CHAPTER OBJECTIVES

- Apply energy methods to solve problems involving deflection
- Discuss work and strain energy, and development of the principle of conservation of energy
- Use principle of conservation of energy to determine stress and deflection of a member subjected to impact
- Develop the method of virtual work and Castigliano's theorem

CHAPTER OBJECTIVES

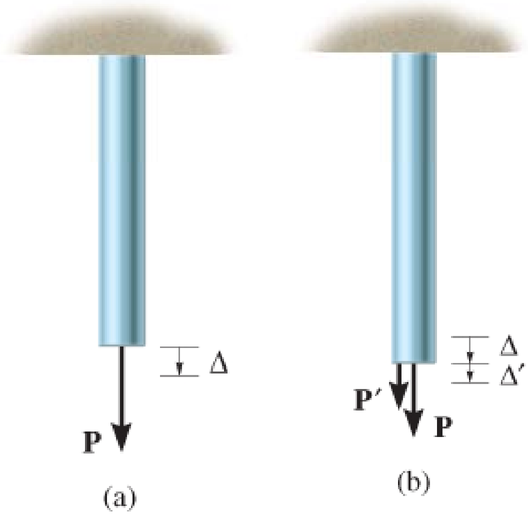
- Use method of virtual and Castigliano's theorem to determine displacement and slope at pts on structural members and mechanical elements

CHAPTER OUTLINE

1. External Work and Strain Energy
2. Elastic Strain Energy for Various Types of Loading
3. Conservation of Energy
4. Impact Loading
5. *Principle of Virtual Work
6. *Method of Virtual Forces Applied to Trusses
7. *Method of Virtual Forces Applied to Beams
8. *Castigliano’s Theorem
9. *Castigliano’s Theorem Applied to Trusses
10. *Castigliano’s Theorem Applied to Beams

14.1 EXTERNAL WORK AND STRAIN ENERGY

Work of a force:



- A force does work when it undergoes a displacement dx in same direction as the force.
- Work done is a scalar, defined as $dU_e = F\,dx$.
- If total displacement is x, work becomes

$$U_e = \int_0^x F\,dx \qquad (14\text{-}1)$$

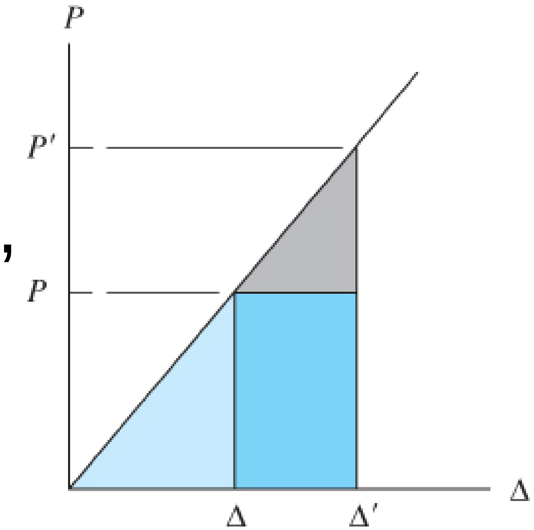
- As magnitude of **F** is gradually increased from zero to limiting value $F = P$, final displacement of end of bar becomes Δ.

14.1 EXTERNAL WORK AND STRAIN ENERGY

Work of a force:

- For linear-elastic behavior of material, $F = (P/\Delta)x$. Substitute into Eqn 14-1

$$U_e = \frac{1}{2}P\Delta \qquad (14\text{-}2)$$



Work done is the average force magnitude (P/2) times the total displacement Δ

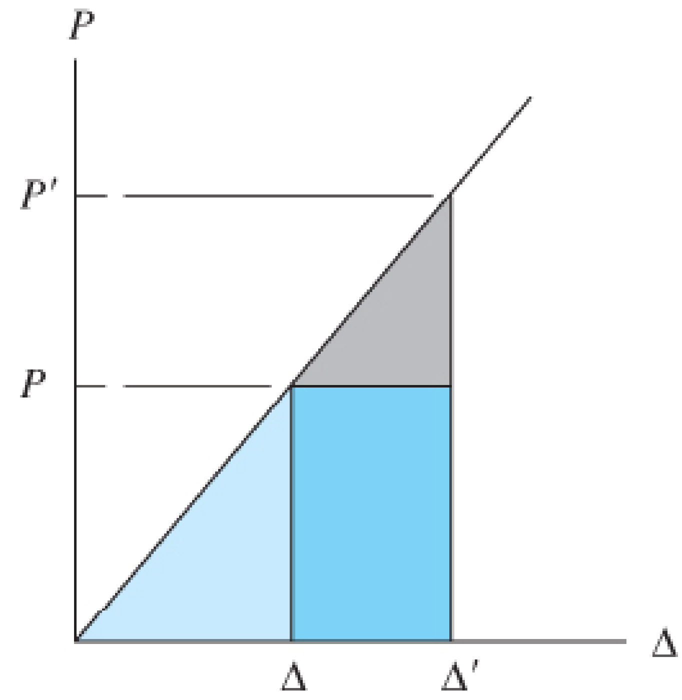
- Suppose that P is already applied to the bar and another force P' is now applied, so end of bar is further displaced by an amount Δ'.
- Work done by P (not P') is then

$$U'_e = P\Delta' \qquad (14\text{-}3)$$

14.1 EXTERNAL WORK AND STRAIN ENERGY

Work of a force:

- When a force P is applied to the bar, followed by the force P', total work done by both forces is represented by the area of the entire triangle in graph shown.



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14.1 EXTERNAL WORK AND STRAIN ENERGY

Work of a couple moment:

- A couple moment M does work when it undergoes a rotational displacement $d\theta$ along its line of action.
- Work done is defined as $dU_e = M d\theta$. If total angle of rotational displacement is θ radians, then work

$$U_e = \int_0^{\theta} M \, d\theta \qquad (14\text{-}4)$$

- If the body has linear-elastic behavior, and its magnitude increases gradually from zero at $\theta = 0$ to M at θ, then work is

$$U_e = \frac{1}{2} M\theta \qquad (14\text{-}5)$$

14.1 EXTERNAL WORK AND STRAIN ENERGY

Work of a couple moment:

- However, if couple moment already applied to the body and other loadings further rotate the body by an amount θ', then work done is

$$U'_e = M\theta'$$

14.1 EXTERNAL WORK AND STRAIN ENERGY

Strain energy:

- When loads are applied to a body and causes deformation, the external work done by the loads will be converted into internal work called strain energy. This is provided no energy is converted into other forms.

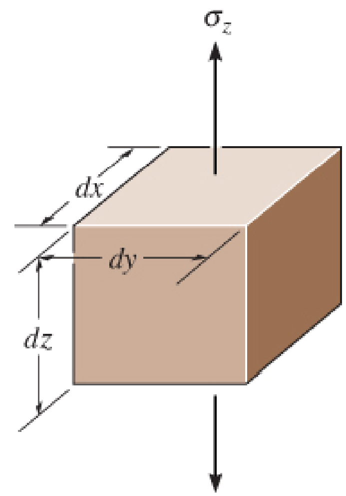
Normal stress

- A volume element subjected to normal stress σ_z.
- Force created on top and bottom faces is $dF_z = \sigma_z \, dA = \sigma_z \, dx \, dy$.

14.1 EXTERNAL WORK AND STRAIN ENERGY

Strain energy:

Normal stress



- This force is increased gradually from zero to dF_z while element undergoes displacement $d\Delta_z = \varepsilon_z\, dz$.
- Work done is $dU_i = 0.5 dF_z\, d\Delta_z = 0.5[\sigma_z\, dx\, dy]\varepsilon_z\, dz$.
- Since volume of element is $dV = dx\, dy\, dz$, we have

$$dU_i = \frac{1}{2}\sigma_z \varepsilon_z\, dV \qquad (14\text{-}6)$$

- Note that dU_i is always positive.

14.1 EXTERNAL WORK AND STRAIN ENERGY

Strain energy:

Normal stress

- In general, for a body subjected to a uniaxial normal stress σ, acting in a specified direction, strain energy in the body is then

$$U_i = \int_V \frac{\sigma \varepsilon}{2} \, dV \qquad (14\text{-}7)$$

- If material behaves linear-elastically, then Hooke's law applies and we express it as

$$\boxed{U_i = \int_V \frac{\sigma^2}{2E} \, dV} \qquad (14\text{-}8)$$

14.1 EXTERNAL WORK AND STRAIN ENERGY

Strain energy:

Shear stress

- Shear stress cause element to deform such that shear force $dF = \tau(dx\ dy)$ acts on top face of element.
- Resultant displacement if $\gamma\, dz$ relative to bottom face.
- Vertical faces only rotate, thus shear forces on these faces do no work.

14.1 EXTERNAL WORK AND STRAIN ENERGY

Strain energy:

Shear stress

- Hence, strain energy stored in the element is

$$dU_i = \frac{1}{2}\tau\gamma\, dV \qquad (14\text{-}9)$$

- Integrating over body's entire volume to obtain strain energy stored in it

$$U_i = \int_V \frac{\tau\gamma}{2}\, dV \qquad (14\text{-}10)$$

- Shear strain energy is always positive.

