Polynomial Interpolation and Approximation

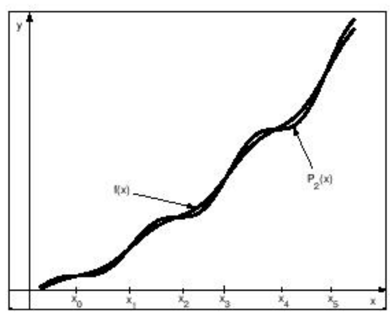
In this chapter we describe the numerical methods for the approximation of functions

We sometimes know the value of a function $f(x)$ at a set of points (say, $x_0 < x_1 < x_2 \cdots < x_n$)

but we do not have an analytic expression for $f(x)$

x	x_0	x_1	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_n)$

The task now is to estimate $f(x)$ for an arbitrary point x by, drawing a smooth curve through the data points x_i.



If the desired x is between the largest and smallest of the data point, then the problem is called *interpolation*;

if x is outside that range, it is called *extrapolation*.

Theorem 4.1 (Weierstrass Approximation Theorem)

If $f(x)$ is a continuous function in the closed interval $[a, b]$ then for every $\epsilon > 0$ there exists a polynomial $p_n(x)$, where the value of n depends on the value of ϵ, such that for all x in $[a, b]$,

$$|f(x) - p_n(x)| < \epsilon. \tag{4.2}$$

Consequently, any continuous function can be approximated to any accuracy by a polynomial of high enough degree. •

The general form of a *nth*-degree polynomial is $p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

where n denotes the degree of the polynomial; and $a_0, a_1, \ldots, a_n$ are constants coefficients.

Polynomial Interpolation

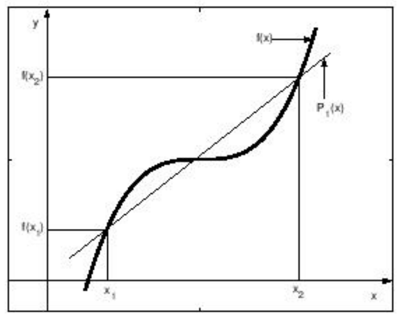
Suppose we have given a set of $(n + 1)$ data points relating a dependent variables $f(x)$ to an independent variable x as follows

x	x_0	x_1	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_n)$

Lagrange Interpolating Polynomials

It is one of the popular and well known interpolation method to approximate the functions at an arbitrary point x.

Linear Lagrange Interpolating Polynomial



Let us consider the construction of a linear polynomial $p_1(x)$ passing through two data points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

$$f(x) \approx p_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1), \tag{4.5}$$

where

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}. \tag{4.6}$$

Note that when $x = x_0$, then $L_0(x_0) = 1$ and $L_1(x_0) = 0$. Similarly, when $x = x_1$, then $L_0(x_1) = 0$ and $L_1(x_1) = 1$. The polynomial (4.5) is known as *linear Lagrange interpolating polynomial* and (4.6) is called the *Lagrange coefficient polynomials*.

Quadratic Lagrange Interpolating Polynomial

When $p_2(x)$ passes through three points $(x_0, f(x_0)), (x_1, f(x_1))$ and $(x_2, f(x_2))$, we have quadratic Lagrange polynomial as follows

$$f(x) \approx p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{4.7}$$

where the Lagrange coefficients are define as follows:

$$\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \\ L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}, \\ L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}. \end{aligned} \tag{4.8}$$

Example 4.5 *Consider the following table:*

x	0	3	7
$f(x)$	2	4	19

(a) Construct quadratic Lagrange polynomial $p_2(x) = ax^2 + bx + c$ *to approximate* $f(x)$.
(b) Use the polynomial in part (a) to interpolate $f(x)$ *at* $x = 4$.

Solution. *(a) Obviously, a quadratic polynomial can be determined so that it passes through the three points. Consider the quadratic Lagrange interpolating polynomial as follows:*

$$p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{4.20}$$

or

$$p_2(x) = 2L_0(x) + 4L_1(x) + 19L_2(x). \tag{4.21}$$

The Lagrange coefficients can be calculate as follows:

$$\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{1}{21}(x^2-10x+21), \\ L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -\frac{1}{12}(x^2-7x), \\ L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{1}{28}(x^2-3x). \end{aligned}$$

Putting these values of the Lagrange coefficients in (4.21), we have

$$p_2(x) = \frac{1}{84}(37x^2 - 55x + 168),$$

(with $a = 37/84, b = -55/84, c = 2$) which is the required quadratic Lagrange polynomial.
(b) Now take $x = 4$ in the above polynomial, we obtain

$$p_2(4) = \frac{1}{84}\Big[37(4)^2 - 55(4) + 168\Big] = 6.4286,$$

which is the required estimate value of $f(4)$. •

Nth Degree Lagrange Interpolating Polynomial

To generalize the concept of the Lagrange interpolation, consider the construction of a polynomial $p_n(x)$ of degree at most n that passes through $(n+1)$ distinct points $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$ and satisfy the interpolation conditions $\quad p_n(x_k) = f(x_k); \qquad k = 0, 1, 2, \ldots, n.$

Assume that there exists polynomial $L_k(x)$ $(k = 0, 1, 2, \ldots, n)$ of degree n having the property

$$L_k(x_j) = \begin{cases} 0 & \text{for} \quad k \neq j \\ 1 & \text{for} \quad k = j \end{cases} \tag{4.12}$$

and

$$\sum_{k=0}^{n} L_k(x) = 1. \tag{4.13}$$

The polynomial $p_n(x)$ is given by

$$\begin{aligned} f(x) \approx p_n(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + \cdots + L_{i-1}(x)f(x_{i-1}) \\ &+ L_i(x)f(x_i) + \cdots + L_n(x)f(x_n) = \sum_{k=0}^{n} L_k(x)f(x_k). \end{aligned} \tag{4.14}$$

$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} = \prod_{k=0}^{n}\left(\frac{x-x_k}{x_i-x_k}\right), \quad i \neq k.$$

the Lagrange interpolation formula of degree n

$$f(x) \approx p_n(x) = \sum_{i=0}^{n}\prod_{k=0}^{n}\left(\frac{x-x_k}{x_i-x_k}\right) f(x_i), \qquad i \neq k.$$

Uniqueness of Lagrange Interpolating Polynomial

To show the *uniqueness* of the interpolating polynomial $p_n(x)$, we suppose that in addition to the polynomial $p_n(x)$ the interpolation problem has another solution $q_n(x)$ of degree $\leq n$ whose graph passes through (x_i, y_i), $i = 0, 1, \ldots, n$. Then define

$$r_n(x) = p_n(x) - q_n(x),$$

of the degree not greater than n. Since

$$r_n(x_i) = p_n(x_i) - q_n(x_i) = f(x_i) - f(x_i) = 0,$$

the polynomial $r_n(x)$ vanishes at $n+1$ point. But by using the following well known result from the theory of equations: *"If a polynomial of degree n vanishes at $n+1$ distinct points, then the polynomial is identically zero"*. Hence $r_n(x)$ vanishes identically, or equivalently, $p_n(x) = q_n(x)$.

Error Formula of Lagrange Polynomial

All can be said with certainty is that $f(x) - p_n(x) = 0$ at $x = x_0, x_1, \ldots, x_n$.

However, it is sometimes possible to obtain a bound on the error $f(x) - p_n(x)$ at an intermediate point x using the following theorem.

