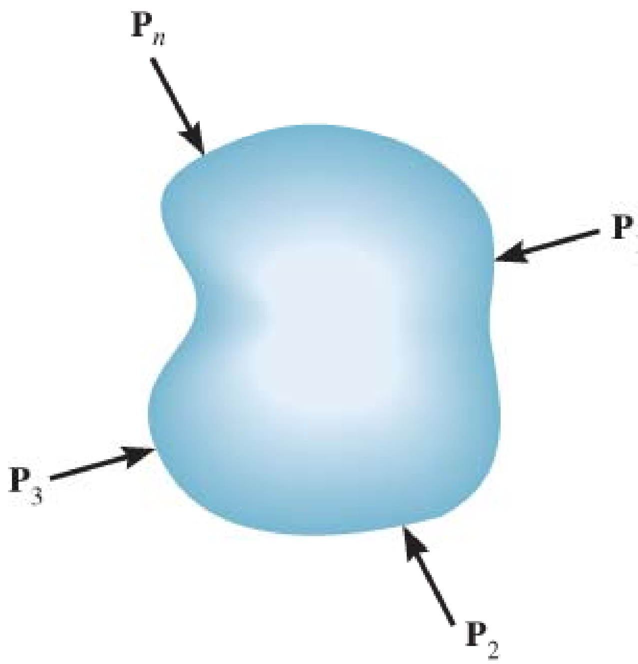
*14.8 CASTIGLIANO'S THEOREM

- Consider a body of arbitrary shape subjected to a series of n forces $\mathbf{P}_1$, $\mathbf{P}_2$, … $\mathbf{P}_n$.
- Since external work done by forces is equal to internal strain energy stored in body, by conservation of energy, $U_e = U_i$.
- However, external work is a function of external loads $U_e = \sum \int P\,dx$.



*14.8 CASTIGLIANO'S THEOREM

- So, internal work is also a function of the external loads. Thus $$U_i = U_e = f\left(P_1, P_2, \ldots, P_n\right) \qquad (14\text{-}44)$$
- Now, if any one of the external forces say P_j is increased by a differential amount dP_j. Internal work increases, so strain energy becomes

$$U_i + dU_i = U_i + \frac{\delta U_i}{\delta P_j} dP_j \qquad (14\text{-}45)$$

- Further application of the loads cause dP_j to move through displacement Δ_j, so strain energy becomes $$U_i + dU_j = U_i + dP_j \Delta_i \qquad (14\text{-}46)$$

*14.8 CASTIGLIANO'S THEOREM

- $dU_j = dP_j\Delta_i$ is the additional strain energy caused by dP_j.
- In summary, Eqn 14-45 represents the strain energy in the body determined by first applying the loads $\mathbf{P}_1$, $\mathbf{P}_2$, …, $\mathbf{P}_n$, then dP_j.
- Eqn 14-46 represents the strain energy determined by first applying dP_j, then the loads $\mathbf{P}_1$, $\mathbf{P}_2$, …, $\mathbf{P}_n$.
- Since theses two eqns are equal, we require

$$\Delta_i = \frac{\delta U_i}{\delta P_j} \qquad (14\text{-}47)$$

*14.8 CASTIGLIANO'S THEOREM

- Note that Eqn 14-47 is a statement regarding the body's compatibility requirements, since it's related to displacement.
- The derivation requires that only conservative forces be considered for analysis.

*14.9 CASTIGLIANO'S THEOREM APPLIED TO TRUSSES

- Since a truss member is subjected to an axial load, strain energy is given by Eqn 14-16, $U_i = N^2L/2AE$.
- Substitute this eqn into Eqn 14-47 and omitting the subscript i, we have

$$\Delta = \frac{\delta}{\delta P}\sum \frac{N^2 L}{2AE}$$

- It is easier to perform differentiation prior to summation. Also, L, A and E are constant for a given member, thus

$$\Delta = \sum N\left(\frac{\delta}{\delta P}\right)\frac{L}{AE} \qquad (14\text{-}48)$$

*14.9 CASTIGLIANO'S THEOREM APPLIED TO TRUSSES

Δ = joint displacement of the truss.

P = external force of variable magnitude applied to the truss joint in direction of Δ.

N = internal axial force in member caused by both force P and loads on the truss.

L = length of a member.