14.1 EXTERNAL WORK AND STRAIN ENERGY

Strain energy:

Shear stress

- Apply Hooke's law $\gamma = \tau/G$,

$$U_i = \int_V \frac{\tau^2}{2G}\, dV \qquad (14\text{-}11)$$

14.1 EXTERNAL WORK AND STRAIN ENERGY

Strain energy:

Mutilaxial stress

- Total strain energy in the body is therefore

$$U_i = \int_V \left[\begin{array}{l} \frac{1}{2}\sigma_x \varepsilon_x + \frac{1}{2}\sigma_y \varepsilon_y + \frac{1}{2}\sigma_z \varepsilon_z \\ + \frac{1}{2}\tau_{xy}\gamma_{xy} + \frac{1}{2}\tau_{yz}\gamma_{yz} + \frac{1}{2}\tau_{xz}\gamma_{xz} \end{array} \right] dV \qquad (14\text{-}12)$$

14.1 EXTERNAL WORK AND STRAIN ENERGY

Strain energy:

Mutilaxial stress

- Eliminate the strains using generalized form of Hooke's law given by Eqns 10-18 and 10-19. After substituting and combining terms, we have

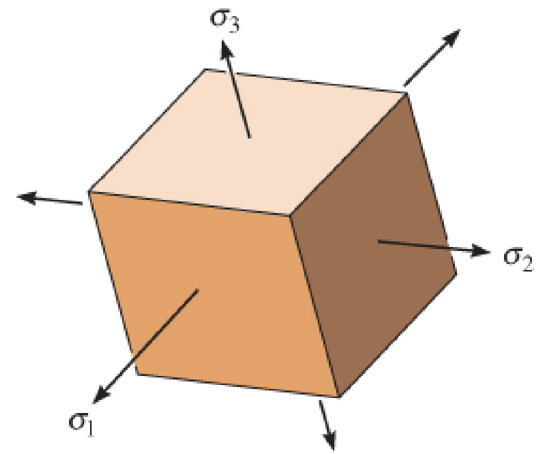
$$U_i = \int_V \begin{bmatrix} \frac{1}{2}\left(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2\right) \\ -\frac{\upsilon}{E}\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z\right) \\ +\frac{1}{2G}\left(\tau_{xy}{}^2 + \tau_{yz}{}^2 + \tau_{xz}{}^2\right) \end{bmatrix} dV \quad (14\text{-}13)$$

14.1 EXTERNAL WORK AND STRAIN ENERGY

Strain energy:

Mutilaxial stress

- If only principal stresses σ_1, σ_2, σ_3 act on the element, this eqn reduces to a simpler form,

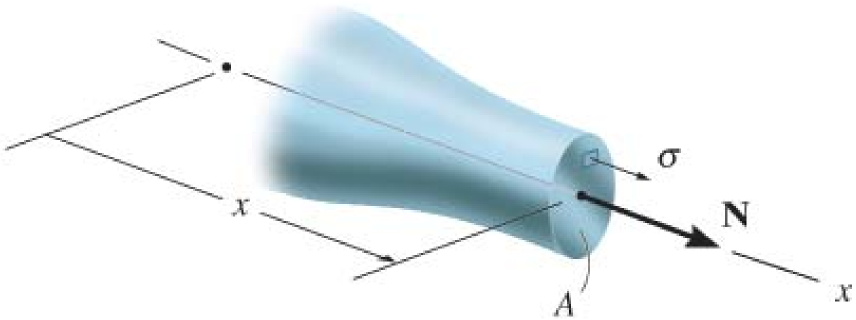


(b)

$$U_i = \int_V \left[\frac{1}{2E}\left(\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2\right) - \frac{\upsilon}{E}\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3\right) \right] dV \qquad (14\text{-}14)$$

14.2 ELASTIC STRAIN ENERGY FOR VARIOUS TYPES OF LOADING

Axial load:



- Consider a bar of variable and slightly tapered x-section, subjected to axial load coincident with bar's centroidal axis.
- Internal axial force at section located from one end is N.
- If x-sectional area at this section is A, then normal stress $\sigma = N/A$.
- Apply Eqn 14-8, we have

$$U_i = \int_V \frac{\sigma_x^2}{2E}\, dV = \int_V \frac{N^2}{2EA^2}\, dV$$

14.2 ELASTIC STRAIN ENERGY FOR VARIOUS TYPES OF LOADING

Axial load:

- Choose element or differential slice having volume $dV = A dx$, general formula for strain energy in bar is

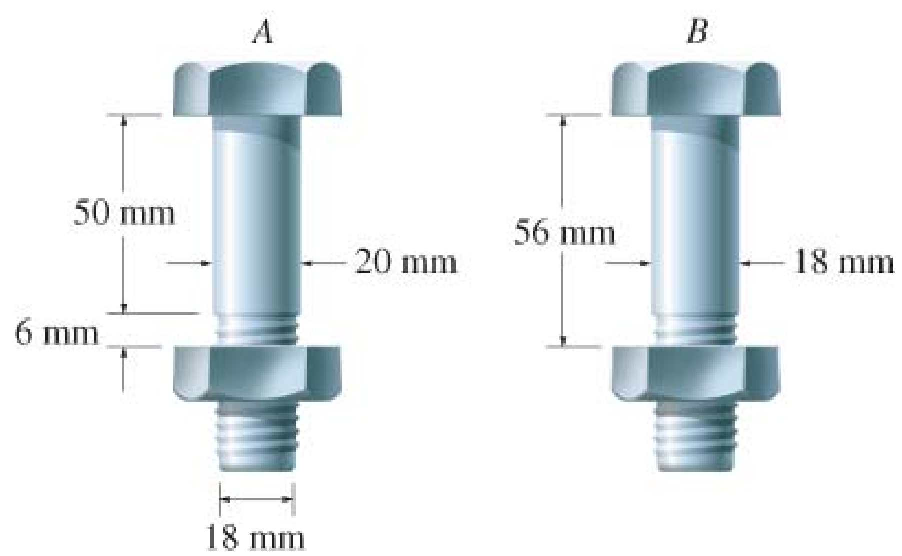
$$U_i = \int_0^L \frac{N^2}{2AE}\,dx \qquad (14\text{-}15)$$

- For a prismatic bar of constant x-sectional area A, length L and constant axial load N, integrating Eqn 14-15 gives

$$U_i = \frac{N^2 L}{2AE} \qquad (14\text{-}16)$$

EXAMPLE 14.1

Choose one of the 2 high-strength steel bolts to support a tensile loading. Determine the greatest amount of elastic strain energy that each bolt can absorb. Bolt A has a diameter of 20 mm for 50 mm of its length and root diameter of 18 mm within 6 mm threaded region. Bolt B has the same diameter throughout its 56 mm length and can be taken as 18 mm. For both cases, neglect extra material that makes up the thread.



Take E_{st} = 210(10^3) MPa, σ_Y = 310 MPa.

EXAMPLE 14.1 (SOLN)

Bolt A:

For bolt subjected to maximum tension, σ_Y will occur within the 6-mm region. This tension is

$$P_{\max} = \sigma_Y A = 310\ \text{N/mm}^2 \left[\pi\left(18\ \text{mm}/2\right)^2\right]$$

$$= 78886\ \text{N} = 78.89\ \text{kN}$$

EXAMPLE 14.1 (SOLN)

Bolt A:

Apply Eqn 14-16 to each region of the bolt,

$$U_i = \sum \frac{N^2 L}{2AE}$$

$$= \frac{\left(78.89 \times 10^3 \text{ N}\right)^2 (50 \text{ mm})}{2\left[\pi (20 \text{ mm}/2)^2\right]\left[210\left(10^3\right) \text{N/mm}^2\right]}$$

$$+ \frac{\left(78.89 \times 10^3 \text{ N}\right)^2 (6 \text{ mm})}{2\left[\pi (18 \text{ mm}/2)^2\right]\left[210\left(10^3\right) \text{N/mm}^2\right]}$$

$$= 2707.8 \text{ N} \cdot \text{mm} = 2.708 \text{ N} \cdot \text{m} = 2.708 \text{ J}$$

EXAMPLE 14.1 (SOLN)

Bolt B:

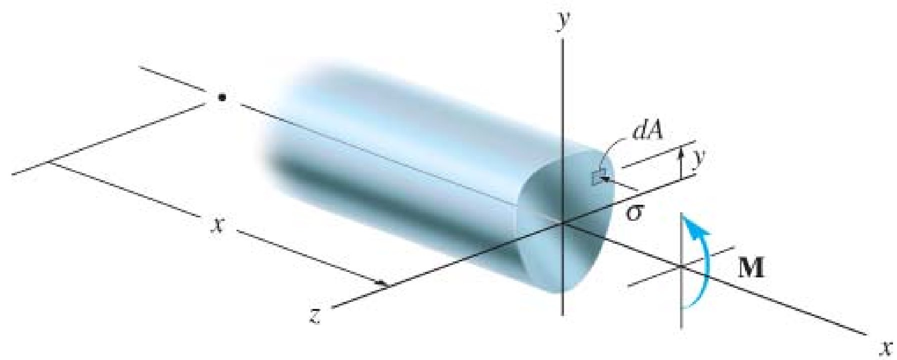
From calculation above, it can also support a maximum tension force of P_{max} = 78.89 kN. Thus,

$$U_i = \frac{N^2 L}{2AE} = \frac{\left(78.89 \times 10^3\ \text{N}\right)^2 (56\ \text{mm})}{2\left[\pi(18\ \text{mm}/2)^2\right]\left[210\left(10^3\right)\text{N/mm}^2\right]}$$
$$= 3261.0\ \text{N}\cdot\text{mm} = 3.26\ \text{N}\cdot\text{m} = 3.26\ \text{J}$$

By comparison, bolt B can absorb 20% more elastic energy than bolt A, even though it has a smaller x-section along its shank.