Theorem 4.2 (Error Formula of Nth Degree Lagrange Polynomial)

If $f(x)$ has $(n+1)$ derivatives on interval I and if it is approximated by a polynomial $p_n(x)$ passing through $(n+1)$ data points on I, then the error E_n is given by

$$E_n = f(x) - p_n(x) = \frac{f^{(n+1)}(\eta(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n), \quad \eta(x) \in I, \tag{4.30}$$

where $p_n(x)$ is Lagrange interpolating polynomial (4.14) and a unknown point $\eta(x) \in (x_0, x_n)$. •

Error Formulas of Linear, Quadratic and Cubic Lagrange Polynomials

$$E_1 = f(x) - p_1(x) = \frac{f''(\eta(x))}{2!}(x - x_0)(x - x_1), \quad \eta(x) \in I,$$

where $p_1(x)$ is the linear Lagrange polynomial (4.5) and a unknown point $\eta(x) \in (x_0, x_1)$.

$$E_2 = f(x) - p_2(x) = \frac{f'''(\eta(x))}{3!}(x - x_0)(x - x_1)(x - x_2), \quad \eta(x) \in I,$$

where $p_2(x)$ is the quadratic Lagrange polynomial (4.7) and a unknown point $\eta(x) \in (x_0, x_2)$.

$$E_3 = f(x) - p_3(x) = \frac{f^{(4)}(\eta(x))}{4!}(x - x_0)(x - x_1)(x - x_2)(x - x_3), \quad \eta(x) \in I,$$

where $p_3(x)$ is the cubic Lagrange polynomial (4.9) and a unknown point $\eta(x) \in (x_0, x_3)$.

Example 4.14 *Use the quadratic Lagrange interpolating polynomial by selecting the best points from $x = -2, 0, 1, 2$ and $x = 2.5$ on the function defined by $f(x) = (x+1)^{1/3}$ to estimate the cube root of $\frac{3}{2}$ and compute an error bound and absolute error.*

Solution. *Since the given function is a cube root of $(x+1)$, so by taking $x+1 = \frac{3}{2}$, we have $x = \frac{1}{2}$, therefore, the best points for the quadratic polynomial are $x_0 = 0, x_1 = 1$, and $x_2 = 2$. Consider a quadratic Lagrange interpolating polynomial as*

$$f(x) \approx p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{4.32}$$

and at $x = 0.5$, gives

$$f(0.5) \approx p_2(0.5) = (1)^{1/3}L_0(0.5) + (2)^{1/3}L_1(0.5) + (3)^{1/3}L_2(0.5). \tag{4.33}$$

The Lagrange coefficients can be calculate as follows:

$$\begin{aligned} L_0(0.5) &= \frac{(0.5-1)(0.5-2)}{(0-1)(0-2)} = 0.375, \\ L_1(0.5) &= \frac{(0.5-0)(0.5-2)}{(1-0)(1-2)} = 0.75, \\ L_2(0.5) &= \frac{(0.5-0)(0.5-1)}{(2-0)(2-1)} = -0.125. \end{aligned}$$

Putting these values of the Lagrange coefficients in (4.33), we have

$$f(0.5) \approx p_2(0.5) = (1)^{1/3}(0.375) + (2)^{1/3}(0.75) - (3)^{1/3}(0.125) = 1.1396,$$

which is the required approximation of the $\left(\frac{3}{2}\right)^{1/3}$.

To compute an error bound for the approximation of the given function in the interval $[0,2]$, *u use the following quadratic error formula*

$$|f(x) - p_2(x)| = \frac{|f^{(3)}(\eta(x))|}{3!}|(x - x_0)(x - x_1)(x - x_2)|.$$

As

$$|f^{(3)}(\eta(x))| \leq M = \max_{0 \leq x \leq 2} |f^{(3)}(x)|,$$

and

$$f'(x) = \frac{1}{3}(x+1)^{-2/3}, \quad f''(x) = -\frac{2}{9}(x+1)^{-5/3}, \quad f^{(3)}(x) = \frac{10}{27}(x+1)^{-8/3},$$

so

$$M = \max_{0 \leq x \leq 2} \left| \frac{10}{27}(x+1)^{-8/3} \right| = \frac{10}{27}.$$

Hence

$$|f(0.5) - p_2(0.5)| \leq \frac{10/27}{6}|(0.5 - 0)(0.5 - 1)(0.5 - 2)|,$$

and it gives

$$|f(0.5) - p_2(0.5)| \leq \frac{10(0.375)}{162} = 0.0232,$$

which is desired error bound. Also, we have the absolute error is given as

$$|f(0.5) - p_2(0.5)| = |(1.5)^{1/3} - 1.1396| = |1.1447 - 1.1396| = 0.0051,$$

Example 4.18 *Consider the following table having the data for* $f(x) = e^{3x}\cos 2x$*:*

x	0.1	0.2	0.4	0.5
$f(x)$	1.32295	1.67828	2.31315	2.42147

Find the approximation of $f(0.3)$ *using the best Lagrange interpolation formula and also estimate an error bound for the approximation.*

Solution. *Using the given data points, the best Lagrange formula to find the interpolating polynomial to approximate the function is the cubic polynomial*

$$p_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3),$$

which implies that

$$\begin{aligned} p_3(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1) \\ &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3), \end{aligned}$$

or

$$\begin{aligned} p_3(x) &= (-110.2458)[(x-0.2)(x-0.4)(x-0.5)] + (279.7133)[(x-0.1)(x-0.4)(x-0.5)] \\ &+ (-385.5250)[(x-0.1)(x-0.2)(x-0.5)] + (201.7892)[(x-0.1)(x-0.2)(x-0.4)]. \end{aligned}$$

Thus

$$p_3(x) = -14.2683x^3 + 8.7246x^2 + 1.9347x + 1.0565. \tag{4.34}$$

Take $x = 0.3$ in the above polynomial (4.34), we have

$$f(0.3) \approx p_3(0.3) = 2.0369.$$

The exact value of $f(0.3) = 2.029998$, so, the actual error is 0.00754. Now to compute an error bound of the approximation, we use the following formula

$$|f(x) - p_3(x)| = \frac{|f^{(4)}(\eta(x))|}{4!}|(x - x_0)(x - x_1)(x - x_2)(x - x_3)|. \quad (4.35)$$

Taking the fourth derivative of the given function, we have

$$f'(x) = e^{3x}(3\cos 2x - 2\sin 2x), \qquad f''(x) = e^{3x}(5\cos 2x - 12\sin 2x),$$

$$f'''(x) = e^{3x}(-9\cos 2x - 46\sin 2x), \qquad f^{(4)}(x) = -e^{3x}(119\cos 2x + 120\sin 2x).$$

Thus

$$|f^{(4)}(\eta(x))| = |-e^{3\eta(x)}(119\cos 2(\eta(x)) + 120\sin 2(\eta(x)))|, \quad \textit{for} \quad \eta(x) \in (0.1, 0.5),$$

and it gives

$$|f^{(4)}(0.1)| = 189.61229 \quad \textit{and} \quad |f^{(4)}(0.5)| = 740.69991.$$

$$M = \max_{0.1 \le x \le 0.5} |f^{(4)}(x)| = 740.69991,$$

and so for $|f^{(4)}(\eta(x))| \le M$, we have (4.35) as follows

$$|f(x) - p_3(x)| \le (740.69991)(0.0004)/24 = 0.01235,$$

which is the required error bound for the approximation. •

Theorem 4.3 (Error Bounds for Lagrange Interpolation at Equally Spaced Points)

Assume that $f(x)$ *is defined on the interval* $[a, b]$, *which contains equally spaced points* $x_k = x_0 + hk$. *Additionally, assume that* $f(x)$ *and the derivatives of* $f(x)$ *up to the order* $(n+1)$, *are continuous and bounded on the special intervals* $[x_0, x_1]$, $[x_0, x_2]$ *and* $[x_0, x_3]$, *respectively; that is*

$$|f^{(n+1)}(x)| \leq M \quad \text{for} \quad x_0 \leq x \leq x_n,$$

for $n = 1, 2, 3$. *Then error bounds for linear, quadratic and cubic polynomials are:*

$$\begin{aligned} |E_1(x)| &\leq \frac{h^2}{8}M && \text{for} && x_0 \leq x \leq x_1, \\ |E_2(x)| &\leq \frac{h^3}{9\sqrt{3}}M && \text{for} && x_0 \leq x \leq x_2, \\ |E_3(x)| &\leq \frac{h^4}{24}M && \text{for} && x_0 \leq x \leq x_3. \end{aligned}$$