A = x-sectional area of a member.

E = modulus of elasticity of the material.

$$\Delta = \sum N\left(\frac{\delta}{\delta P}\right)\frac{L}{AE} \qquad (14\text{-}48)$$

*14.9 CASTIGLIANO'S THEOREM APPLIED TO TRUSSES

- In order to determine the partial derivative $\delta N/\delta P$, we need to treat P as a variable, not numeric qty. Thus, each internal axial force N must be expressed as a function of P.
- By comparison, Eqn 14-48 is similar to that used for method of virtual work, Eqn 14-39, except that n is replaced by $\delta N/\delta P$.
- These terms; n and $\delta N/\delta P$, are the same, since they represent the rate of change of internal axial force w.r.t. the load P.

*14.9 CASTIGLIANO'S THEOREM APPLIED TO TRUSSES

Procedure for analysis

External force P.

- Place a force P on truss at the joint where the desired displacement is to be determined.
- This force is assumed to have a variable magnitude and should be directed along the line of action of the displacement.

Internal forces N.

- Determine the force N in each member caused by both the real (numerical) loads and the variable force P. Assume that tensile forces are +ve and compressive forces are –ve.

*14.9 CASTIGLIANO'S THEOREM APPLIED TO TRUSSES

Procedure for analysis

Internal forces N.

- Find the respective partial derivative $\delta N/\delta P$ for each member.
- After N and $\delta N/\delta P$ have been determined, assign P its numerical value if it has actually replaced a real force on the truss. Otherwise, set P equal to zero.

Castigliano's Second Theorem.

- Apply Castigliano's second theorem to determine the desired displacement Δ.

*14.9 CASTIGLIANO'S THEOREM APPLIED TO TRUSSES

Procedure for analysis

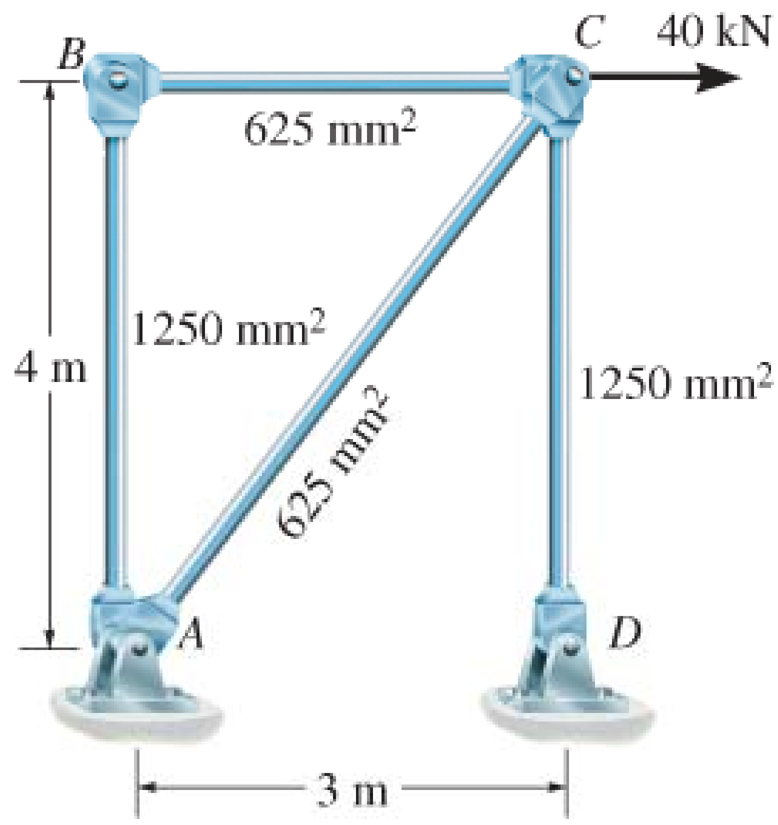
Castigliano's Second Theorem.

- It is important to retain the algebraic signs for corresponding values of N and $\delta N/\delta P$ when substituting these terms into the eqn.
- If the resultant sum $\sum N\,(\delta N/\delta P)\,L/AE$ is +*ve*, Δ is in the same direction as P. If a –ve value results, Δ is opposite to P.

