine the strength of a mechanical element or component. The example may also clear up any confusion existing between the phrases *strength of a machine part, strength of a material,* and *strength of a part at a point.*

A certain force F applied at D near the end of the 380-mm lever shown in Fig. 5–16, which is quite similar to a socket wrench, results in certain stresses in the cantilevered bar $OABC$. This bar ($OABC$) is of AISI 1035 steel, forged and heat-treated so that it has a minimum (ASTM) yield strength of 560 MPa. We presume that this component would be of no value after yielding. Thus the force F required to initiate yielding can be regarded as the strength of the component part. Find this force.

Solution We will assume that lever DC is strong enough and hence not a part of the problem. A 1035 steel, heat-treated, will have a reduction in area of 50 percent or more and hence is a ductile material at normal temperatures. This also means that stress concentration at shoulder A need not be considered. A stress element at A on the top sur-
l be subjected to a tensile bending stress and a torsional stress. This point, on
ım-diameter section, is the weakest section, and governs the strength of the
y. The two stresses are



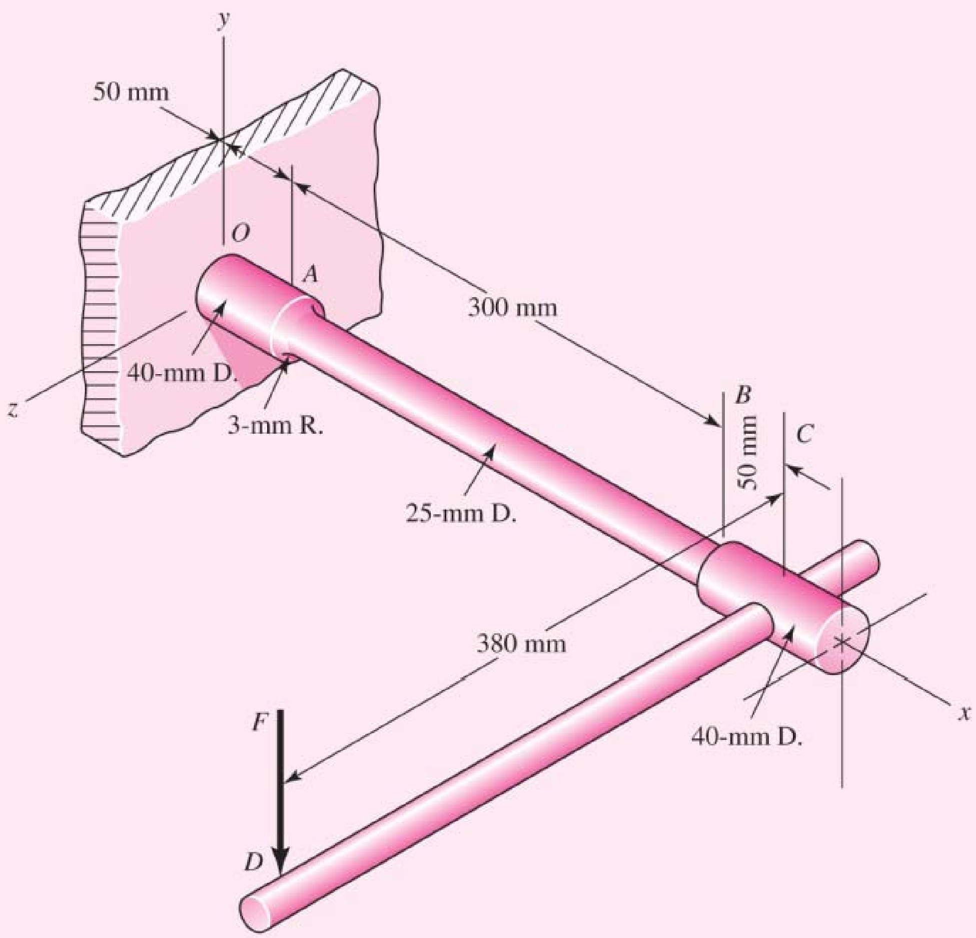
$$\sigma_x = \frac{M}{I/c} = \frac{32M}{\pi d^3} = \frac{32(0.355F)}{\pi(0.025^3)} = 231\,424\,F$$

$$\tau_{zx} = \frac{Tr}{J} = \frac{16T}{\pi d^3} = \frac{16(0.38F)}{\pi(0.025^3)} = 123\,860\,F$$

$$\sigma_x = \frac{M}{I/c} = \frac{32M}{\pi d^3} = \frac{32(0.355F)}{\pi(0.025^3)} = 231\ 424\ F$$

$$\tau_{zx} = \frac{Tr}{J} = \frac{16T}{\pi d^3} = \frac{16(0.38F)}{\pi(0.025^3)} = 123\ 860\ F$$