Continue in the similar manner for the interval $[x_0, x_n]$, *for* $n = 1, 2, \ldots, n$, *we have*

$$|E_n(x)| \leq \frac{M}{4(n+1)} \left(\frac{b-a}{n}\right)^{n+1}, \quad \text{for} \quad x_0 \leq x \leq x_n, \tag{4.36}$$

the general error bound formula. •

Example 4.20 *Find an error bound if $f(x) = \sin x$ is approximated by an interpolation polynomial with ten equally spaced data points in $[0, 1.6875]$.*

Solution. *Given $n = 9$ and $a = 0, b = 1.6875$,*

Note that $f^{(n)}(x) = \pm \sin x$ for even n and $f^{(n)}(x) = \pm \cos x$ for odd n, so we have a uniform bound on $f^{(n)}(x)$ for all n. That is $|f^{(n)}(x)| \leq 1$ for all x and for all n.

$$M = \max_{0 \leq x \leq 1.6875} |f^{(10)}(x)| = \max_{0 \leq x \leq 1.6875} |-\sin x| \leq 1, \quad \forall\, x \in [0, 1.6875].$$

Hence, the interpolation error (use Theorem 4.36) can be bounded by

$$|E_9(x)| = |\sin x - p_9(x)| \leq \frac{1}{40}\left(\frac{1.6875}{9}\right)^{10} \approx 1.34 \times 10^{-9},$$

for all $x \in [0, 1.6875]$. •

EXAMPLE 3 Table 3.2 lists values of a function at various points. The approximations to $f(1.5)$ obtained by various Lagrange polynomials will be compared.

Table 3.2

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

Since 1.5 is between 1.3 and 1.6, the most appropriate linear polynomial uses $x_0 = 1.3$ and $x_1 = 1.6$. The value of the interpolating polynomial at 1.5 is

$$P_1(1.5) = \frac{(1.5-1.6)}{(1.3-1.6)}(0.6200860) + \frac{(1.5-1.3)}{(1.6-1.3)}(0.4554022) = 0.5102968.$$

Two polynomials of degree 2 can reasonably be used, one by letting $x_0 = 1.3$, $x_1 = 1.6$, and $x_2 = 1.9$, which gives

$$\begin{aligned} P_2(1.5) &= \frac{(1.5-1.6)(1.5-1.9)}{(1.3-1.6)(1.3-1.9)}(0.6200860) + \frac{(1.5-1.3)(1.5-1.9)}{(1.6-1.3)(1.6-1.9)}(0.4554022) \\ &\quad + \frac{(1.5-1.3)(1.5-1.6)}{(1.9-1.3)(1.9-1.6)}(0.2818186) \\ &= 0.5112857, \end{aligned}$$

and the other by letting $x_0 = 1.0$, $x_1 = 1.3$, and $x_2 = 1.6$, which gives

$$\hat{P}_2(1.5) = 0.5124715.$$

In the third-degree case, there are also two reasonable choices for the polynomial. One is with $x_0 = 1.3$, $x_1 = 1.6$, $x_2 = 1.9$, and $x_3 = 2.2$, which gives

$$P_3(1.5) = 0.5118302.$$

The other is obtained by letting $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, and $x_3 = 1.9$, which gives

$$\hat{P}_3(1.5) = 0.5118127.$$

The fourth-degree Lagrange polynomial uses all the entries in the table. With $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$, and $x_4 = 2.2$, the approximation is

$$P_4(1.5) = 0.5118200.$$

value at 1.5 is known to be 0.5118277. Therefore,

$$|P_1(1.5) - f(1.5)| \approx 1.53 \times 10^{-3},$$

$$|P_2(1.5) - f(1.5)| \approx 5.42 \times 10^{-4},$$

$$|\hat{P}_2(1.5) - f(1.5)| \approx 6.44 \times 10^{-4},$$

$$|P_3(1.5) - f(1.5)| \approx 2.5 \times 10^{-6},$$

$$|\hat{P}_3(1.5) - f(1.5)| \approx 1.50 \times 10^{-5},$$

$$|P_4(1.5) - f(1.5)| \approx 7.7 \times 10^{-6}.$$

Although $P_3(1.5)$ is the most accurate approximation, if we had no knowledge of the actual value of $f(1.5)$, we would accept $P_4(1.5)$ as the best approximation since it includes the most data about the function. The Lagrange error term derived in Theorem 3.3 cannot be applied here since no knowledge of the fourth derivative of f is available. Unfortunately, this is generally the case.

Neville's Algorithm

Neville's algorithm is equivalent to a Lagrange polynomial. It is based on a series of linear interpolations. The data do not have to be in monotonic order, or in any structured order. However, the most accurate results are obtained if the data are arranged in order of closeness to the point to be interpolated.

Consider the following set of data:

x_i	f_i
x_1	f_1
x_2	f_2
x_3	f_3
x_4	f_4

Recall the linear Lagrange interpolating polynomial, Eq. (4.43):

$$f(x) = \frac{(x-b)}{(a-b)}f(a) + \frac{(x-a)}{(b-a)}f(b) \tag{4.50}$$

which can be written in the following form:

$$f(x) = \frac{(x-a)f(b) - (x-b)f(a)}{(b-a)} \tag{4.51}$$

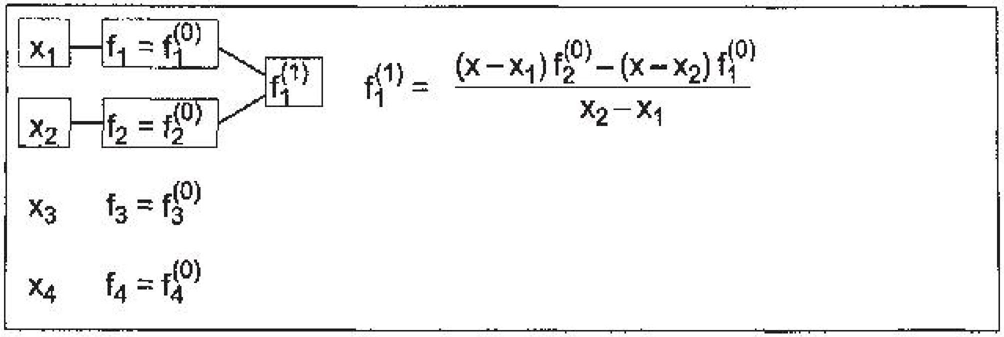
In terms of general notation, Eq. (4.51) yields

$$\boxed{f_i^{(n)} = \frac{(x-x_i)f_{i+1}^{(n-1)} - (x-x_{i+n})f_i^{(n-1)}}{x_{i+n} - x_i}} \tag{4.52}$$

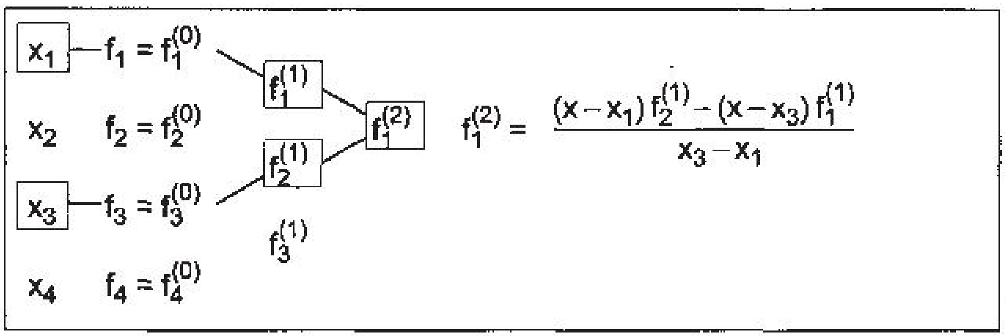
where the subscript i denotes the base point of the value (e.g., $i, i+1$, etc.) and the superscript (n) denotes the degree of the interpolation (e.g., zeroth, first, second, etc.).