EXAMPLE 14.17

Determine the horizontal displacement of joint C of steel truss shown. The x-sectional area of each member is also indicated.

Take E_{st} = 210(10^3) N/mm^2.

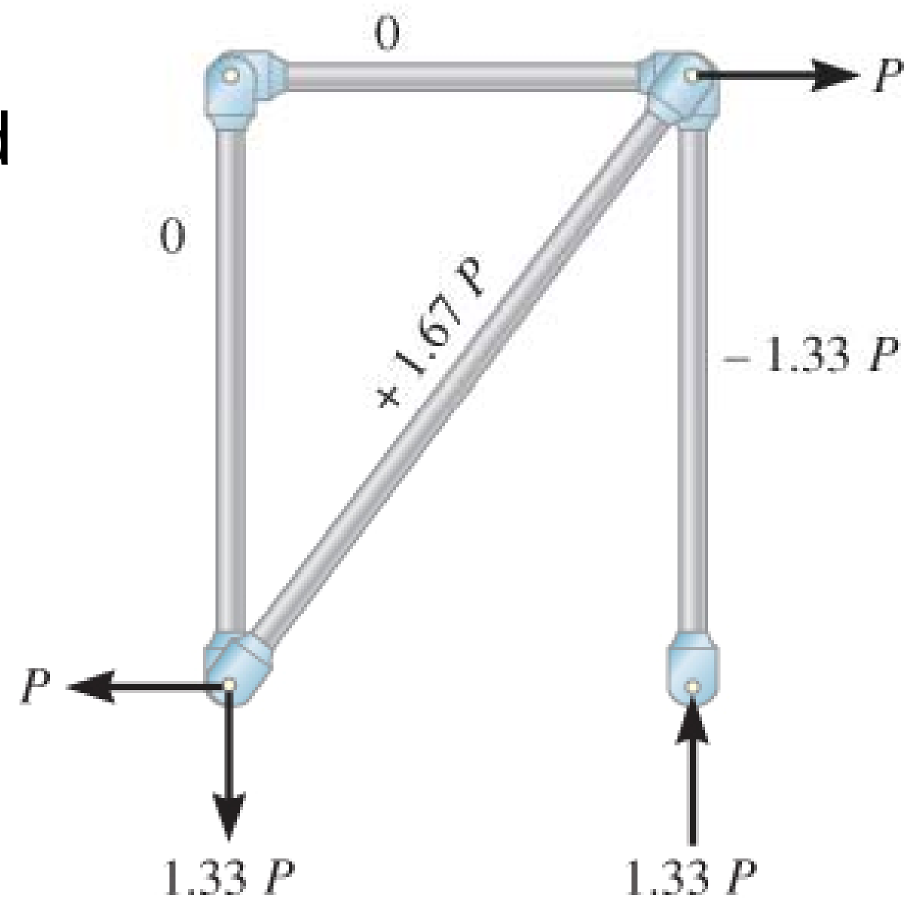


(a)

EXAMPLE 14.17 (SOLN)

External force P.

Since horizontal displacement of C is to be determined, a horizontal variable force P is applied to joint C. Later this force will be set equal to the fixed value of 40 kN.



(b)

EXAMPLE 14.17 (SOLN)

Internal forces N.

Using method of joints, force N in each member is found. Results are shown in table:

Member	N	$\delta N/\delta P$	N (P = 40 kN)	L	$N(\delta N/\delta P)L$
AB	0	0	0	4000	0
BC	0	0	0	3000	0
AC	1.67P	1.67	$66.67(10^3)$	5000	$556.7(10^6)$
CD	–1.33P	–1.33	$-53.33(10^3)$	4000	$283.7(10^6)$

EXAMPLE 14.12 (SOLN)

Castigliano's Second Theorem

Applying Eqn 14-8 gives

$$\Delta_{C_h} = \sum N\left(\frac{\delta N}{\delta P}\right)\frac{L}{AE}$$

$$= 0 + 0 + \frac{556.7\left(10^6\right)\ \mathrm{N\cdot m}}{\left[625\ \mathrm{mm}^2\right]\left[210\left(10^3\right)\ \mathrm{N/mm}^2\right]}$$

$$+ \frac{283.7\left(10^3\right)\ \mathrm{N\cdot m}}{\left[1250\ \mathrm{mm}^2\right]\left[210\left(10^3\right)\ \mathrm{N/mm}^2\right]}$$

$$\Delta_{C_h} = 4.24 + 1.08 = 5.32\ \mathrm{mm}$$

EXAMPLE 14.12 (SOLN)

b)