Figure 5–16



1

Employing the distortion-energy theory, we find, from Eq. (5–15), that

$$\sigma' = \left(\sigma_x^2 + 3\tau_{zx}^2\right)^{1/2} = [(231\,424F)^2 + 3(123\,860F)^2]^{1/2} = 315\,564\,F$$

Equating the von Mises stress to S_y, we solve for F and get

$$F = \frac{S_y}{315\,564} = \frac{560 \times 10^6}{315\,564} = 1.77\text{ kN}$$

In this example the strength of the material at point A is $S_y = 560$ MPa. The strength of the assembly or component is $F = 1.8$ kN.

Let us see how to apply the MSS theory. For a point undergoing plane stress with only one non-zero normal stress and one shear stress, the two nonzero principal stresses σ_A and σ_B will have opposite signs and hence fit case 2 for the MSS theory. From Eq. (3–14),

$$\sigma_A - \sigma_B = 2\left[\left(\frac{\sigma_x}{2}\right)^2 + \tau_{zx}^2\right]^{1/2} = \left(\sigma_x^2 + 4\tau_{zx}^2\right)^{1/2}$$

For case 2 of the MSS theory, Eq. (5–3) applies and hence

$$\tau_{\max} = \frac{S_y}{2n} \qquad \text{or} \qquad \sigma_1 - \sigma_3 = \frac{S_y}{n}$$

$$\left(\sigma_x^2 + 4\tau_{zx}^2\right)^{1/2} = S_y$$

$$[(231\,424F)^2 + 4(123\,860F)^2]^{1/2} = 339\,002\,F = 560 \times 10^6$$

$$F = 1.65\text{ kN}$$

which is about 7 percent less than found for the DE theory. As stated earlier, the MSS theory is more conservative than the DE theory.

*10.7 THEORIES OF FAILURE

IMPORTANT

- If material is ductile, failure is specified by the initiation of yielding, whereas if it is brittle, it is specified by fracture.
- Ductile failure can be defined when slipping occurs between the crystals that compose the material.
- This slipping is due to shear stress and the maximum-shear-stress theory is based on this idea.
- Strain energy is stored in a material when subjected to normal stress.

*10.7 THEORIES OF FAILURE

IMPORTANT

- The maximum-distortion-energy theory depends on the strain energy that distorts the material, and not the part that increases its volume.
- The fracture of a brittle material is caused by the maximum tensile stress in the material, and not the compressive stress.
- This is the basis of the maximum-normal-stress theory, and it is applicable if the stress-strain diagram is similar in tension and compression.