A table of linearly interpolated values is constructed for the original data, which are denoted as $f_i^{(0)}$. For the first interpolation of the data,

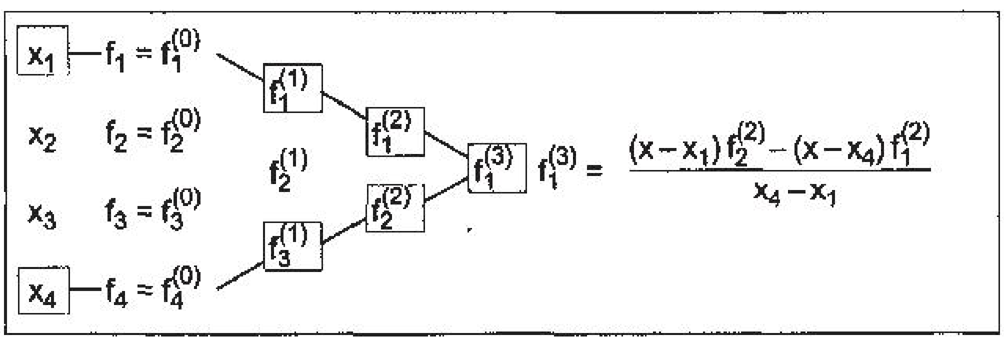
$$f_i^{(1)} = \frac{(x - x_i) f_{i+1}^{(0)} - (x - x_{i+1}) f_i^{(0)}}{x_{i+1} - x_i} \tag{4.53}$$



(a) First set of linear interpolations.



(b) Second set of linear interpolation.



(c) Third set of linear interpolations

as illustrated in Figure 4.6a. This creates a column of $n-1$ values of $f_i^{(1)}$. A second column of $n-2$ values of $f_i^{(2)}$ is obtained by linearly interpolating the column of $f_i^{(1)}$ values. Thus,

$$f_i^{(2)} = \frac{(x - x_i)f_{i+1}^{(1)} - (x - x_{i+2})f_i^{(1)}}{x_{i+2} - x_i} \tag{4.54}$$

which is illustrated in Figure 4.6b. This process is repeated to create a third column of $f_i^{(3)}$ values, as illustrated in Figure 4.6c, and so on. The form of the resulting table is illustrated in Table 4.1.

It can be shown by direct substitution that each specific value in Table 4.1 is identical to a Lagrange polynomial based on the data points used to calculate the specific value. For example, $f_1^{(2)}$ is identical to a second-degree Lagrange polynomial based on points 1, 2, and 3.

The advantage of Neville's algorithm over direct Lagrange polynomial interpolation is now apparent. The third-degree Lagrange polynomial based on points 1 to 4 is obtained simply by applying the linear interpolation formula, Eq. (4.52), to $f_1^{(2)}$ and $f_2^{(2)}$ to obtain

Table 4.1. Table for Neville's Algorithm

x_i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$
x_1	$f_1^{(0)}$			
		$f_1^{(1)}$		
x_2	$f_2^{(0)}$		$f_1^{(2)}$	
		$f_2^{(1)}$		$f_1^{(3)}$
x_3	$f_3^{(0)}$		$f_2^{(2)}$	
		$f_3^{(1)}$		
x_4	$f_4^{(0)}$			

$f_1^{(3)}$. None of the prior work must be redone, as it would have to be redone to evaluate a third-degree Lagrange polynomial. If the original data are arranged in order of closeness to the interpolation point, each value in the table, $f_i^{(n)}$, represents a centered interpolation.

Example 4.4. Neville's algorithm.

Consider the four data points given in Example 4.3. Let's interpolate for $f(3.44)$ using linear, quadratic, and cubic interpolation using Neville's algorithm. Rearranging the data in order of closeness to $x = 3.44$ yields the following set of data:

x	$f(x)$
3.40	0.294118
3.50	0.285714
3.35	0.298507
3.60	0.277778

Applying Eq. (4.52) to the values of $f_i^{(0)}$ gives

$$f_1^{(1)} = \frac{(x - x_1)f_2^{(0)} - (x - x_2)f_1^{(0)}}{x_2 - x_1} = \frac{(3.44 - 3.40)0.285714 - (3.44 - 3.50)0.294118}{3.50 - 3.40}$$

$$= 0.290756 \tag{4.55a}$$

Thus, the result of linear interpolation is $f(3.44) = f_1^{(1)} = 0.290756$. To evaluate $f_1^{(2)}$, $f_2^{(1)}$ must first be evaluated. Thus,

$$f_2^{(1)} = \frac{(x - x_2)f_3^{(0)} - (x - x_3)f_2^{(0)}}{x_3 - x_2} = \frac{(3.44 - 3.50)0.298507 - (3.44 - 3.35)0.285714}{3.35 - 3.50}$$

$$= 0.290831 \tag{4.55b}$$

Evaluating $f_1^{(2)}$ gives

$$f_1^{(2)} = \frac{(x - x_1)f_2^{(1)} - (x - x_3)f_1^{(1)}}{x_3 - x_1} = \frac{(3.44 - 3.40)0.290831 - (3.44 - 3.35)0.290756}{3.35 - 3.40}$$

$$= 0.290696 \tag{4.56}$$

Table 4.2. Neville's Algorithm

x_i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$
$x_1 = 3.40$	0.294118			
		0.290756		
$x_2 = 3.50$	0.285714		0.290697	
		0.290831		0.290698
$x_3 = 3.35$	0.298507		0.290703	
		0.291045		
$x_4 = 3.60$	0.277778			

Thus, the result of quadratic interpolation is $f(3.44) = f_1^{(2)} = 0.290696$. To evaluate $f_1^{(3)}, f_3^{(1)}$ and $f_2^{(2)}$ must first be evaluated. Then $f_1^{(3)}$ can be evaluated. These results, and the results calculated above, are presented in Table 4.2.

These results are the same as the results obtained by Lagrange polynomials in Example 4.3.

The advantage of Neville's algorithm over a Lagrange interpolating polynomial, if the data are arranged in order of closeness to the interpolated point, is that none of the work performed to obtain a specific degree result must be redone to evaluate the next higher degree result.

Neville's algorithm has a couple of minor disadvantages. All of the work must be redone for each new value of x. The amount of work is essentially the same as for a Lagrange polynomial. The divided difference polynomial presented in Section 4.5 minimizes these disadvantages.

Newton's General Interpolating Formula

Since we noted in the previous section that for a small number of data point one can easily use the Lagrange formula of the interpolating polynomial. However, for a large number of data points there will be many multiplication and more significantly, whenever a new data point is added to an existing set, the interpolating polynomial has to be completely recalculated. Here, we describe an efficient way of organizing the calculations so as to overcome these disadvantages.
Let us consider the *nth*-degree polynomial $p_n(x)$ that agrees with the function $f(x)$ at the distinct numbers $x_0, x_1, \ldots, x_n$. The divided differences of $f(x)$ with respect to $x_0, x_1, \ldots, x_n$ are derived to express $p_n(x)$ in the form

$$\begin{aligned} p_n(x) = a_0 &+ a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \\ &+ a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}), \end{aligned} \tag{4.41}$$

for appropriate constants $a_0, a_1, \ldots, a_n$.
Now to determine the constants, firstly, by evaluating $p_n(x)$ at x_0, we have

$$p_n(x_0) = a_0 = f(x_0) \tag{4.42}$$

Similarly, when $p_n(x)$ is evaluated at x_1, then

$$p_n(x_1) = a_0 + a_1(x_1 - x_0) = f(x_1),$$

which implies that

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \tag{4.43}$$

Divided Differences

Firstly, we define the *Zeroth divided difference* at the point x_i by

$$f[x_i] = f(x_i), \tag{4.44}$$

which is simply the value of the function $f(x)$ at x_i.