Here, we must apply Eqn 14-41. Realize that member AC is shortened by ΔL = –6 mm, we have

$$1 \cdot \Delta = \sum n \ \Delta L; \qquad 1 \text{ kN} \cdot \Delta_{C_h} = (1.25 \text{ kN})(-6 \text{ mm})$$

$$\Delta_{C_h} = -7.5 \text{ mm} = 7.5 \text{ mm} \leftarrow$$

The –ve sign indicates that joint C is displaced to the left, opposite to the 1-kN load.

*14.10 CASTIGLIANO'S THEOREM APPLIED TO BEAMS

- Internal strain energy for a beam is caused by both bending and shear. As pointed out in Example 14.7, if beam is long and slender, strain energy due to shear can be neglected.
- Thus, internal strain energy for a beam is given by Eqn 14-17; $U_i = \int M^2\, dx/2EI$. We then substitute into $\Delta_i = \delta U_i/\delta P_i$, Eqn 14-47 and omitting subscript i, we have

$$\Delta = \frac{\delta}{\delta P}\int_0^L \frac{M^2 dx}{2EI}$$

*14.10 CASTIGLIANO'S THEOREM APPLIED TO BEAMS

- It is easier to differentiate prior to integration, thus provided E and I are constant, we have

$$\Delta = \int_0^L M\left(\frac{\delta M}{\delta P}\right)\frac{dx}{EI} \qquad (14\text{-}49)$$

Δ = displacement of the pt caused by the real loads acting on the beam.

P = external force of variable magnitude applied to the beam in the direction of Δ.

*14.10 CASTIGLIANO'S THEOREM APPLIED TO BEAMS

M = internal moment in the beam, expressed as a function of x and caused by both the force P and the loads on the beam.

E = modulus of elasticity of the material.

I = moment of inertia of x-sectional area computed about the neutral axis.

$$\Delta = \int_0^L M\left(\frac{\delta M}{\delta P}\right)\frac{dx}{EI} \qquad (14\text{-}49)$$

*14.10 CASTIGLIANO'S THEOREM APPLIED TO BEAMS

- If slope of tangent θ at a pt on elastic curve is to be determined, the partial derivative of internal moment M w.r.t. an external couple moment M' acting at the pt must be found.
- For this case

$$\theta = \int_0^L M\left(\frac{\delta M}{\delta M'}\right)\frac{dx}{EI} \qquad (14\text{-}50)$$

- The eqns above are similar to those used for the method of virtual work, Eqns 14-42 and 14-43, except m and m_0 replace $\delta M/\delta P$ and $\delta M/\delta M'$, respectively.

*14.10 CASTIGLIANO'S THEOREM APPLIED TO BEAMS

- If the loading on a member causes significant strain energy within the member due to axial load, shear, bending moment, and torsional moment, then the effects of all these loadings should be included when applying Castigliano's theorem.

$$\Delta = \sum N\left(\frac{\delta N}{\delta P}\right)\frac{L}{AE} + \int_0^L f_s V\left(\frac{\delta V}{\delta P}\right)\frac{dx}{GA} + \int_0^L M\left(\frac{\delta M}{\delta P}\right)\frac{dx}{EI} + \int_0^L T\left(\frac{\delta T}{\delta P}\right)\frac{dx}{GJ} \qquad (14\text{-}51)$$

*14.10 CASTIGLIANO'S THEOREM APPLIED TO BEAMS

Procedure for analysis

External force P or couple moment M'.

- Place force P on the beam at the pt and directed along the line of action of the desired displacement.
- If the slope of the tangent is to be determined, place a couple moment M' at the pt.
- Assume that both P and M' have a variable magnitude.

*14.10 CASTIGLIANO'S THEOREM APPLIED TO BEAMS

Procedure for analysis

Internal moment M.

- Establish appropriate x coordinates that are valid within regions of the beam where there is no discontinuity of force, distributed load, or couple moment.
- Calculate the internal moments M as a function of P or M' and the partial derivatives $\delta M/\delta P$ or $\delta M/\delta M'$ for each coordinate of x.