14.2 ELASTIC STRAIN ENERGY FOR VARIOUS TYPES OF LOADING

Bending moment:



- For the axisymmetric beam shown.
- Internal moment M, normal stress acting on element a distance y from neutral axis is $\sigma = My/I$.
- If volume of element is $dV = dA\,dx$, where dA is area of exposed face and dx its length, elastic strain energy in beam is

$$U_i = \int_0^L \frac{M^2}{2EI^2}\left(\int_A y^2\, dA\right) dx$$

14.2 ELASTIC STRAIN ENERGY FOR VARIOUS TYPES OF LOADING

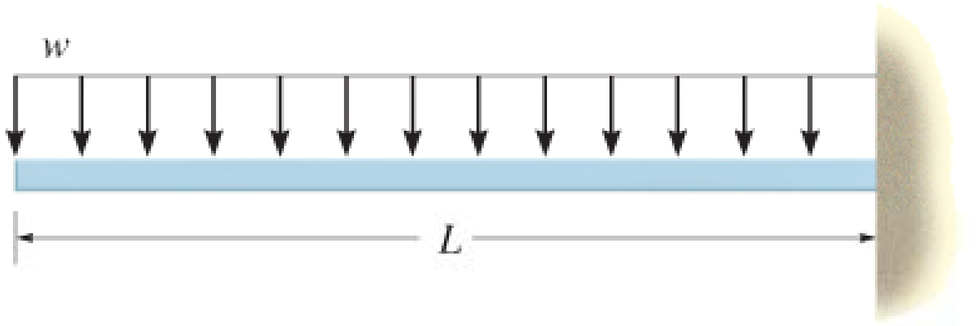
Bending moment:

- Realize that area integral represents the moment of inertia of beam about neutral axis, thus

$$U_i = \int_0^L \frac{M^2 dx}{2EI} \qquad (14\text{-}7)$$

EXAMPLE 14.2

Determine the elastic strain energy due to bending of the cantilevered beam if beam is subjected to uniform distributed load w. EI is constant.



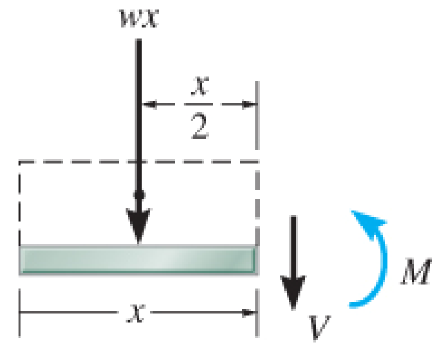
(a)

EXAMPLE 14.2 (SOLN)

Establish the x coordinate with origin at the left side. Thus, internal moment is

$$+\sum M_{NA} = 0; \qquad M + wx\left(\frac{x}{2}\right) = 0$$

$$M = -w\left(\frac{x^2}{2}\right)$$



(b)

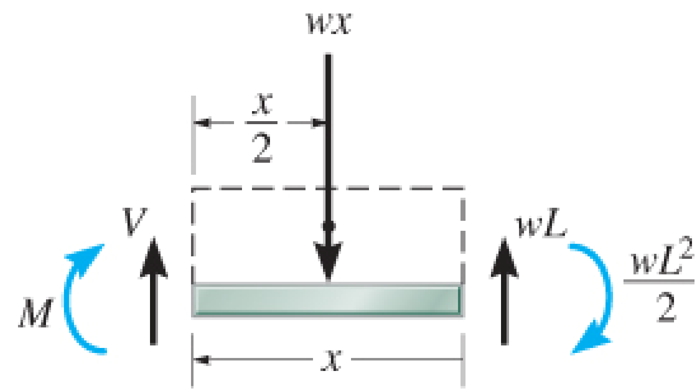
Applying Eqn 14-17 yields

$$U_i = \int_0^L \frac{M^2 dx}{2EI} = \int_0^L \frac{\left[-w\left(x^2/2\right)\right]^2 dx}{2EI} = \frac{w^2}{8EI}\int_0^L x^4\, dx$$

$$U_i = \frac{w^2 L^5}{40EI}$$

EXAMPLE 14.2 (SOLN)

For x coordinate with origin on the the right side and extending +ve to the left. Thus, in this case



(c)

$$\circlearrowleft + \sum M_{NA} = 0;$$

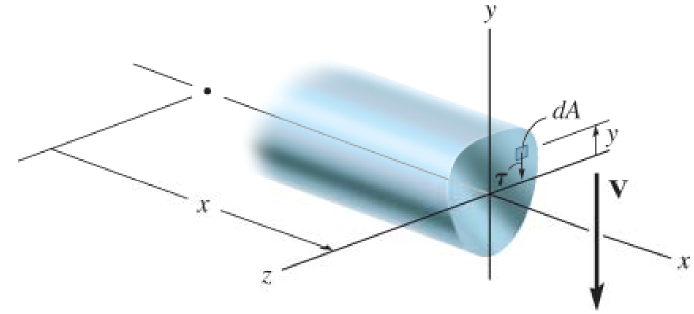
$$-M - wx\left(\frac{x}{2}\right) + wL(x) - \frac{wL^2}{2} = 0$$

$$M = -\frac{wL^2}{2} + wLx - w\left(\frac{x^2}{2}\right)$$

Applying Eqn 14-17, we obtain the same result.

14.2 ELASTIC STRAIN ENERGY FOR VARIOUS TYPES OF LOADING

Transverse shear:



- Consider prismatic beam with axis of symmetry about the y axis.
- Internal shear V at section x results in shear stress acting on the volume element, having length dx and area dA, is $\tau = VQ/It$.
- Substitute into Eqn 14-11,

$$U_i = \int_0^L \frac{V^2}{2GI^2}\left(\int_A \frac{Q^2}{t^2}\, dA\right) dx$$

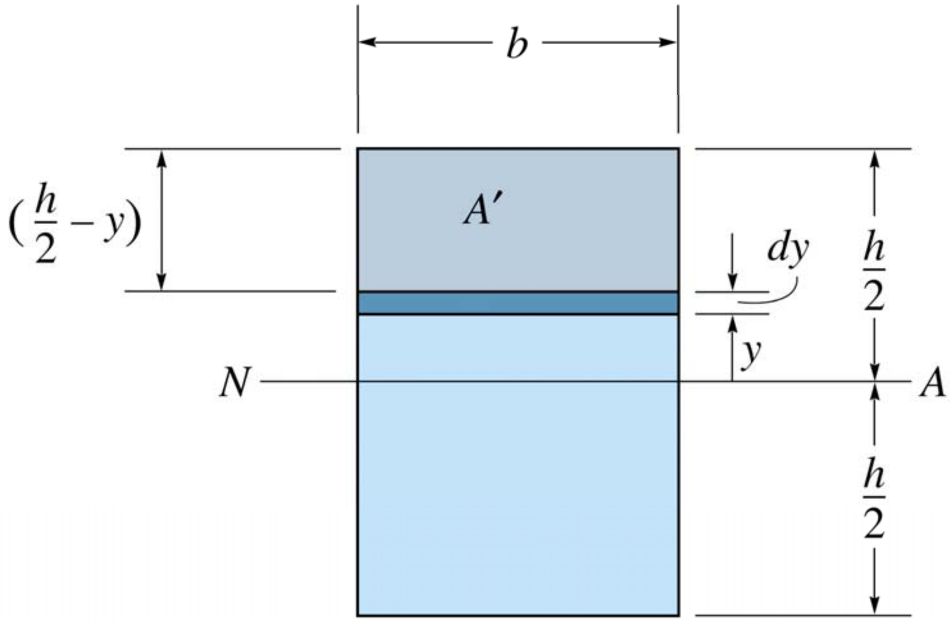
Transverse shear:

- Realize that integral in parentheses is evaluated over beam's x-sectional area.
- To simplify, we define the form factor for shear as

$$f_s = \frac{A}{I^2}\int_A \frac{Q^2}{t^2}\,dA \qquad (14\text{-}18)$$

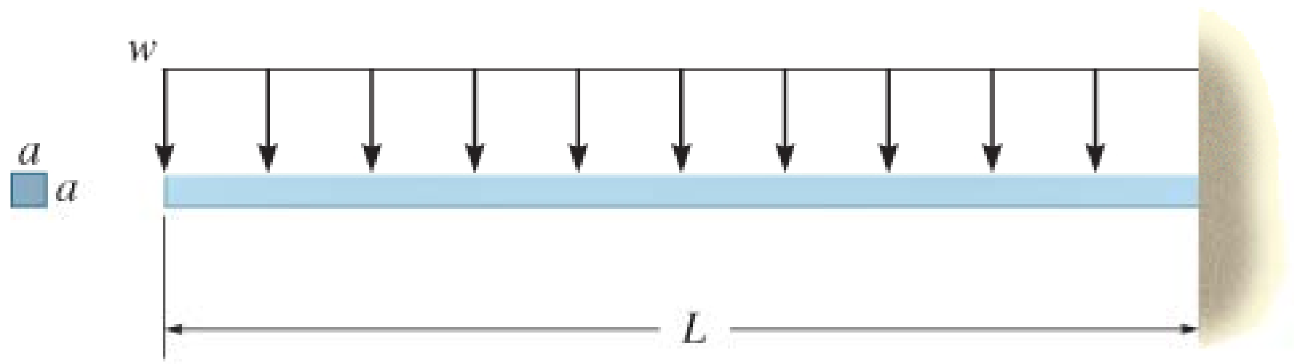
- Form factor is dimensionless and unique for each specific x-sectional area.
- Substitute Eqn 14-18 into above eqn,

$$\boxed{U_i = \int_0^L \frac{f_s V^2 dx}{2GA}} \qquad (14\text{-}19)$$


b
$(\frac{h}{2} - y)$
A′
dy
$\frac{h}{2}$
y
N
A
$\frac{h}{2}$

EXAMPLE 14.4

Determine the strain energy in cantilevered beam due to shear if beam has a square x-section and is subjected to a uniform distributed load w. EI and G is constant.



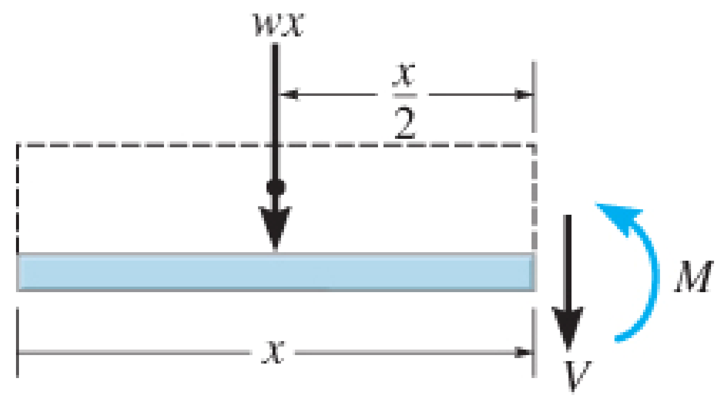
(a)

EXAMPLE 14.4 (SOLN)

From free-body diagram of arbitrary section, we have

$$+\uparrow \sum F_y = 0; \qquad -V - wx = 0$$

$$V = -wx$$



(b)

Since x-section is square,
form factor f_s = 6/5 and therefore Eqn 14-19 becomes

$$(U_i)_s = \int_0^L \frac{6/5(-wx)^2 dx}{2GA} = \frac{3w^2}{5GA}\int_0^L x^2\ dx$$

$$(U_i)_s = \frac{w^2 L^3}{5GA}$$

EXAMPLE 14.4 (SOLN)

Using results of Example 14.2, with $A = a^2$, $I = 1/12a^4$, ratio of shear to bending strain energy is

$$\frac{(U_i)_s}{(U_i)_b} = \frac{w^3L^3/5Ga^2}{w^2L^5/40E(1/12)a^4} = \frac{2}{3}\left(\frac{a}{L}\right)^2\frac{E}{G}$$

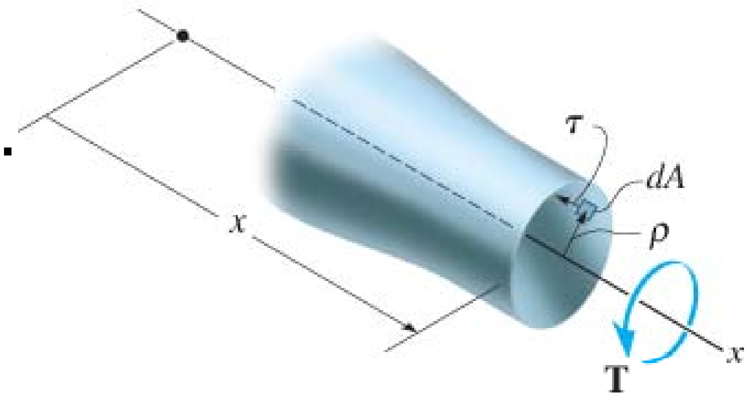
Since $G = E/2(1 + \nu)$ and $\nu \leq 0.5$ (sec 10.6), then as an upper bound, $E = 3G$, so that

$$\frac{(U_i)_s}{(U_i)_b} = 2\left(\frac{a}{L}\right)^2$$

For $L = 5a$, contributions due to shear strain energy is only 8% of bending strain energy. Thus, shear strain energy is usually neglected in engineering analysis.

14.2 ELASTIC STRAIN ENERGY FOR VARIOUS TYPES OF LOADING

Torsional moment:



- Consider slightly tapered shaft.
- Section of shaft taken distance x from one end subjected to internal torque T.
- On arbitrary element of length dx and area dA, stress is $\tau = T\rho/J$.
- Strain energy stored in shaft is

$$U_i = \int_0^L \frac{T^2}{2GJ^2}\left(\int_A \rho^2 \, dA\right) dx$$

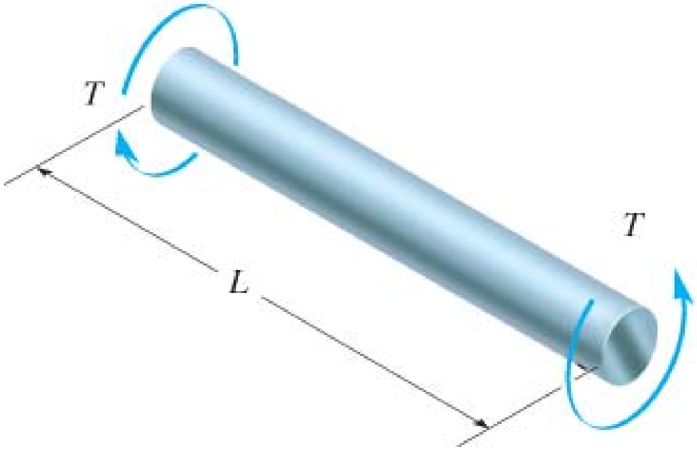
14.2 ELASTIC STRAIN ENERGY FOR VARIOUS TYPES OF LOADING

Torsional moment:

- Since area integral represents the polar moment of inertia J for shaft at section,

$$U_i = \int_0^L \frac{T^2 dx}{2GJ} \qquad (14\text{-}21)$$

- Most common case occurs when shaft has constant x-sectional area and applied torque is constant, integrating
Eqn 14-21 gives



$$U_i = \frac{T^2 L}{2GJ} \qquad (14\text{-}22)$$

Torsional moment:

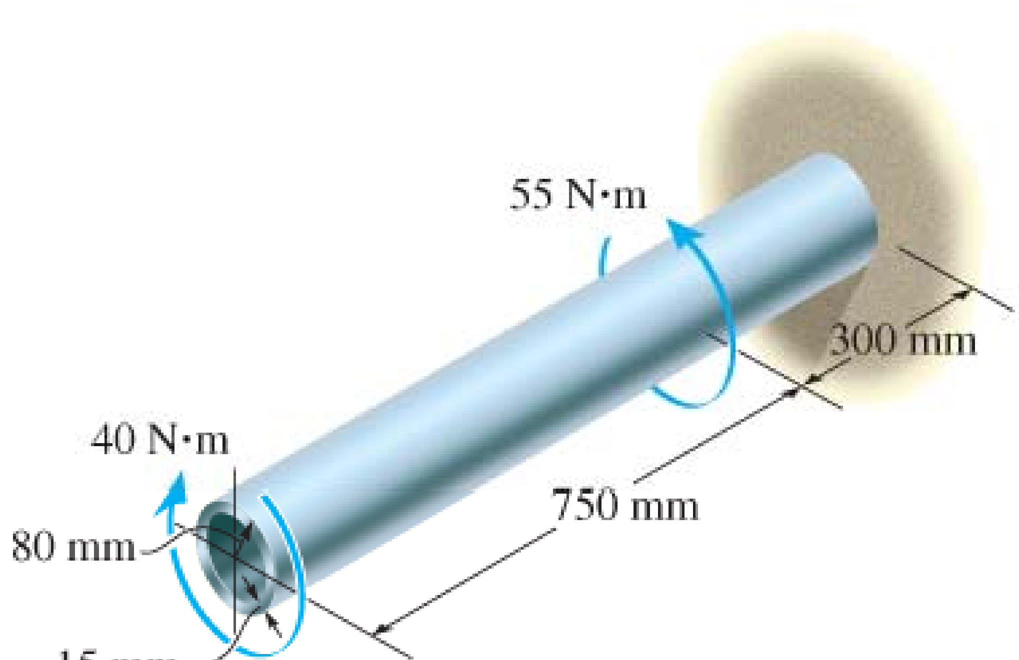
- If x-section is of other shapes than circular or tubular, Eqn 14-22 is modified.
- For example, for a rectangular shaft with dimensions $h > b$,

$$U_i = \frac{T^2 L}{2Cb^3 hG} \qquad (14\text{-}23)$$

$$C = \frac{hb^3}{16}\left[\frac{16}{3} - 3.336\frac{b}{h}\left(1 - \frac{b^4}{12h^4}\right)\right] \qquad (14\text{-}24)$$