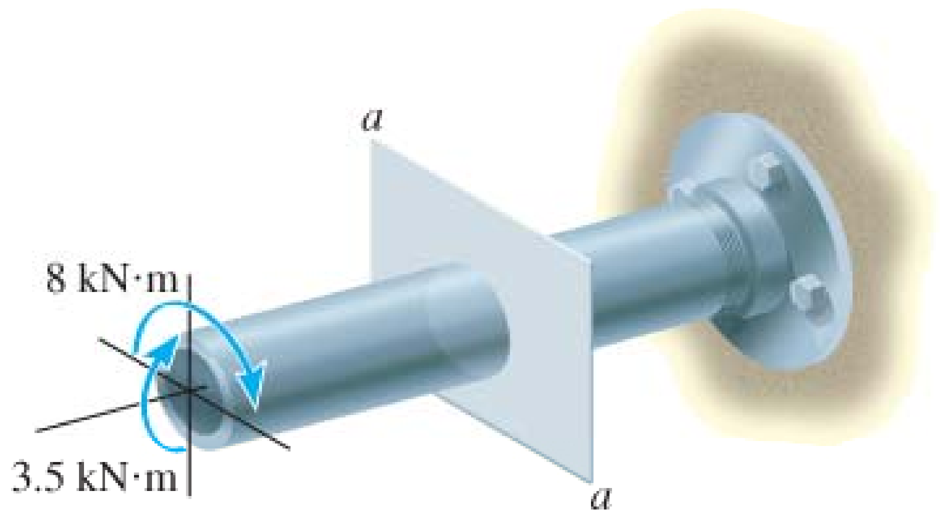
*10.7 THEORIES OF FAILURE

IMPORTANT

- If a brittle material has a stress-strain diagram that is different in tension and compression, then Mohr's failure criterion may be used to predict failure.
- Due to material imperfections, tensile fracture of a brittle material is difficult to predict, and so theories of failure for brittle materials should be used with caution.

*EXAMPLE 10.12

Steel pipe has inner diameter of 60 mm and outer diameter of 80 mm. If it is subjected to a torsional moment of 8 kN·m and a bending moment of 3.5 kN·m, determine if these loadings cause failure as defined by the maximum-distortion-energy theory. Yield stress for the steel found from a tension test is σ_Y = 250 MPa.



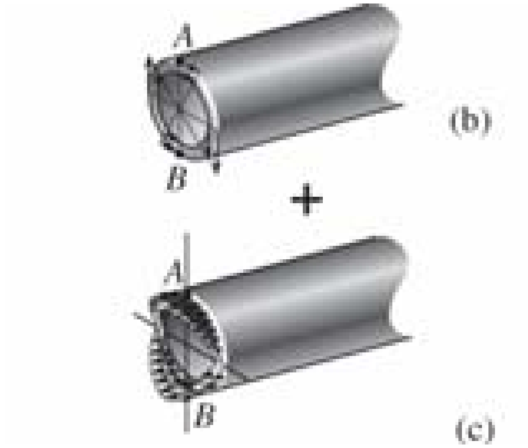
(a)

*EXAMPLE 10.12 (SOLN)

Investigate a pt on pipe that is subjected to a state of maximum critical stress.

Torsional and bending moments are uniform throughout the pipe's length.

At arbitrary section *a*-*a*, loadings produce the stress distributions shown.

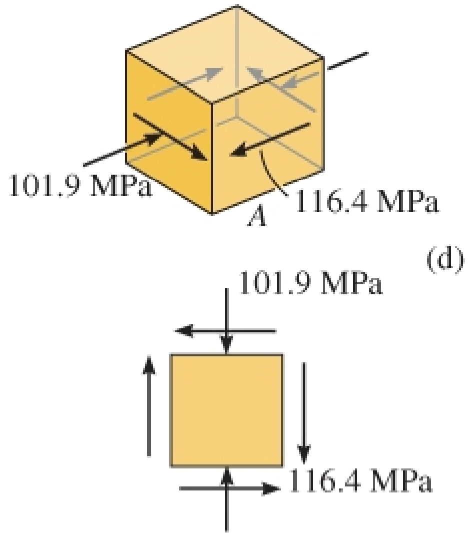


*EXAMPLE 10.12 (SOLN)

By inspection, pts A and B subjected to same state of critical stress. Stress at A,

$$\tau_A = \frac{Tc}{J} = \frac{(8000\ \mathrm{N\cdot m})(0.04\ \mathrm{m})}{(\pi/2)\left[(0.04\ \mathrm{m})^4 - (0.03\ \mathrm{m})^4\right]} = 116.4\ \mathrm{MPa}$$

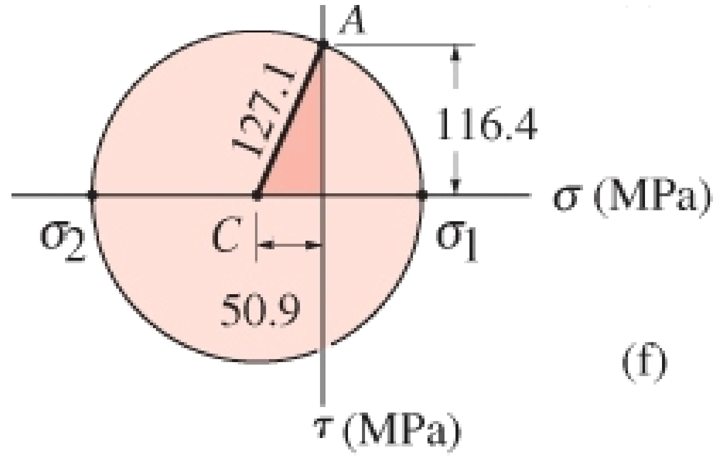
$$\sigma_\mathrm{A} = \frac{Mc}{I} = \frac{(3500\ \mathrm{N\cdot m})(0.04\ \mathrm{m})}{(\pi/4)\left[(0.04\ \mathrm{m})^4 - (0.03\ \mathrm{m})^4\right]} = 101.9\ \mathrm{MPa}$$