*14.10 CASTIGLIANO'S THEOREM APPLIED TO BEAMS

Procedure for analysis

Internal moment M.

- After M and $\delta M/\delta P$ or $\delta M/\delta M'$ have been determined, assign P or M' its numerical value if it has actually replaced a real force or couple moment. Otherwise, set P or M' equal to zero.

Castigliano's second theorem.

- Apply Eqn 14-49 or 14-50 to determine the desired displacement Δ or θ. It is important to retain the algebraic signs for corresponding values of M and $\delta M/\delta P$ *or* $\delta M/\delta M'$.

*14.10 CASTIGLIANO'S THEOREM APPLIED TO BEAMS

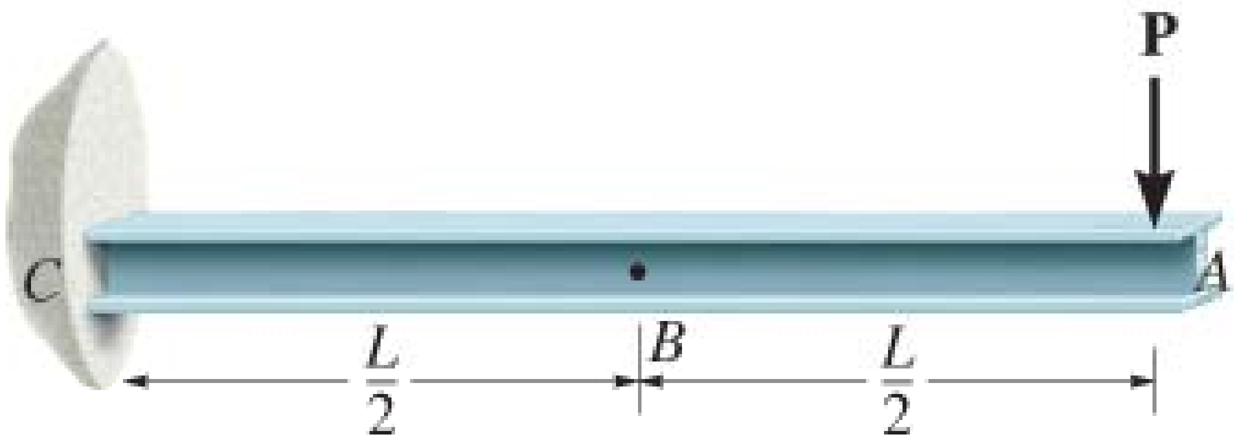
Procedure for analysis

Castigliano's second theorem.

- If the resultant sum of all the definite integrals is +ve, Δ or θ is in the same direction as P or M'. If a –ve value results, Δ or θ is opposite to P or M'.

EXAMPLE 14.20

Determine the slope at pt B of the beam shown. EI is a constant.



(a)

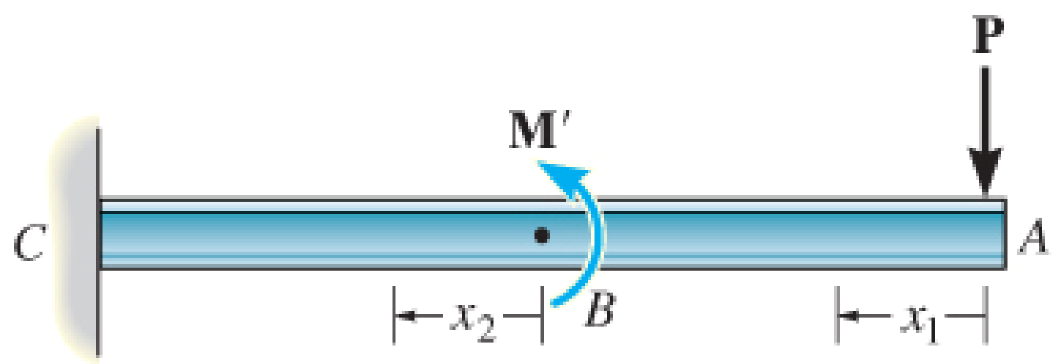
EXAMPLE 14.20 (SOLN)

External couple moment M'.

Since slope at pt B is to be determined, an external couple moment M' is placed on the beam at this pt.

Internal moments M.