The *first-order* or *first divided difference* at the points x_i and x_{i+1} can be defined by

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}. \tag{4.45}$$

In general, the *nth divided difference* $f[x_i, x_{i+1}, \ldots, x_{i+n}]$ is defined by

$$f[x_i, x_{i+1}, \ldots, x_{i+n}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+n}] - f[x_i, x_{i+1}, \ldots, x_{i+n-1}]}{x_{i+n} - x_i}. \tag{4.46}$$

By using this definition, (4.42) and (4.43) can be written as

$$a_0 = f[x_0]; \qquad a_1 = f[x_0, x_1],$$

respectively. Similarly, one can have the values of other constants involving in (4.41) such as

$$\begin{aligned} a_2 &= f[x_0, x_1, x_2], \\ a_3 &= f[x_0, x_1, x_2, x_3], \\ \cdots &= \quad \cdots \\ \cdots &= \quad \cdots \\ a_n &= f[x_0, x_1, \ldots, x_n]. \end{aligned}$$

Table 4.1: Divided difference table for a function $y = f(x)$

k	x_k	Zero Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	x_0	$f[x_0]$			
1	x_1	$f[x_1]$	$f[x_0, x_1]$		
2	x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
3	x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$

Linear Newton's Interpolating Polynomial

The linear Newton's interpolating polynomial passing through two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ can be written as

$$f(x) \approx p_1(x) = f[x_0] + (x - x_0)f[x_0, x_1].$$

Quadratic Newton's Interpolating Polynomial

The quadratic Newton's interpolating polynomial passing through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ can be written in terms of divided differences as

$$p_2(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2].$$

This polynomial can also be written as

$$f(x) \approx p_2(x) = p_1(x) + (x - x_0)(x - x_1)f[x_0, x_1, x_2],$$

Nth Degree Newton's Interpolating Polynomial

Repeating this entire process again, $p_3(x), p_4(x)$ and higher degree interpolating polynomials can be consecutively obtained in the same way. In general, the interpolating polynomial $p_n(x)$ passing through the points $(x_i, f(x_i))(i = 0, 1, \ldots, n)$, can be written in terms of divided differences as

$$\begin{aligned} f(x) \approx p_n(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ \cdots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1}), \end{aligned} \tag{4.49}$$

Theorem 4.4 (Newton's Interpolating Polynomial)

Suppose that $x_0, x_1, \ldots, x_n$ are $(n+1)$ distinct points in the interval $[a, b]$. There exists a unique polynomial $p_n(x)$ of degree at most n with the property that

$$f(x_i) = p_n(x_i), \quad \textit{for} \quad i = 0, 1, \ldots, n.$$

The Newton's form of this polynomial is

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}),$$

where

$$a_k = f[x_0, x_1, x_2, \cdots, x_k], \quad \textit{for} \quad k = 0, 1, 2, \ldots, n.$$

Example 4.29 *Consider the following table of date points*

x	3	1	5	6
$f(x)$	1	-3	2	4

Find the third divided difference $f[3, 1, 5, 6]$ and use it to find the Newton's form of the interpolating polynomial. Find approximation of $f(2)$.

Solution. *The third divided differences for the given data points are listed in Table 4.5. The cubic*

Table 4.5: Divided difference table for a function $y = f(x)$

k	x_k	Zero Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	$x_0 = 3$	$f[x_0] = 1$			
1	$x_1 = 1$	$f[x_1] = -3$	$f[x_0, x_1] = 2$		
2	$x_2 = 5$	$f[x_2] = 2$	$f[x_1, x_2] = 5/4$	$f[x_0, x_1, x_2] = -3/8$	
3	$x_3 = 6$	$f[x_3] = 4$	$f[x_2, x_3] = 2$	$f[x_1, x_2, x_3] = 3/20$	$f[x_0, x_1, x_2, x_3] = 7/40$

Newton's interpolating polynomial passing through the given can be written as

$$p_3(x) = f[x_0]+(x-x_0)f[x_0, x_1]+(x-x_0)(x-x_1)f[x_0, x_1, x_2]+(x-x_0)(x-x_1)(x-x_2)f[x_0, x_1, x_2, x_3],$$

so using Table 4.5, we have

$$p_3(x) = 1 + 2(x - x_0) - \frac{3}{8}(x - x_0)(x - x_1) + \frac{7}{40}(x - x_0)(x - x_1)(x - x_2),$$

or

$$p_3(x) = \frac{1}{40}[7x^3 - 78x^2 + 301x - 350].$$

Thus at $x = 2$, we get

$$f(2) \approx p_3(2) = \frac{1}{40}[7(2)^3 - 78(2)^2 + 301(2) - 350] = -\frac{1}{10},$$

Example 4.33 *(a) Construct the divided difference table for the function* $f(x) = \ln(x+2)$ *in the interval* $0 \leq x \leq 3$ *for the stepsize* $h = 1$.

(b) Use Newton divided difference interpolation formula to construct the interpolating polynomials of degree 2 and degree 3 to approximate $\ln(3.5)$.

(c) Compute error bounds for the approximations in part (b).

Solution. *(a) The results of the divided differences are listed in Table 4.9.*

(b) Firstly, we construct the second degree polynomial $p_2(x)$ *by using the quadratic Newton interpolation formula as follows*

$$p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1),$$

then with the help of the divided differences Table 4.9, we get

$$p_2(x) = 0.6932 + 0.4055(x - 0) - 0.0589(x - 0)(x - 1),$$

which implies that $p_2(x) = -0.0568x^2 + 0.4644x + 0.6932$ *and* $p_2(1.5) = 1.2620,$

with possible actual error $f(1.5) - p_2(1.5) = 1.2528 - 1.2620 = -0.0072. 1.2620,$

Now to construct the cubic interpolatory polynomial $p_3(x)$ *that fits at all four points. We only have to add one more term to the polynomial* $p_2(x)$:

$$p_3(x) = p_2(x) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2),$$

and this gives

$$p_3(x) = p_2(x) + 0.0089(x^3 - 3x^2 + 2x) \quad \text{and} \quad p_3(1.5) = 1.2620 - 0.0033 = 1.2587,$$

Table 4.9: Divide differences table for the Example 4.33

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	0	0.6932			
1	1	1.0986	0.4055		
2	2	1.3863	0.2877	- 0.0589	
3	3	1.6094	0.2232	- 0.0323	0.0089

with possible actual error $f(1.5) - p_3(1.5) = 1.2528 - 1.2587 = -0.0059.$

(c) Now to compute the error bounds for the approximations in part (b), we use the error formula (4.30). For the polynomial $p_2(x)$*, we have*

$$|f(x) - p_2(x)| = \frac{|f'''(\eta(x))|}{3!}|(x - x_0)(x - x_1)(x - x_2)|.$$

The third derivative of the given function is given as

$$f'''(x) = \frac{2}{(x+2)^3} \quad \text{and} \quad |f'''(\eta(x))| = \left|\frac{2}{(\eta(x)+2)^3}\right|, \quad \text{for} \quad \eta(x) \in (0,2).$$

Then $M = \max_{0 \le x \le 2}\left|\frac{2}{(x+2)^3}\right| = 0.25,$ *and* $|f(1.5) - p_2(1.5)| \le (0.375)(0.25)/6 = 0.0156,$

the error bound for the cubic polynomial $p_3(x)$ *is* $|f(x) - p_3(x)| = \frac{|f^{(4)}(\eta(x))|}{4!}|(x - x_0)(x - x_1)(x - x_2)(x - x_3)|$

$$f^{(4)}(x) = \frac{-6}{(x+2)^4} \quad \text{and} \quad |f^{(4)}(\eta(x))| = \left|\frac{-6}{(\eta(x)+2)^4}\right|, \quad \text{for} \quad \eta(x) \in (0,3).$$

Since $|f^{(4)}(0)| = 0.375$ *and* $|f^{(4)}(3)| = 0.0096,$

so $|f^{(4)}(\eta(x))| \le \max_{0 \le x \le 3}\left|\frac{-6}{(x+2)^4}\right| = 0.375$ *and* $|f(1.5) - p_3(1.5)| \le (0.5625)(0.375)/24 = 0.0088,$