*EXAMPLE 10.12 (SOLN)

Mohr's circle for this state of stress has center located at

$$\sigma_{avg} = \frac{0 - 101.9}{2} = -50.9 \text{ MPa}$$



(f)

The radius is calculated from the shaded triangle to be R = 127.1 and the in-plane principal stresses are

$$\sigma_1 = -50.9 + 127.1 = 76.2 \text{ MPa}$$

$$\sigma_2 = -50.9 - 127.1 = -178.0 \text{ MPa}$$

*EXAMPLE 10.12 (SOLN)

Using Eqn 10-30, we have

$$\left(\sigma_1{}^2 - \sigma_1\sigma_2 + \sigma_2{}^2\right) \leq \sigma_Y{}^2$$

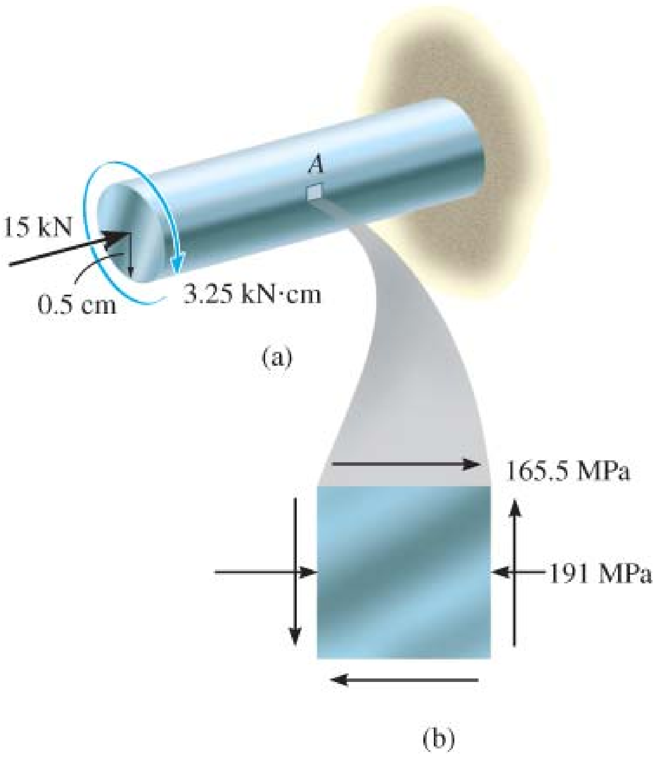
$$\text{Is } \left[(76.2)^2 - (76.2)(-178.0) + (-178.0)^2\right] \leq \sigma_Y{}^2 \ \ ?$$

$$51{,}100 < 62{,}500 \text{ OK!}$$

Since criterion is met, material within the pipe will not yield (“fail”) according to the maximum-distortion-energy theory.

*EXAMPLE 10.14

Solid shaft has a radius of 0.5 cm and made of steel having yield stress of σ_Y = 360 MPa. Determine if the loadings cause the shaft to fail according to the maximum-shear-stress theory and the maximum-distortion-energy theory.