Two coordinates x_1 and x_2 is used to determine the internal moments within beam since there is a discontinuity, M' at B. x_1 ranges from A to B, and x_2 ranges from B to C.

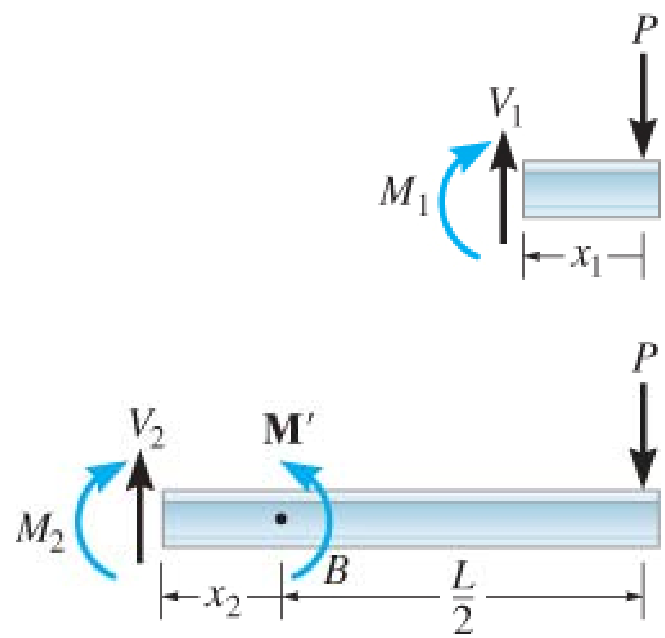


(b)

EXAMPLE 14.20 (SOLN)

Internal moments *M*.

Using method of sections, internal moments and partial derivatives are determined.



(c)

For x_1,

$$\circlearrowleft + \sum M_{NA} = 0; \qquad -M_1 - Px_1 = 0$$

$$M_1 = -Px_1$$

$$\frac{\delta M_1}{\delta M'} = 0$$

EXAMPLE 14.20 (SOLN)

Internal moments M.

For x_2,

$$\circlearrowleft + \sum M_{NA} = 0; \qquad -M_2 + M' - P\left(\frac{L}{2} + x_2\right) = 0$$

$$M_2 = M' - P\left(\frac{L}{2} + x_2\right)$$

$$\frac{\delta M_2}{\delta M'} = 1$$

EXAMPLE 14.20 (SOLN)

Castigliano's second theorem.

Setting M' = 0 and applying Eqn 14-50, we have,

$$\theta_B = \int_0^L M\left(\frac{\delta M}{\delta M'}\right)\frac{dx}{EI}$$

$$= \int_0^{L/2} \frac{(-Px_1)(0)dx_1}{EI} + \int_0^{L/2} \frac{-P[(L/2)+x_2]dx_2}{EI}$$