EXAMPLE 14.5

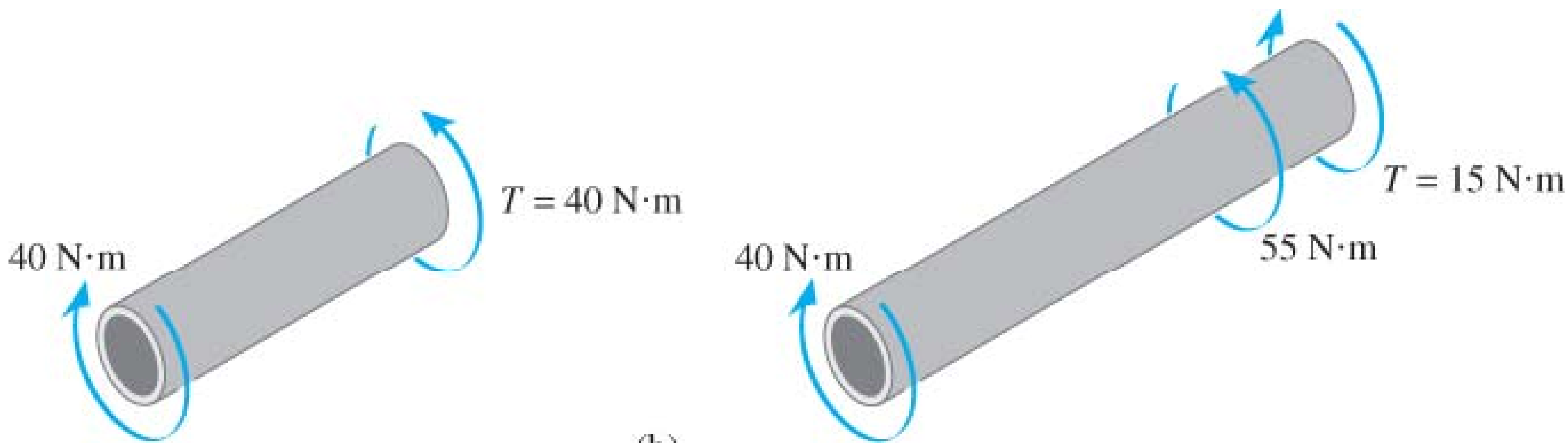
Tubular shaft fixed at the wall and subjected to two torques as shown. Determine the strain energy stored in shaft due to this loading. G = 75 GPa.



(a)

EXAMPLE 14.5 (SOLN)

Using method of sections, internal torque first determined within the two regions of shaft where it is constant. Although torques are in opposite directions, this will not affect the value of strain energy, since torque is squared in Eqn 14-22.



(b)

EXAMPLE 14.5 (SOLN)

Polar moment of inertia for shaft is

$$J = \frac{\pi}{2}\left[(0.08\ \text{m})^4 - (0.065\ \text{m})^4\right] = 36.30\left(10^{-6}\right)\text{m}^4$$

Applying Eqn 14-22, we have

$$U_i = \sum \frac{T^2 L}{2GJ} = \frac{(40\ \text{N}\cdot\text{m})^2(0.750\ \text{m})}{2\left[75\left(10^9\right)\text{N/m}^2\right]36.30\left(10^{-6}\right)\text{m}^4}$$

$$+\frac{(15\ \text{N}\cdot\text{m})^2(0.300\ \text{m})}{2\left[75\left(10^9\right)\text{N/m}^2\right]36.30\left(10^{-6}\right)\text{m}^4}$$

$$= 233\ \mu\text{J}$$

14.2 ELASTIC STRAIN ENERGY FOR VARIOUS TYPES OF LOADING

IMPORTANT

- A force does work when it moves through a displacement.
- If force is increased gradually in magnitude from zero to F, the work is $U = (F/2)\Delta$, whereas if force is constant when the displacement occurs then $U = F\Delta$.
- A couple moment does work when it moves through a rotation.
- Strain energy is caused by the internal work of the normal and shear stresses. It is always a positive quantity.

14.2 ELASTIC STRAIN ENERGY FOR VARIOUS TYPES OF LOADING

IMPORTANT

- The strain energy can be related to the resultant internal loadings N, V, M, and T.
- As the beam becomes longer, the strain energy due to bending becomes much larger than strain energy due to shear.
- For this reason, shear strain energy in beams can generally be neglected.

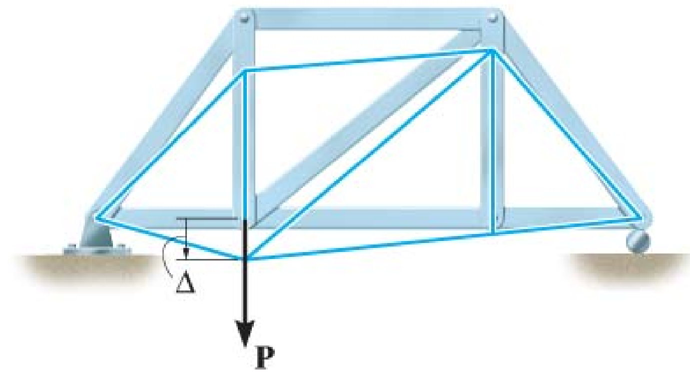
14.3 CONSERVATION OF ENERGY

- A loading is applied slowly to a body, so that kinetic energy can be neglected.
- Physically, the external loads tend to deform the body as they do external work U_e as they are displaced.
- This external work is transformed into internal work or strain energy U_i, which is stored in the body.
- Thus, assuming material's elastic limit not exceeded, conservation of energy for body is stated as

$$\boxed{U_e = U_i} \qquad (14\text{-}25)$$

14.3 CONSERVATION OF ENERGY

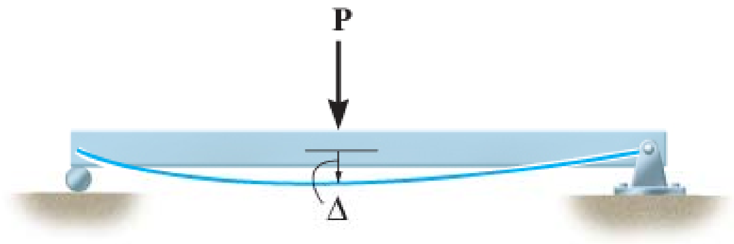
- Consider a truss subjected to load P. P applied gradually thus $U_e = 0.5P\Delta$, where Δ is vertical displacement of truss at pt where P is applied.



- Assume that P develops an axial force N in a particular member, and strain energy stored is $U_i = N^2L/2AE$.
- Summing strain energies for all members of the truss, we write Eqn 14-25 as

$$\frac{1}{2}P\Delta = \sum \frac{N^2 L}{2AE} \qquad (14\text{-}26)$$

14.3 CONSERVATION OF ENERGY

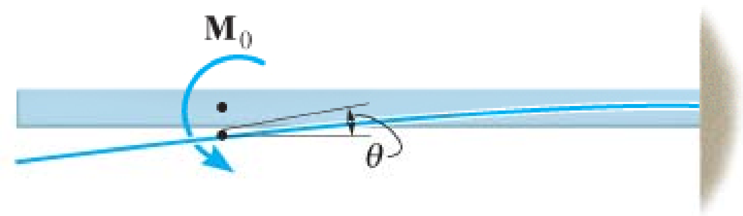


- Consider a beam subjected to load P. External work is $U_e = 0.5P\Delta$.
- Strain energy in beam can be neglected.
- Beam's strain energy determined only by the moment M, thus with Eqn 14-17, Eqn 14-25 written as

$$\frac{1}{2}P\Delta = \int_0^L \frac{M^2}{2EI}dx \qquad (14-27)$$

14.3 CONSERVATION OF ENERGY

- Consider a beam loaded by a couple moment M_0. A rotational displacement θ is caused. Using Eqn 14-5, external work done is $U_e = 0.5M_0\theta$.