*EXAMPLE 10.14 (SOLN)

State of stress in shaft caused by axial force and torque. Since maximum shear stress caused by torque occurs in material at outer surface, we have

$$\sigma_x = -\frac{P}{A} = \frac{15\ \text{kN}}{\pi(0.5\ \text{cm})^2} = -19.10\ \text{kN/cm}^2 = 191\ \text{MPa}$$

$$\tau_{xy} = \frac{Tc}{J} = \frac{3.25\ \text{kN}\cdot\text{cm}(0.5\ \text{cm})}{\pi / 2(0.5\ \text{cm})^4}$$

$$\tau_{xy} = 16.55\ \text{kN/cm}^2 = 165.5\ \text{MPa}$$

*EXAMPLE 10.14 (SOLN)

Stress components acting on an element of material at pt A. Rather than use Mohr's circle, principal stresses are obtained using stress-transformation eqns 9-5:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$= \frac{-191+0}{2} \pm \sqrt{\left(\frac{-191+0}{2}\right)^2 + (165.5)^2}$$

$$= -95.5 \pm 191.1$$

$$\sigma_1 = 95.6 \text{ MPa}$$

$$\sigma_2 = -286.6 \text{ MPa}$$

*EXAMPLE 10.14 (SOLN)

Maximum-shear-stress theory

Since principal stresses have opposite signs, absolute maximum shear stress occur in the plane, apply Eqn 10-27,

$$|\sigma_1 - \sigma_2| \leq \sigma_Y$$

$$\text{Is } |95.6 - (-286.6)| \leq 360 \quad ?$$

$$382.2 > 360 \text{ Fail!}$$

Thus, shear failure occurs by maximum-shear-stress theory.

*EXAMPLE 10.14 (SOLN)

Maximum-distortion-energy theory

Applying Eqn 10-30, we have

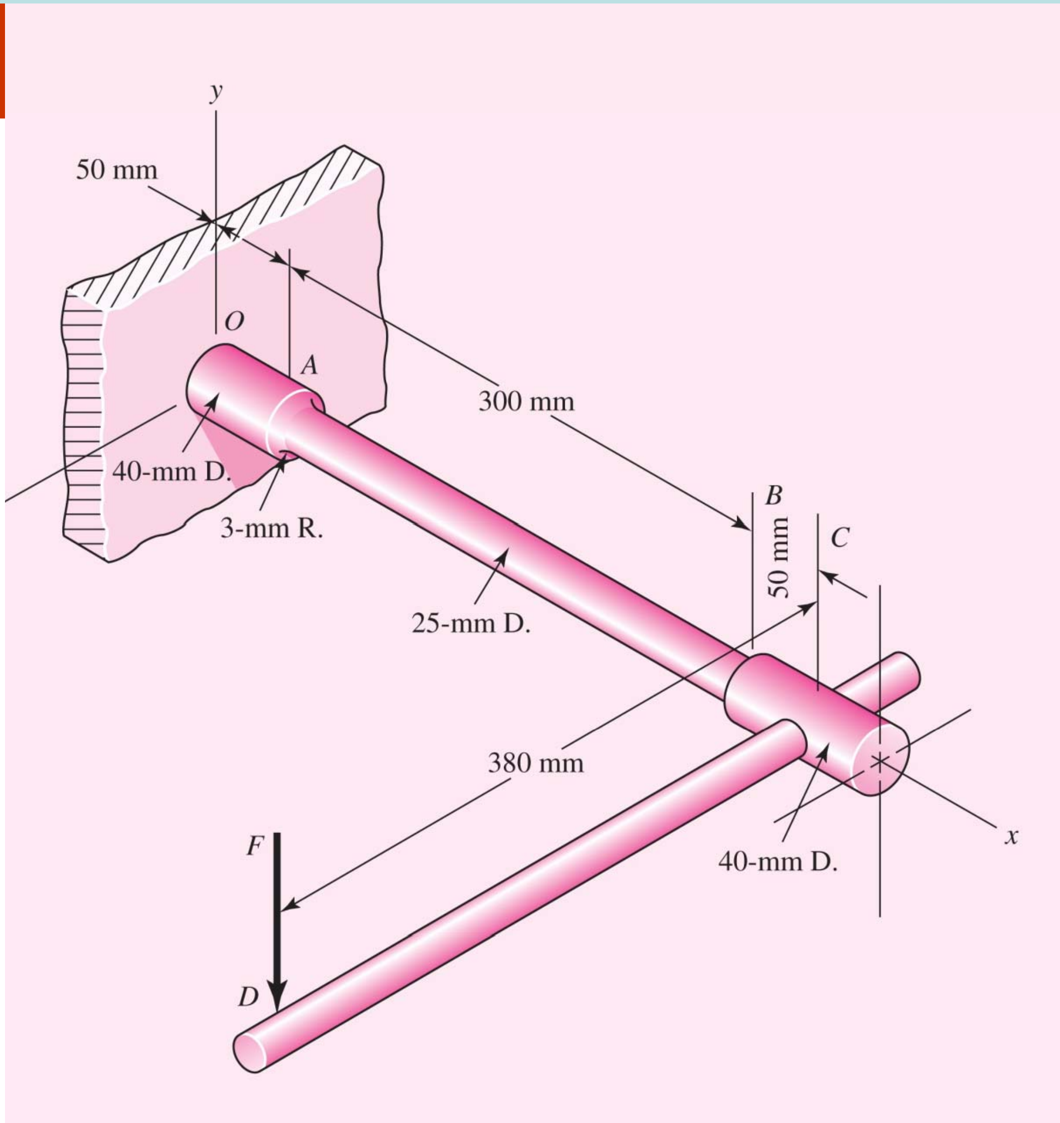
$$\left(\sigma_1{}^2 - \sigma_1\sigma_2 + \sigma_2{}^2\right) \leq \sigma_Y$$