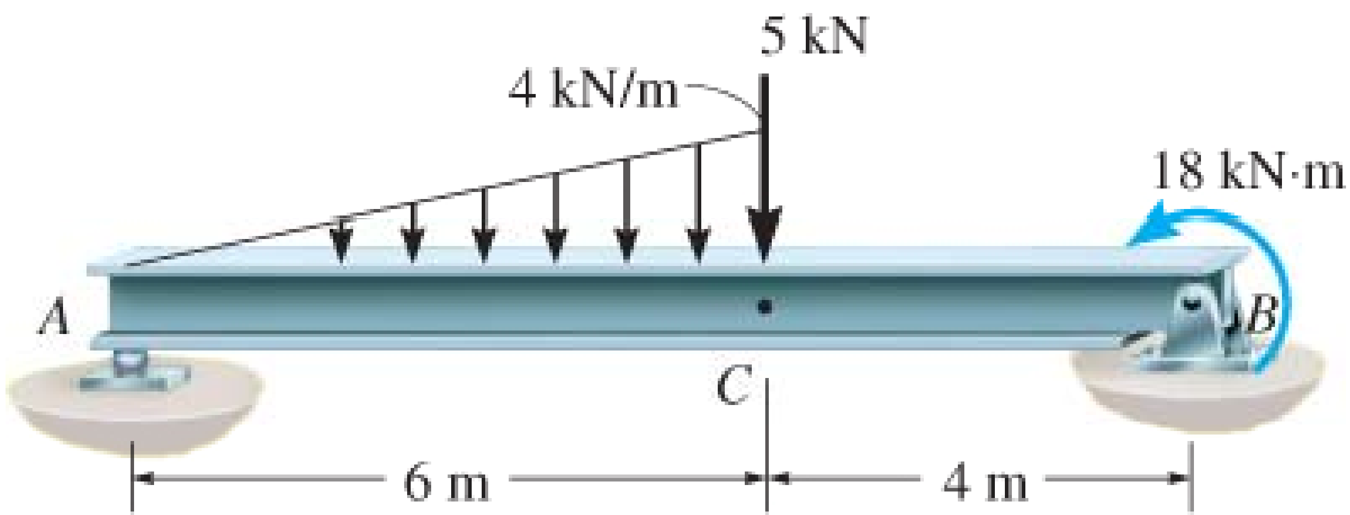
$$= -\frac{3PL^2}{8EI}$$

Negative sign indicates that θ_B is opposite to direction of couple moment M'.

EXAMPLE 14.21

Determine the vertical displacement of pt C of the steel beam shown.

Take E_{st} = 200 GPa, I = 125(10^{-6}) m^4.

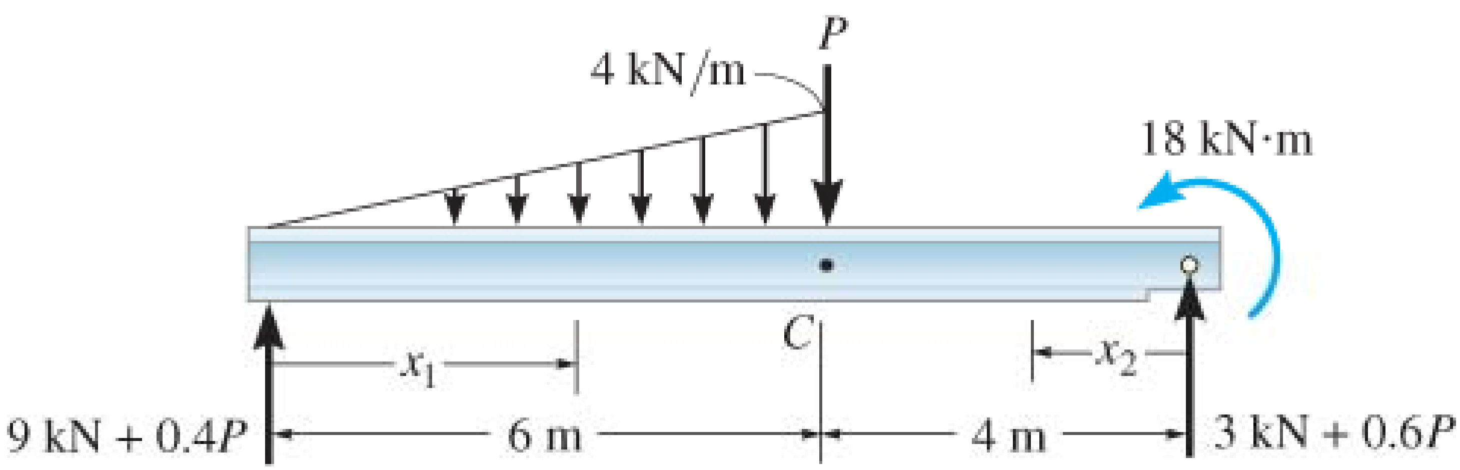


(a)

EXAMPLE 14.21 (SOLN)

External force P.

A vertical force P is applied at pt C. Later this force will be set equal to the fixed value of 5 kN.