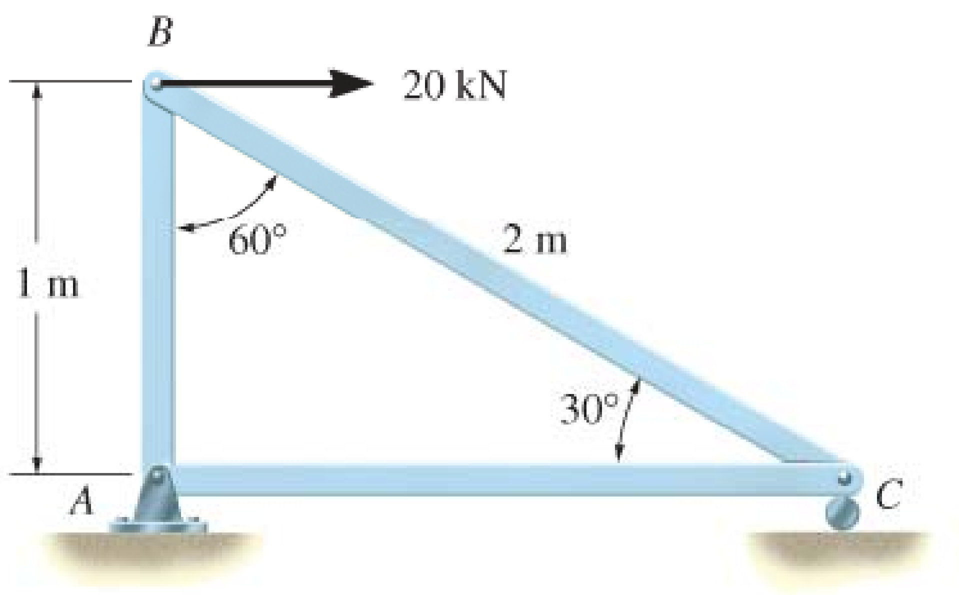
- Thus Eqn 14-25 becomes

$$\frac{1}{2}M_0\theta = \int_0^L \frac{M^2}{2EI}dx \qquad (14\text{-}28)$$

- Note that Eqn 14-25 is only applicable for a single external force or external couple moment acting on structure or member.

EXAMPLE 14.6

The three-bar truss is subjected to a horizontal force of 20 kN. If x-sectional area of each member is 100 mm^2, determine the horizontal displacement at pt B. E = 200 GPa.



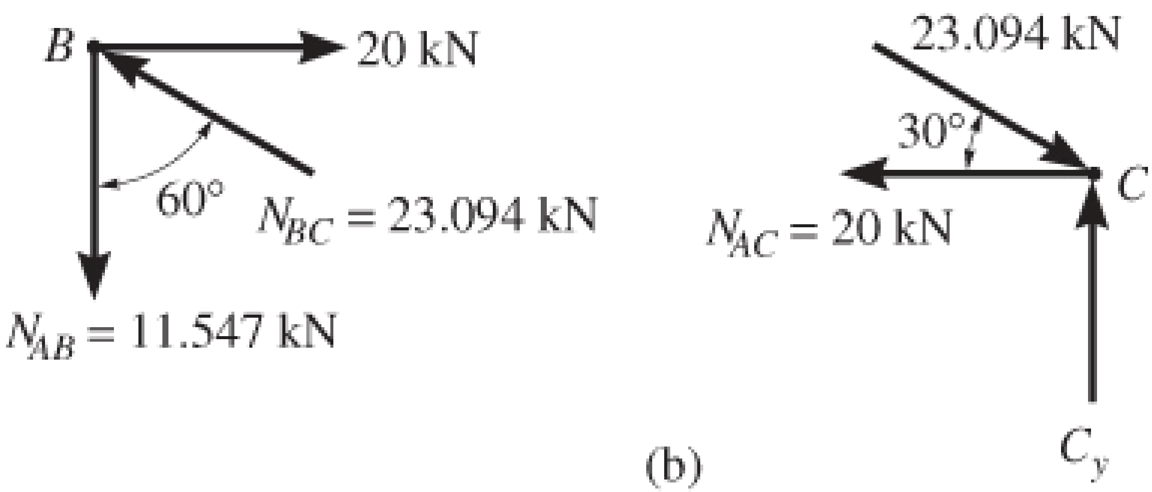
(a)

EXAMPLE 14.6 (SOLN)

Since only a single external force acts on the truss and required displacement is in same direction as the force, we use conservation of energy.

Also, the reactive force on truss do no work since they are not displaced.

Using method of joints, force in each member is determined as shown on free-body diagrams of pins at B and C.



(b)

EXAMPLE 14.6 (SOLN)

Applying Eqn 14-26,

$$\frac{1}{2}P\Delta = \sum \frac{N^2 L}{2AE}$$

$$\frac{1}{2}\left(20\times10^3\ \text{N}\right)\left(\Delta_B\right)_h = \frac{\left(11.547\times10^3\ \text{N}\right)^2(1\ \text{m})}{2AE}$$

$$+\frac{\left(-23.094\times10^3\ \text{N}\right)^2(2\ \text{m})}{2AE}+\frac{\left[-20\left(10^3\right)\text{N}\right]^2(1.732\ \text{m})}{2AE}$$

$$\left(\Delta_B\right)_h = \frac{94640.0\ \text{N}\cdot\text{m}}{AE}$$

EXAMPLE 14.6 (SOLN)

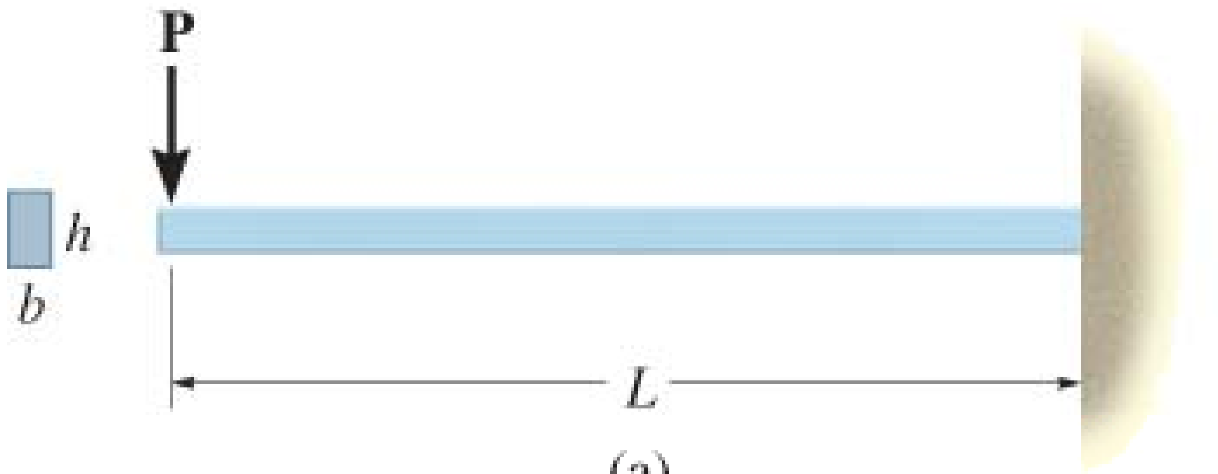
Substituting in numerical data for A and E and solving, we get

$$(\Delta_B)_h = \frac{94640.0 \text{ N}\cdot\text{m}}{\left(100 \text{ mm}^2\right)\left(1 \text{ m}/1000 \text{ mm}\right)^2 200\left(10^9\right) \text{N/mm}^2}$$

$$= 4.73\times10^{-3} \text{ m} = 4.73 \text{ mm} \rightarrow$$

EXAMPLE 14.7

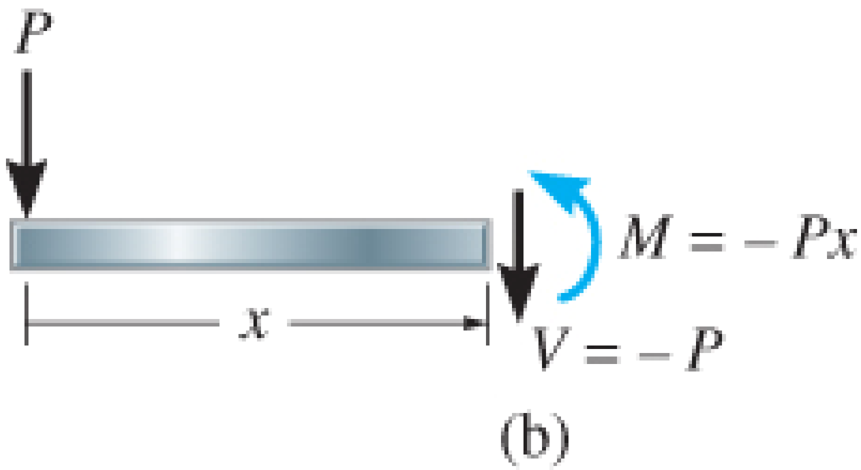
Cantilevered beam has a rectangular x-section and subjected to a load P at its end. Determine the displacement of the load. EI is a constant.