$$\text{Is } \left[(95.6)^2 - (95.6)(-286.6) - (-286.6)^2\right] \leq (360)^2 \ \ ?$$

$$118{,}677.9 \leq 129{,}600 \quad \text{OK!}$$

However, using the maximum-distortion-energy theory, failure will not occur. Why?

10. Strain Transformation



EXAMPLE 5–5 Consider the wrench in Ex. 5–3, Fig. 5–16, as made of cast iron, machined to dimension. The force F required to fracture this part can be regarded as the strength of the component part. If the material is ASTM grade 30 cast iron, find the force F with

(*a*) Coulomb-Mohr failure model.

(*b*) Modified Mohr failure model.

Solution We assume that the lever DC is strong enough, and not part of the problem. Since grade 30 cast iron is a brittle material *and* cast iron, the stress-concentration factors K_t and K_{ts} are set to unity. From Table A–24, the tensile ultimate strength is 210 MPa and the compressive ultimate strength is 750 MPa. The stress element at A on the top surface will be subjected to a tensile bending stress and a torsional stress. This location, on the 1-in-diameter section fillet, is the weakest location, and it governs the strength of the assembly. The normal stress σ_x and the shear stress at A are given by

- Stress concentration is a highly localized effect.
- Geometric (theoretical) stress-concentration factor for normal stress K_t and shear stress K_{ts} is defined as

$$\sigma_{\max} = K_t \sigma_{\text{nom}}$$

$$\tau_{\max} = K_{ts} \tau_{\text{nom}}$$

$$\sigma_x = K_t \frac{M}{I/c} = K_t \frac{32M}{\pi d^3} = (1)\frac{32(0.355)}{\pi(0.025)^3} = 231\ 424F$$

$$\tau_{xy} = K_{ts} \frac{Tr}{J} = K_{ts} \frac{16T}{\pi d^3} = (1)\frac{16(0.38)}{\pi(0.025)^3} = 123\ 860F$$

From Eq. (3–13) the nonzero principal stresses σ_A and σ_B are

$$\sigma_A, \sigma_B = \frac{231\ 424F + 0}{2} \pm \sqrt{\left(\frac{231\ 424F - 0}{2}\right)^2 + (123\ 860F)^2} = 285\ 213F, -53\ 789F$$

This puts us in the fourth-quadrant of the σ_A, σ_B plane.
(*a*) For BCM, Eq. (5–31*b*) applies with $n = 1$ for failure.

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{285\ 213F}{210 \times 10^6} - \frac{(-53\ 789F)}{750 \times 10^6} = 1$$

Solving for F yields

Answer $$F = 699\ \text{N}$$

(*b*) For MM, the slope of the load line is $|\sigma_B/\sigma_A| = 53\ 789/285\ 213 = 0.189 < 1$. Obviously, Eq. (5–32*a*) applies.