Theorem 4.5 *Let $p_n(x)$ be the polynomial of degree at most n that interpolates a function $f(x)$ at a set of $n+1$ distinct points $x_0, x_1, \ldots, x_n$. If x is a point different from the points $x_0, x_1, \ldots, x_n$, then*

$$f(x) - p_n(x) = f[x_0, x_1, \ldots, x_n, x] \prod_{j=0}^{n} (x - x_j). \tag{4.55}$$

Theorem 4.6 (Divided Differences and Derivatives)

Suppose that $f \in C^n[a,b]$ and $x_0, x_1, \ldots, x_n$ are distinct number in $[a,b]$. Then for some point $\eta(x)$ in the interval (a,b) spanned by $x_0, \ldots, x_n$ exists with

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\eta(x))}{n!}. \tag{4.56}$$

Example 4.37 *Let $f(x) = x \ln x$, and the points $x_0 = 1.1, x_1 = 1.2, x_2 = 1.3$. Compute the best approximate value for unknown point $\eta(x)$ by using the relation (4.56).*

Solution. *Given $f(x) = x \ln x$, then*

$$\begin{aligned} f(1.1) &= 1.1 \ln(1.1) = 0.1048, \\ f(1.2) &= 1.2 \ln(1.2) = 0.2188, \\ f(1.3) &= 1.3 \ln(1.3) = 0.3411. \end{aligned}$$

Since the relation (4.56) for the given data points is

$$f[x_0, x_1, x_2] = \frac{f''(\eta(x))}{2!}. \tag{4.57}$$

To compute the value of the left-hand side of the relation (4.57), we have to find the values of the first-order divided differences

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{0.2188 - 0.1048}{1.2 - 1.1} = 1.1400,$$

and

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0.3411 - 0.2188}{1.3 - 1.2} = 1.2230.$$

Using these values, we can compute the second-order divided difference as

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1.2230 - 1.1400}{1.3 - 1.1} = 0.4150.$$

Now we calculate the right-hand side of the relation (4.57) for the given points and which gives us

$$\frac{f''(x_0)}{2} = \frac{1}{2x_0} = 0.4546, \quad \frac{f''(x_1)}{2} = \frac{1}{2x_1} = 0.4167, \quad \frac{f''(x_2)}{2} = \frac{1}{2x_2} = 0.3846.$$

We note that the left-hand side of (4.57) is nearly equal to the right-hand side when $x_1 = 1.2$. *Hence the best approximate value of* $\eta(x)$ *is* 1.2. •

Properties of Divided Differences

1. Divided difference of a constant is zero. Let $f(x) = a$, then

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{a - a}{x_1 - x_0} = 0.$$

2. Divided difference of $h(x) = af(x)$, a is constant, is the divided difference of $f(x)$ multiplied by a. Let $h(x) = af(x)$, then

$$h[x_0, x_1] = \frac{h(x_1) - h(x_0)}{x_1 - x_0} = \frac{af(x_1) - af(x_0)}{x_1 - x_0} = a\frac{f(x_1) - f(x_0)}{x_1 - x_0} = af[x_0, x_0].$$

3. Divided difference obeys linear property.
   Let $F(x) = af_1(x) + bf_2(x)$, then $\quad F[x_0, x_1] = af_1[x_0, x_1] + bf_2[x_0, x_1].$

4. If $p_n(x)$ is a polynomial of degree n, then the divided differences of order n is always constant and $(n+1), (n+2), \ldots$ are identically zero.

5. The divided difference is a symmetric function of its arguments. Thus if $(t_0, t_1, \ldots, t_n)$ is a permutation of $(x_0, x_1, \ldots, x_n)$, then

$$f[t_0, t_1, \ldots, t_n] = f[x_0, x_1, \ldots, x_n],$$

6. The interpolating polynomial of degree n can be obtained by adding a single term to the polynomial of degree $(n-1)$ expressed in the Newton form.

$$p_n(x) = p_{n-1}(x) + f[x_0, \ldots, x_n] \prod_{j=0}^{n-1} (x - x_j).$$

7. The divided difference $f[x_0, \ldots, x_{n-1}]$ is the coefficient of x^{n-1} in the polynomial that interpolates $(x_0, f_0), (x_1, f_1), \ldots, (x_{n-1}, f_{n-1})$.

8. A sequence of divided differences may be constructed recursively from the formula

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0},$$

and the zeroth-order divided difference is defined by

$$f[x_i] = f(x_i), \quad i = 0, 1, \ldots, n.$$

9. $$f[x_0, x_0] = f'(x_0). \qquad f[x_0, x_0, x_0] = \frac{f''(x_0)}{2}. \qquad f[x_0, x_0, \ldots, x_0] = \frac{f^{(n)}(x_0)}{n!},$$

Example 4.39 *Let* $f(x) = \ln(x+2)$.
(a) Compute $f[0,0,1], f[0,1,1]$ *and* $f[0,0,1,1]$.
(b) Compute the approximation of $\ln 2.5$ *by using cubic Newton's interpolating polynomial.*

Solution. *(a) Using* $f(x) = \ln(x+2)$ *and* $x_0 = 0, x_1 = 1$, *we find the third-order divided difference* $f[0,0,1]$ *as follows:*

$$f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0} = \frac{f[x_0, x_1] - f'(x_0)}{x_1 - x_0},$$

$$f[0,0,1] = \frac{f[0,1] - f'(0)}{1-0} = f(1) - f(0) - f'(0) = 1.0986 - 0.6932 - 0.5 = -0.0946.$$

$$f[0,1,1] = \frac{f[1,1] - f[0,1]}{1-0} = f'(1) - f(1) + f(0) = 0.3333 - 1.0986 + 0.6932 = -0.0721,$$

$$f[0,0,1,1] = \frac{f[0,1,1] - f[0,0,1]}{1-0} = -0.0721 + 0.0946 = 0.0225.$$

(b) The cubic Newton's interpolating polynomial has the following form

$$p_3(x) = f[x_0]+(x-x_0)f[x_0,x_0]+(x-x_0)(x-x_0)f[x_0,x_0,x_1]+(x-x_0)(x-x_0)(x-x_1)f[x_0,x_0,x_1,x_1],$$

so using values of part (a) and $x = 0.5$*, we get*

$$\ln 2.5 \approx p_3(0.5) = f(0)+(0.5-0)f'(0)+(0.5-0)(0.5-0)f[0,0,1]+(0.5-0)(0.5-0)(0.5-1)f[0,0,1,1],$$

or

$$\ln 2.5 \approx p_3(0.5) = 0.6932 + 0.25 - 0.0237 + 0.0090 = 0.9286,$$

the required approximation of $\ln 3.5$ *and*

$$|\ln 2.5 - p_3(0.5)| = |0.9163 - 0.9285| = 0.0122,$$

the possible absolute error in the approximation. •

Newton forward-difference formula

Newton's interpolatory divided-difference formula can be expressed in a simplified form when $x_0, x_1, \ldots, x_n$ are arranged consecutively with equal spacing. In this case, we introduce the notation $h = x_{i+1} - x_i$, for each $i = 0, 1, \ldots, n-1$ and let $x = x_0 + sh$. Then the difference $x - x_i$ can be written as $x - x_i = (s - i)h$. So Eq. (3.10) becomes

$$\begin{aligned} P_n(x) = P_n(x_0 + sh) &= f[x_0] + shf[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2] \\ &\quad + \cdots + s(s-1)(s-n+1)h^n f[x_0, x_1, \ldots, x_n] \\ &= \sum_{k=0}^{n} s(s-1)\cdots(s-k+1)h^k f[x_0, x_1, \ldots, x_k]. \end{aligned}$$

Using binomial-coefficient notation,

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!},$$

we can express $P_n(x)$ compactly as

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k! h^k f[x_0, x_i, \ldots, x_k]. \qquad (3.11)$$

by making use of the forward difference notation Δ

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}\Delta f(x_0)$$

$$f[x_0, x_1, x_2] = \frac{1}{2h}\left[\frac{\Delta f(x_1) - \Delta f(x_0)}{h}\right] = \frac{1}{2h^2}\Delta^2 f(x_0),$$

and, in general,

$$f[x_0, x_1, \dots, x_k] = \frac{1}{k!h^k}\Delta^k f(x_0).$$

Then, Eq. (3.11) has the following formula.