(a)

EXAMPLE 14.7 (SOLN)

Internal moment and moment in beam as a function of x are determined using the method of sections.



When applying Eqn 14-25 we will consider the strain energy due to shear and bending.

EXAMPLE 14.7 (SOLN)

Using Eqns 14-19 and 14-17, we have

$$\frac{1}{2}P\Delta = \int_0^L \frac{f_s V^2\, dx}{2GA} + \int_0^L \frac{M^2\, dx}{2EI}$$

$$= \int_0^L \frac{(6/5)(-P)^2\, dx}{2GA} + \int_0^L \frac{(-Px)^2\, dx}{2EI}$$

$$= \frac{3P^2L}{5GA} + \frac{P^2L^3}{6EI} \qquad (1)$$

First term on the right side represents strain energy due to shear, while the second is due to bending. As stated in Example 14.4, the shear strain energy in most beams is much smaller than the bending strain energy.

EXAMPLE 14.7 (SOLN)

To show this is the case, we require

$$\frac{3}{5}\frac{P^2L}{GA} << \frac{P^2L^3}{6EI}$$

$$\frac{3}{5}\frac{P^2L}{G(bh)} << \frac{P^2L^3}{6E\left[\frac{1}{12}\left(bh^3\right)\right]}$$

$$\frac{3}{5G} << \frac{2L^2}{Eh^2}$$

Since $E \leq 3G$ (see Example 14.4) then $0.9 << \left(\frac{L}{h}\right)^2$

EXAMPLE 14.7 (SOLN)

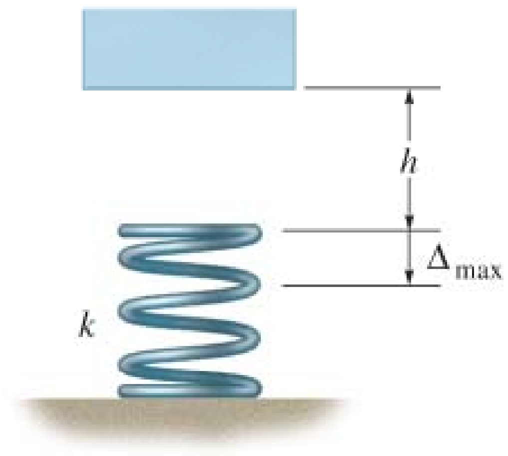
Hence, if h is small and L relatively long, beam becomes slender and shear strain energy can be neglected. Shear strain energy is only important for short, deep beams. Beams for which $L = 5h$ have more than 28 times more bending energy than shear strain energy, so neglecting only incurs an error of about 3.6%. Eqn (1) can be simplified to

$$\frac{1}{2}P\Delta = \frac{P^2L^3}{6EI}$$

$$\Delta = \frac{PL^3}{3EI}$$

14.4 IMPACT LOADING

- An impact occurs when one object strikes another, such that large forces are developed between the objects during a very short period of time.



$$U_e = U_i$$

$$W\left(h + \Delta_{max}\right) = \frac{1}{2}\left(k\Delta_{max}\right)\Delta_{max}$$

$$W\left(h + \Delta_{max}\right) = \frac{1}{2}k\Delta_{max}^2 \qquad (14\text{-}29)$$

$$\Delta_{max}^2 - \frac{2W}{k}\Delta_{max} - 2\left(\frac{W}{k}\right)h = 0$$

14.4 IMPACT LOADING

- Solving and simplifying ($\Delta_{st} = W/k$),

$$\Delta_{\max} = \Delta_{st} + \sqrt{(\Delta_{st})^2 + 2\Delta_{st}h}$$

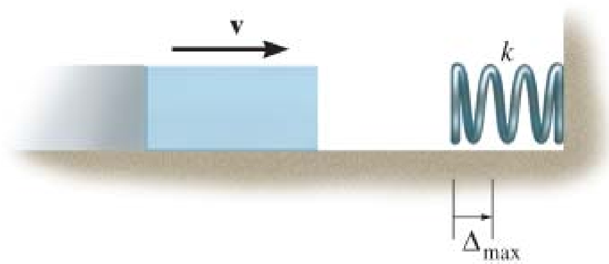
$$\Delta_{\max} = \Delta_{st}\left[1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)}\right] \qquad (14\text{-}30)$$

- Once $\Delta_{\max}$ is computed, maximum force applied to the spring is

$$F_{\max} = k\Delta_{\max} \qquad (14\text{-}31)$$

14.4 IMPACT LOADING

- For a case where the block is sliding on a smooth horizontal surface with known velocity v just before it collides with the spring.



- The block's kinetic energy, $0.5(W/g)v^2$ is transformed into stored energy in the spring.

$$U_e = U_i$$

$$\frac{1}{2}\left(\frac{W}{g}\right)\upsilon^2 = \frac{1}{2}k\Delta_{\max}^2$$

$$\Delta_{\max} = \sqrt{\frac{W\upsilon^2}{gk}} \qquad (14\text{-}32)$$

14.4 IMPACT LOADING

- Ratio of equivalent static load $P_{\max}$ to the load W is called the impact factor, n. Since $P_{\max} = k\Delta_{\max}$ and $W = k\Delta_{st}$, then from Eqn. 14-30, we express it as

$$n = 1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)} \qquad (14\text{-}34)$$

- This factor represents the magnification of a statically applied load so that it can be treated dynamically.
- Using Eqn 13-34, n can be computed for any member that has a linear relationship between load and deflection.

14.4 IMPACT LOADING

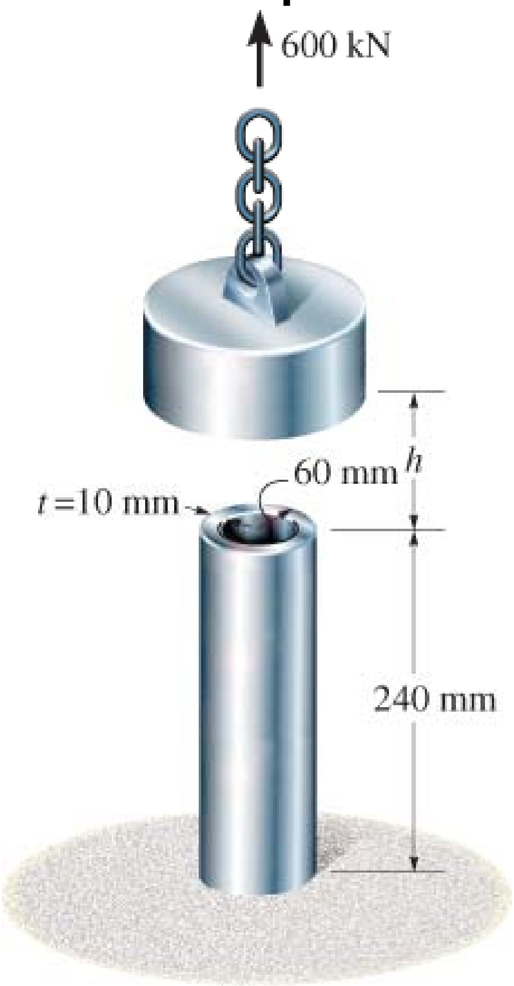
IMPORTANT

- Impact occurs when a large force is developed between two objects which strike one another during a short period of time.
- We can analyze the effects of impact by assuming the moving body is rigid, the material of the stationary body is linearly elastic, no energy is lost in the collision, the bodies remain in contact during collision, and inertia of elastic body is neglected.
- The dynamic load on a body can be treated as a statically applied load by multiplying the static load by a magnification factor.

EXAMPLE 14.8

Aluminum pipe is used to support a load of 600 kN. Determine the maximum displacement at the top of the pipe if load is (a) applied gradually, and (b) applied suddenly by releasing it from the top of the pipe at h = 0.

Take $E_{al} = 70(10^3)$ N/mm^2 and assume that the aluminum behaves elastically.