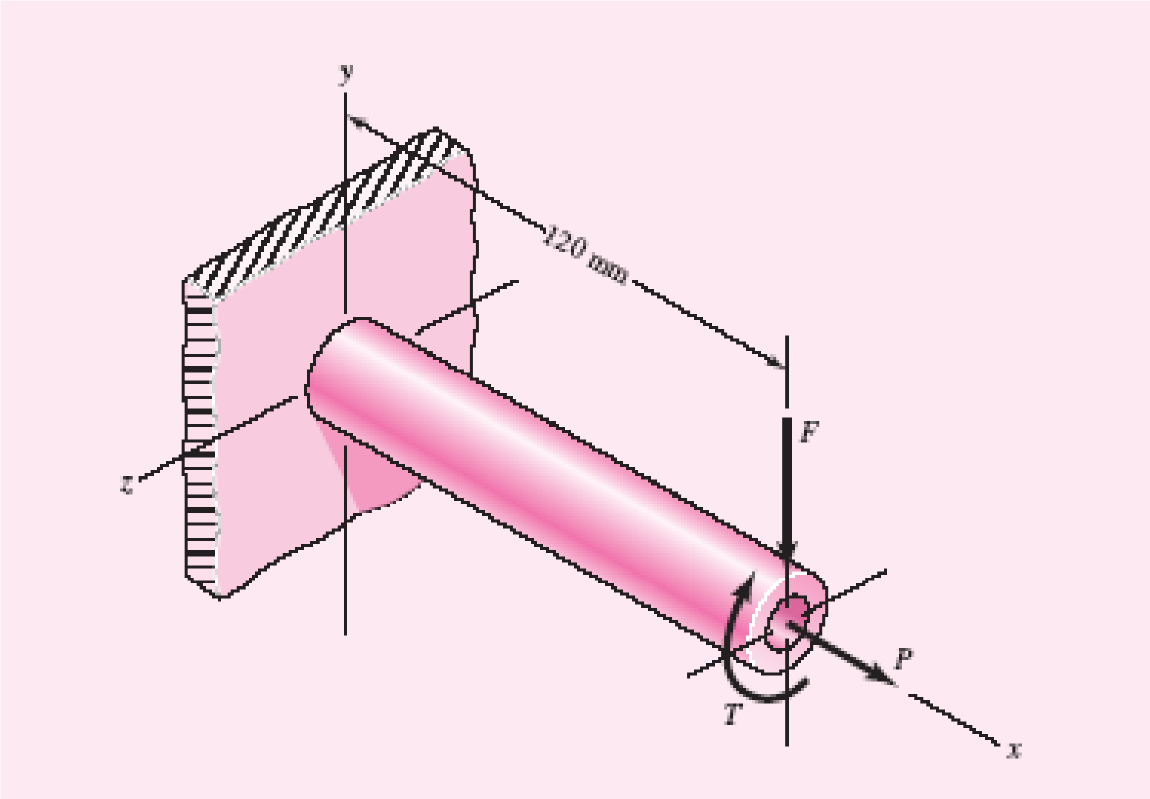
$$\frac{\sigma_A}{S_{ut}} = \frac{285\ 213F}{210 \times 10^6} = 1$$

Answer $$F = 736\ \text{N}$$

As one would expect from inspection of Fig. 5–19, Coulomb-Mohr is more conservative.

EXAMPLE 5–4

The cantilevered tube shown in Fig. 5–17 is to be made of 2014 aluminum alloy treated to obtain a specified minimum yield strength of 276 MPa. We wish to select a stock-size tube from Table A–8 using a design factor $n_d = 4$. The bending load is $F = 1.75$ kN, the axial tension is $P = 9.0$ kN, and the torsion is $T = 72$ N · m. What is the realized factor of safety?



10.

Table A-8

Properties of Round Tubing

w_a = unit weight of aluminum tubing, lbf/ft
w_s = unit weight of steel tubing, lbf/ft
m = unit mass, kg/m
A = area, in^2 (cm^2)
I = second moment of area, in^4 (cm^4)
J = second polar moment of area, in^4 (cm^4)
k = radius of gyration, in (cm)
Z = section modulus, in^3 (cm^3)
d, t = size (OD) and thickness, in (mm)