Newton Forward-Difference Formula

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0) \tag{3.12}$$

If the interpolating nodes are reordered as $x_n, x_{n-1}, \dots, x_0$, a formula similar to Eq. (3.10) results:

$$P_n(x) = f[x_n] + f[x_n, x_{n-1}](x - x_n) + f[x_n, x_{n-1}, x_{n-2}](x - x_n)(x - x_{n-1})$$
$$+ \cdots + f[x_n, \dots, x_0](x - x_n)(x - x_{n-1}) \cdots (x - x_1).$$

If the nodes are equally spaced with $x = x_n + sh$ and $x = x_i + (s + n - i)h$, then

$$\begin{aligned} P_n(x) &= P_n(x_n + sh) \\ &= f[x_n] + shf[x_n, x_{n-1}] + s(s+1)h^2 f[x_n, x_{n-1}, x_{n-2}] + \cdots \\ &\quad + s(s+1)\cdots(s+n-1)h^n f[x_n, \ldots, x_0]. \end{aligned}$$

which gives the following result.

Newton Backward–Difference Formula

$$P_n(x) = f[x_n] + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n)$$

by letting

$$\binom{-s}{k} = \frac{-s(-s-1)\cdots(-s-k+1)}{k!} = (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!},$$

$$\nabla p_n = p_n - p_{n-1}, \quad \text{for } n \geq 1.$$

Higher powers are defined recursively by

$$\nabla^k p_n = \nabla(\nabla^{k-1} p_n), \quad \text{for } k \geq 2.$$

EXAMPLE Table 3.2 lists values of a function at various points.

Table 3.2

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

The divided-difference Table

		First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

If an approximation to $f(1.1)$ is required, the reasonable choice for the nodes would be $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$, and $x_4 = 2.2$ since this choice makes the earliest possible use of the data points closest to $x = 1.1$, and also makes use of the fourth divided difference. This implies that $h = 0.3$ and $s = \frac{1}{3}$, so the Newton forward divided-difference formula is used with the divided differences that have a *solid* underscore in Table 3.9:

$$\begin{aligned}
P_4(1.1) &= P_4\left(1.0 + \frac{1}{3}(0.3)\right) \\
&= 0.7651997 + \frac{1}{3}(0.3)(-0.4837057) + \frac{1}{3}\left(-\frac{2}{3}\right)(0.3)^2(-0.1087339) \\
&\quad + \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(0.3)^3(0.0658784) \\
&\quad + \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)(0.3)^4(0.0018251) \\
&= 0.7196480.
\end{aligned}$$

To approximate a value when x is close to the end of the tabulated values, say, $x = 2.0$, we would again like to make the earliest use of the data points closest to x. This requires using the Newton backward divided-difference formula with $s = -\frac{2}{3}$ and the divided differences in Table 3.9 that have a *dashed* underscore:

$$\begin{aligned}
P_4(2.0) &= P_4\left(2.2 - \frac{2}{3}(0.3)\right) \\
&= 0.1103623 - \frac{2}{3}(0.3)(-0.5715210) - \frac{2}{3}\left(\frac{1}{3}\right)(0.3)^2(0.0118183) \\
&\quad - \frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)(0.3)^3(0.0680685) - \frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)(0.3)^4(0.0018251) \\
&= 0.2238754.
\end{aligned}$$

■

The Newton formulas are not appropriate for approximating $f(x)$ when x lies near the center of the table. A number of divided-difference formulas are available for this case

These methods are known as **centered-difference formulas.**

For the centered-difference formulas, we choose x_0 near the point being approximated and label the nodes directly below x_0 as $x_1, x_2, \ldots$ and those directly above as $x_{-1}, x_{-2}, \ldots$. With this convention, **Stirling's formula** is given by

$$P_n(x) = P_{2m+1}(x) = f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2h^2 f[x_{-1}, x_0, x_1] \quad (3.14)$$

$$+ \frac{s(s^2-1)h^3}{2} f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2])$$

$$+ \cdots + s^2(s^2-1)(s^2-4)\cdots(s^2-(m-1)^2)h^{2m} f[x_{-m}, \ldots, x_m]$$

$$+ \frac{s(s^2-1)\cdots(s^2-m^2)h^{2m+1}}{2}(f[x_{-m-1}, \ldots, x_m] + f[x_{-m}, \ldots, x_{m+1}]),$$

if $n = 2m+1$ is odd. If $n = 2m$ is even, we use the same formula but delete the last line. The entries used for this formula are underlined in Table 3.10

x	$f(x)$	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
x_{-2}	$f[x_{-2}]$				
		$f[x_{-2}, x_{-1}]$			
x_{-1}	$f[x_{-1}]$		$f[x_{-2}, x_{-1}, x_0]$		
		$\underline{f[x_{-1}, x_0]}$		$\underline{f[x_{-2}, x_{-1}, x_0, x_1]}$	
x_0	$\underline{f[x_0]}$		$\underline{f[x_{-1}, x_0, x_1]}$		$\underline{f[x_{-2}, x_{-1}, x_0, x_1, x_2]}$
		$\underline{f[x_0, x_1]}$		$\underline{f[x_{-1}, x_0, x_1, x_2]}$	
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$			
x_2	$f[x_2]$				

EXAMPLE Consider the table of data that was given in the previous examples. To use Stirling's formula to approximate $f(1.5)$ with $x_0 = 1.6$, we use the *underlined* entries in the difference Table 3.11.

Table 3.11

x	$f(x)$	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

The formula, with $h = 0.3$, $x_0 = 1.6$, and $s = -\frac{1}{3}$, becomes

$$
\begin{aligned}
f(1.5) &\approx P_4\left(1.6 + \left(-\frac{1}{3}\right)(0.3)\right) \\
&= 0.4554022 + \left(-\frac{1}{3}\right)\left(\frac{0.3}{2}\right)((-0.5489460) + (-0.5786120)) \\
&\quad + \left(-\frac{1}{3}\right)^2 (0.3)^2(-0.0494433) \\
&\quad + \frac{1}{2}\left(-\frac{1}{3}\right)\left(\left(-\frac{1}{3}\right)^2 - 1\right)(0.3)^3(0.0658784 + 0.0680685) \\
&\quad + \left(-\frac{1}{3}\right)^2\left(\left(-\frac{1}{3}\right)^2 - 1\right)(0.3)^4(0.0018251) \\
&= 0.5118200.
\end{aligned}
$$

■

Interpolation with Spline Functions

Definition 4.1 (Spline Function)

Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$. A function $s : [a, b] \to \mathbf{R}$ is a spline or spline function of degree m with points $x_0, x_1, \ldots, x_n$ if
1. A function s is a piecewise polynomial such that, on each subinterval $[x_k, x_{k+1}]$, s has degree at most m.
2. A function s is $m-1$ times differentiable everywhere. •

Piecewise Linear Interpolation

It is the one of the simplest piecewise polynomial interpolation for the approximation of the function,

Consider the set of seven data points (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5) and (x_6, y_6)

If we use a straight line on each subinterval (see Figure 4.4) then we can interpolate the data with a piecewise linear function, where

$$s_k(x) = p_k(x) = \frac{(x - x_{k+1})}{(x_k - x_{k+1})} y_k + \frac{(x - x_k)}{(x_{k+1} - x_k)} y_{k+1},$$