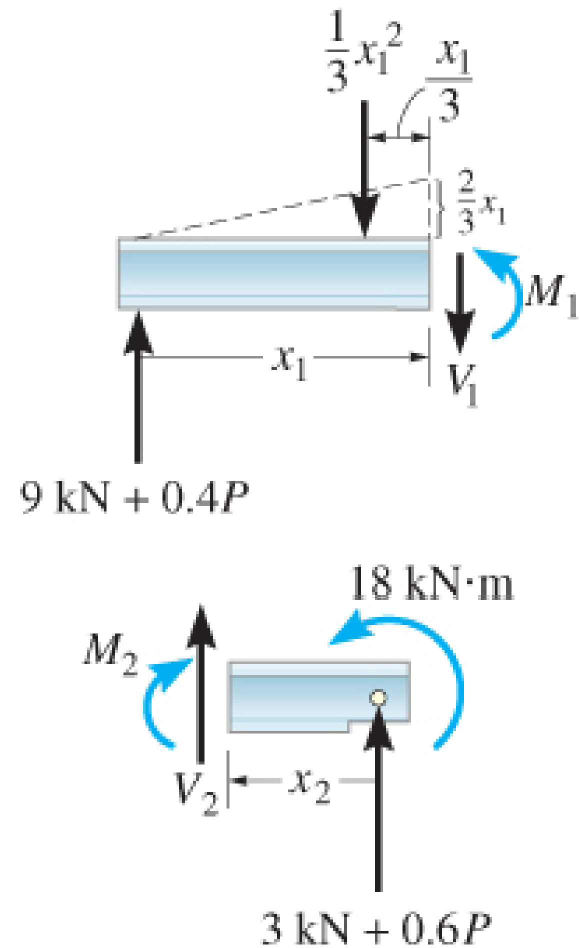


(b)

EXAMPLE 14.21 (SOLN)

Internal moments M.

Two x coordinates are needed for the integration since the load is discontinuous at C. Using method of sections, the internal moments and partial derivatives are determined as follows.



(c)

EXAMPLE 14.21 (SOLN)

Internal moments M.

For x_1,

$$+\sum M_{NA} = 0; \qquad M_1 + \frac{1}{3}x_1^2\left(\frac{x_1}{3}\right) - (9 + 0.4P)x_1 = 0$$

$$M_1 = (9 + 0.4P)x_1 - \frac{1}{9}x_1^3$$

$$\frac{\delta M_1}{\delta P} = 0.4x_1$$

EXAMPLE 14.21 (SOLN)

Internal moments M.

For x_2,

$$\circlearrowleft + \sum M_{NA} = 0; \qquad -M_2 + 18 + (3 + 0.6P)x_2 = 0$$

$$M_2 = 18 + (3 + 0.6P)x_2$$

$$\frac{\delta M_2}{\delta P} = 0.6x_2$$

EXAMPLE 14.21 (SOLN)

Catigliano's second theorem.

Setting P = 5 kN and applying Eqn 14-49, we have

$$\Delta_{C_v} = \int_0^L M\left(\frac{\delta M}{\delta P}\right)\frac{dx}{EI}$$

$$= \int_0^6 \frac{\left(11x_1 - \frac{1}{9}x_1^3\right)(0.4x_1)\,dx_1}{EI} + \int_0^4 \frac{(18 + 6x_2)(0.6x_2)\,dx_2}{EI}$$

$$= \frac{410.9\ \mathrm{kN\cdot m^3}}{\left[200(106)\ \mathrm{kN/m^2}\right]125\left(10^{-6}\right)\mathrm{m}^4}$$

$$= 0.0164\ \mathrm{m} = 16.4\ \mathrm{mm}$$

CHAPTER REVIEW

- When a force (or couple moment) acts on a deformable body it will do external work when it displaces (or rotates).
- The internal stresses produced in the body also undergo displacement, thereby creating elastic strain energy that is stored in the material.
- The conservation of energy states that the external work done by the loading is equal to the internal strain energy produced in the body.

CHAPTER REVIEW

- This principal can be used to solve problems involving elastic impact, which assumes the moving body is rigid and all strain energy is stored in the stationary body.
- The principal of virtual work can be used to determine the displacement of a joint on a truss or the slope and the displacement of pts on a beam or frame.
- It requires placing an entire virtual unit force (or virtual unit couple moment) at the pt where the displacement (or rotation) is to be determined.

CHAPTER REVIEW

- The external virtual work developed is then equated to the internal virtual strain energy in the member or structure.
- Castigliano's theorem can also be used to determine the displacement of a joint on a truss or slope or the displacement of a pt on a beam or truss.
- Here a variable force P (or couple moment M) is placed at the pt where the displacement (or slope) is to be determined.

CHAPTER REVIEW

- The internal loading is then determined as a function of P (or M) and its partial derivative w.r.t. P (or M) is determined.
- Castigliano's theorem is then applied to obtain the desired displacement (or rotation).