Size, in	w_a	w_s	A	I	k	Z	J
$1 \times \frac{1}{8}$	0.416	1.128	0.344	0.034	0.313	0.067	0.067
$1 \times \frac{1}{4}$	0.713	2.003	0.589	0.046	0.280	0.092	0.092
$1\frac{1}{2} \times \frac{1}{8}$	0.653	1.769	0.540	0.129	0.488	0.172	0.257
$1\frac{1}{2} \times \frac{1}{4}$	1.188	3.338	0.982	0.199	0.451	0.266	0.399
$2 \times \frac{1}{8}$	0.891	2.670	0.736	0.325	0.664	0.325	0.650
$2 \times \frac{1}{4}$	1.663	4.673	1.374	0.537	0.625	0.537	1.074
$2\frac{1}{2} \times \frac{1}{8}$	1.129	3.050	0.933	0.660	0.841	0.528	1.319
$2\frac{1}{2} \times \frac{1}{4}$	2.138	6.008	1.767	1.132	0.800	0.906	2.276
$3 \times \frac{1}{4}$	2.614	7.343	2.160	2.059	0.976	1.373	4.117
$3 \times \frac{3}{8}$	3.742	10.51	3.093	2.718	0.938	1.812	5.436
$4 \times \frac{3}{16}$	2.717	7.654	2.246	4.090	1.350	2.045	8.180
$4 \times \frac{3}{8}$	5.167	14.52	4.271	7.090	1.289	3.544	14.180

Size, mm	m	A	I	k	Z	J
12 × 2	0.490	0.628	0.082	0.361	0.136	0.163
16 × 2	0.687	0.879	0.220	0.500	0.275	0.440
16 × 3	0.956	1.225	0.273	0.472	0.341	0.545
20 × 4	1.569	2.010	0.684	0.583	0.684	1.367
25 × 4	2.060	2.638	1.508	0.756	1.206	3.015
25 × 5	2.452	3.140	1.669	0.729	1.336	3.338
30 × 4	2.550	3.266	2.827	0.930	1.885	5.652
30 × 5	3.065	3.925	3.192	0.901	2.128	6.381
42 × 4	3.727	4.773	8.717	1.351	4.151	17.430
42 × 5	4.536	5.809	10.130	1.320	4.825	20.255
50 × 4	4.512	5.778	15.409	1.632	6.164	30.810
50 × 5	5.517	7.065	18.118	1.601	7.247	36.226

Solution Since the maximum bending moment is $M = 120F$, the normal stress, for an element on the top surface of the tube at the origin, is

$$\sigma_x = \frac{P}{A} + \frac{Mc}{I} = \frac{9}{A} + \frac{120(1.75)(d_o/2)}{I} = \frac{9}{A} + \frac{105d_o}{I} \tag{1}$$

where, if millimeters are used for the area properties, the stress is in gigapascals.

The torsional stress at the same point is

$$\tau_{zx} = \frac{Tr}{J} = \frac{72(d_o/2)}{J} = \frac{36d_o}{J} \tag{2}$$

For accuracy, we choose the distortion-energy theory as the design basis. The von Mises stress, as in the previous example, is

$$\sigma' = \left(\sigma_x^2 + 3\tau_{zx}^2\right)^{1/2} \tag{3}$$

On the basis of the given design factor, the goal for σ' is

$$\sigma' \le \frac{S_y}{n_d} = \frac{0.276}{4} = 0.0690 \text{ GPa} \tag{4}$$

where we have used gigapascals in this relation to agree with Eqs. (1) and (2).

Programming Eqs. (1) to (3) on a spreadsheet and entering metric sizes from Table A–8 reveals that a 42- × 5-mm tube is satisfactory. The von Mises stress is found to be $\sigma' = 0.06043$ GPa for this size. Thus the realized factor of safety is

Answer

$$n = \frac{S_y}{\sigma'} = \frac{0.276}{0.06043} = 4.57$$

For the next size smaller, a 42- × 4-mm tube, $\sigma' = 0.07105$ GPa giving a factor of safety of

$$n = \frac{S_y}{\sigma'} = \frac{0.276}{0.07105} = 3.88$$

CHAPTER REVIEW

- Provided the principal stresses for a material are known, then a theory of failure can be used as a basis for design.
- Ductile materials fail in shear, and here the maximum-shear-stress theory or the maximum-distortion-energy theory can be used to predict failure.
- Both theories make comparison to the yield stress of a specimen subjected to uniaxial stress.

CHAPTER REVIEW

- Brittle materials fail in tension, and so the maximum-normal-stress theory or Mohr's failure criterion can be used to predict failure.
- Comparisons are made with the ultimate tensile stress developed in a specimen.