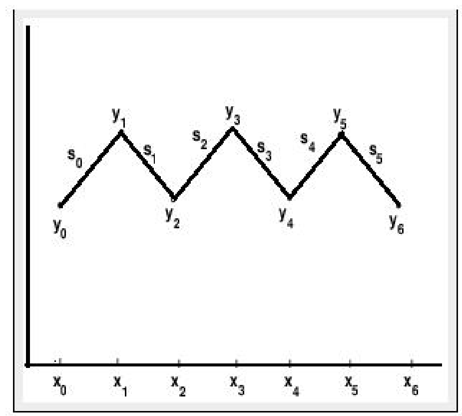


Figure 4.4: Linear spline.

or

$$s_k(x) = y_k + \frac{(y_{k+1} - y_k)}{(x_{k+1} - x_k)}(x - x_k).$$

It gives us

$$s_k(x) = A_k + B_k(x - x_k), \tag{4.59}$$

where the values of the coefficients A_k and B_k are given as

$$A_k = y_k \quad \text{and} \quad B_k = \frac{(y_{k+1} - y_k)}{(x_{k+1} - x_k)}. \tag{4.60}$$

Note that the linear spline must be continuous at given points $x_0, x_1, \ldots, x_n$ and

$$s(x_k) = f(x_k), \qquad \text{for} \quad k = 0, 1, \ldots, n.$$

Example 4.43 *Find the linear splines which interpolates the following data*

x	1	2	3	4
$f(x)$	1.0	0.67	0.50	0.40

Find the approximation of the function $f(x) = \dfrac{2}{x+1}$ *at* $x = 2.9$. *Compute absolute error.*

Solution. *Given* $x_0 = 1.0, x_1 = 2.0, x_2 = 3.0, x_3 = 4.0$, *then using (4.60), we have*

$$A_0 = y_0 = 1.0, \;\; A_1 = y_1 = 0.67, \;\; A_2 = y_2 = 0.50, \;\; A_3 = y_3 = 0.4,$$

and

$$B_0 = \frac{(y_1 - y_0)}{(x_1 - x_0)} = \frac{(0.67 - 1.0)}{(2.0 - 1.0)} = -0.33,$$

$$B_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(0.50 - 0.67)}{(3.0 - 2.0)} = -0.17,$$

$$B_2 = \frac{(y_3 - y_2)}{(x_3 - x_2)} = \frac{(0.40 - 0.50)}{(4.0 - 3.0)} = -0.10.$$

Now using (4.59), the linear splines for three subintervals are define as

$$s(x) = \begin{cases} s_0(x) = 1.0 - 0.33(x - 1.0) = 1.33 - 0.33x, & 1 \leq x \leq 2, \\ s_1(x) = 0.67 - 0.17(x - 2.0) = 1.01 - 0.17x, & 2 \leq x \leq 3, \\ s_2(x) = 0.50 - 0.10(x - 3.0) = 0.80 - 0.10x, & 3 \leq x \leq 4. \end{cases}$$

The value $x = 2.9$ *lies in the interval* $[2, 3]$*, so*

$$f(2.9) \approx s_1(2.9) = 1.01 - 0.17(2.9) = 0.517.$$

Also,

$$|f(2.9) - s_1(2.9)| = |0.513 - 0.517| = 0.004,$$

Cubic Spline

Spline interpolation is a form of interpolation where the interpolant is a special type of piecewise polynomial called a spline

Consider the problem of interpolating between the data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ by means of spline fitting.

Then the cubic spline $f(x)$ is such that

(i) $f(x)$ is a linear polynomial outside the interval (x_0, x_n)

(ii) $f(x)$ is a cubic polynomial in each of the subintervals,

(iii) $f'(x)$ and $f''(x)$ are continuous at each point.

Since $f(x)$ is cubic in each of the subintervals $f''(x)$ shall be linear.

$$f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3 \qquad (i = 1, 2, \ldots, n)$$

defines the cubic spline in interval i, $x_i \le x \le x_{i+1}$ $(i = 1, 2, \ldots, n)$.

$$\begin{aligned} f_i(x) = {} & \frac{f_i''}{6(x_{i+1} - x_i)}(x_{i+1} - x)^3 + \frac{f_{i+1}''}{6(x_{i+1} - x_i)}(x - x_i)^3 \\ & + \left[\frac{f_i}{x_{i+1} - x_i} - \frac{f_i''(x_{i+1} - x_i)}{6}\right](x_{i+1} - x) \\ & + \left[\frac{f_{i+1}}{x_{i+1} - x_i} - \frac{f_{i+1}''(x_{i+1} - x_i)}{6}\right](x - x_i) \end{aligned}$$

$$(x_i - x_{i-1})f_{i-1}'' + 2(x_{i+1} - x_{i-1})f_i'' + (x_{i+1} - x_i)f_{i+1}'' = 6\frac{f_{i+1} - f_i}{x_{i+1} - x_i} - 6\frac{f_i - f_{i-1}}{x_i - x_{i-1}}$$

$$f(x) = \frac{(x_{i+1} - x)^3 M_i}{6h} + \frac{(x - x_i)^3 M_{i+1}}{6h}$$

$$+ \frac{(x_{i+1} - x)}{h}\left(y_i - \frac{h^2}{6} M_i\right) + \frac{(x - x_i)}{h}\left(y_{i+1} - \frac{h^2}{6} M_{i+1}\right)$$

where $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$,

$i = 1, 2, 3, ..., (n-1)$

and $M_0 = 0$, $M_n = 0$, $x_{i+1} - x_i = h$.
which gives $n+1$ equations in $n+1$ unknowns $M_i (i = 0, 1, ..., n)$ which can be solved. Substituting the value of M_i gives the concerned cubic spline.

Ex. Obtain cubic spline for the following data:

x	0	1	2	3
y	2	-6	-8	2

Sol. Since points are equispaced with $h = 1$ and $n = 3$, the cubic spline can be determined from

$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), i = 1, 2$

also $M_0 = 0$, $M_3 = 0$

$\therefore$ for $i = 1$, $M_0 + 4M_1 + M_2 = 6(y_0 - 2y_1 + y_2)$

therefore, $4M_1 + M_2 = 36$; —(1)

for $i = 2$, $M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3)$

$M_1 + 4M_2 = 72$ —(2)

solving these, we get M_1 =4.8 and M_2 =16.8

Now the cubic spline in $(x_i \leq x \leq x_{i+1})$ is

$$f(x) = \frac{(x_{i+1} - x)^3 M_i}{6h} + \frac{(x - x_i)^3 M_{i+1}}{6h}$$

$$+ \frac{(x_{i+1} - x)}{h}\left(y_i - \frac{h^2}{6}M_i\right) + \frac{(x - x_i)}{h}\left(y_{i+1} - \frac{h^2}{6}M_{i+1}\right) \text{ —(3)}$$

Ex. The following values of x and y are given:

x	1	2	3	4
y	1	2	5	11

Find the cubic splines and evaluate $y(1.5)$ and $y'(3)$

Sol. Since points are equispaced with $h = 1$ and $n = 3$, the cubic spline can be determined from

$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), i = 1, 2$

also $M_0 = 0$, $M_3 = 0$

$\therefore$ for $i = 1$, $M_0 + 4M_1 + M_2 = 6(y_0 - 2y_1 + y_2)$

therefore, $4M_1 + M_2 = 12$; —(1)

for $i = 2$, $M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3)$

$M_1 + 4M_2 = 18$ —(2)

solving these, we get M_1 =2 and M_2 =4

Now the cubic spline in $(x_i \leq x \leq x_{i+1})$ is

$$f(x) = \frac{(x_{i+1} - x)^3 M_i}{6h} + \frac{(x - x_i)^3 M_{i+1}}{6h}$$

$$+ \frac{(x_{i+1} - x)}{h}\left(y_i - \frac{h^2}{6}M_i\right) + \frac{(x - x_i)}{h}\left(y_{i+1} - \frac{h^2}{6}M_{i+1}\right) \text{—(3)}$$