EXAMPLE 14.8 (SOLN)

(a) When load applied gradually, work done by weight is transformed into elastic strain energy in pipe. Applying conservation of energy,

$$U_e = U_i$$

$$\frac{1}{2} W \Delta_{st} = \frac{W^2 L}{2AE}$$

$$\Delta_{st} = \frac{WL}{AE} = \frac{600 \text{ kN}(240 \text{ mm})}{\pi\left[(60 \text{ mm})^2 - (50 \text{ mm})^2\right] 70 \text{ kN/mm}^2}$$

$$= 0.5953 \text{ mm}$$

EXAMPLE 14.8 (SOLN)

(b) With h = 0, apply Eqn 14-30. Hence

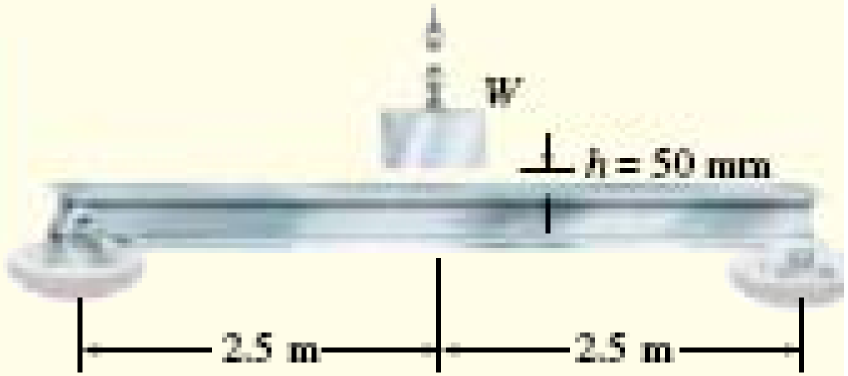
$$\Delta_{\max} = \Delta_{st}\left[1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)}\right]$$

$$= 2\Delta_{st} = 2(0.5953\ \text{mm})$$

$$= 1.1906\ \text{mm}$$

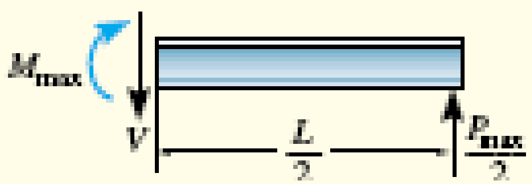
The displacement of the weight is twice as great as when the load is applied statically. In other words, the impact factor is n = 2, Eqn 14-34.

EXAMPLE 14.9



(a)

The A-36 steel beam shown in Fig. 14–27*a* is a W250 × 58. Determine the maximum bending stress in the beam and the beam's maximum deflection if the weight $W = 6000$ N is dropped from a height $h = 50$ mm onto the beam. $E_{st} = 210(10^3)$ N/mm^2.



(b)

Solution I

We will apply Eq. 14–30. First, however, we must calculate Δ_{st}. Using the table in Appendix C, and the data in Appendix B for the properties of a W250 × 58, we have

$$\Delta_{st} = \frac{WL^3}{48EI} = \frac{(6000\text{ N})[5\text{ m}(1000\text{ mm/m})^3}{48[210(10^3)\text{ N/mm}^2]87.3(10^6)\text{ mm}^2} = 0.8523\text{ mm}$$

$$_{max} = \Delta_{st}\left[1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)}\right]$$

$$= 0.8523\text{ mm}\left[1 + \sqrt{1 + 2\left(\frac{50\text{ mm}}{0.8523\text{ mm}}\right)}\right] = 10.124\text{ mm} \quad \textit{Ans.}$$

This deflection is caused by an equivalent static load P_{max}, determined from $P_{max} = (48EI/L^3)\,\Delta_{max}$.

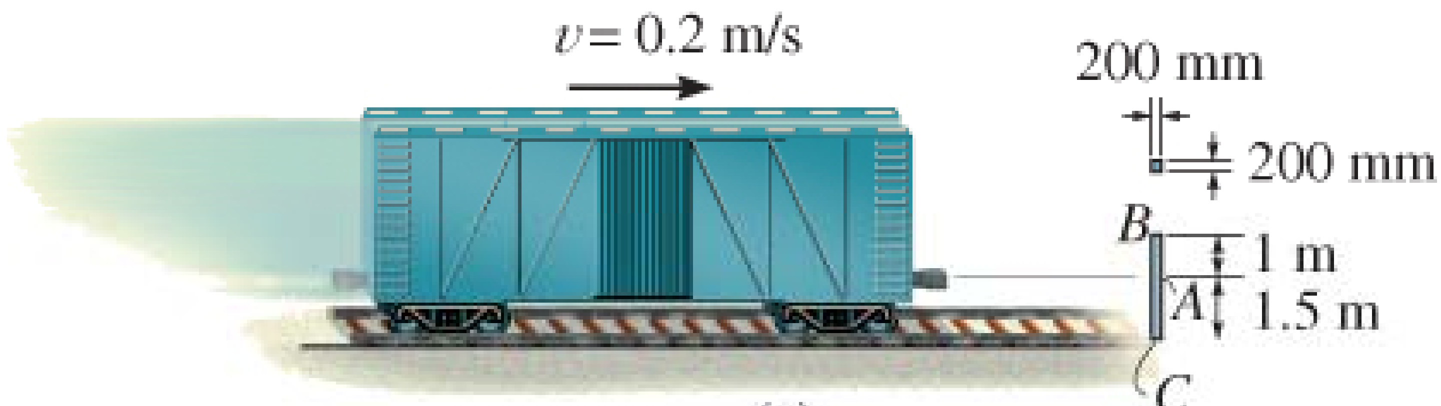
The internal moment caused by this load is maximum at the center of the beam, such that by the method of sections, Fig. 14–27*b*, $M_{max} = P_{max}L/4$. Applying the flexure formula to determine the bending stress, we have

$$\sigma_{max} = \frac{M_{max}c}{I} = \frac{P_{max}Lc}{4I} = \frac{12E\Delta_{max}c}{L^2}$$

$$= \frac{12[210(10^3)\text{ N/mm}^2](10.124\text{ mm})(252\text{ mm/2})}{[(5\text{ m})(1000\text{ mm/m})]^2} = 128.58\text{ N/mm}^2 \quad \textit{Ans.}$$

EXAMPLE 14.10

A railroad car assumed to be rigid and has a mass of 80 Mg is moving forward at a speed of $v = 0.2$ m/s when it strikes a steel 200-mm by 200-mm post at A. If the post is fixed to the ground at C, determine the maximum horizontal displacement of its top B due to the impact. Take E_{st} = 200 GPa.



(a)

EXAMPLE 14.10 (SOLN)

Kinetic energy of the car is transformed into internal bending strain energy only for region AC of the post.. Assume that pt A is displaced $(\Delta_A)_{max}$, then force P_{max} that causes this displacement can be determined from table in Appendix C.

$$P_{max} = \frac{3EI(\Delta_A)_{max}}{L_{AC}^3} \qquad (1)$$

$$Ue = Ui; \qquad \frac{1}{2}m\upsilon^2 = \frac{1}{2}P_{max}(\Delta_A)_{max}$$

$$\frac{1}{2}m\upsilon^2 = \frac{1}{2}\frac{3EI}{L_{AC}^3}(\Delta_A)_{max}^2; \; (\Delta_A)_{max} = \sqrt{\frac{m\upsilon^2 L_{AC}^3}{3EI}}$$

EXAMPLE 14.10 (SOLN)

Substitute in numerical data yields

$$(\Delta_A)_{\max} = \sqrt{\frac{80\left(10^3\right)\text{kg}(0.2\text{ m/s})^2(1.5\text{ m})^3}{3\left[200\left(10^9\right)\text{N/m}^2\right]\left[\frac{1}{12}(0.2\text{ m})^4\right]}}$$

$$= 0.0116\text{ m} = 11.6\text{ mm}$$

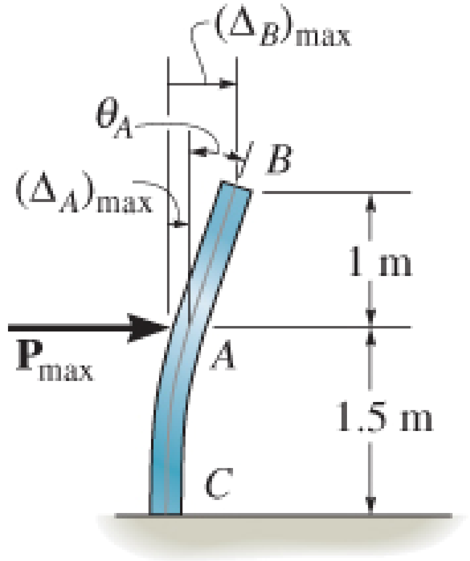
Using Eqn (1), force $P_{\max}$ becomes

$$P_{\max} = \frac{3\left[200\left(10^9\right)\text{N/m}^2\right]\left[\frac{1}{12}(0.2\text{ m})^4\right](0.0116\text{ m})}{(1.5\text{ m})^3}$$

$$= 275.4\text{ kN}$$

EXAMPLE 14.10 (SOLN)

Refer to figure, segment AB of post remains straight. To determine displacement at B, we must first determine slope at A. Using formula from table in Appendix C to determine θ_A, we have



(b)

$$\theta_A = \frac{P_{max} L_{AC}^2}{2EI} = \frac{275.4\left(10^3\right)\text{ N }(1.5\text{ m})^2}{2\left[200\left(10^9\right)\text{ N/m}^2\right]\left[\frac{1}{12}(0.2\text{ m})^4\right]}$$

$$= 0.01162\text{ rad}$$

EXAMPLE 14.10 (SOLN)

The maximum displacement at B is thus

$$\begin{aligned}(\Delta_B)_{\max} &= (\Delta_A)_{\max} + \theta_A L_{AB} \\ &= 11.62\text{ mm} + (0.01162\text{ rad})1\left(10^3\right)\text{mm} \\ &= 23.2\text{ mm}\end{aligned}$$