Chapter 4

Polynomial Interpolation and Approximation

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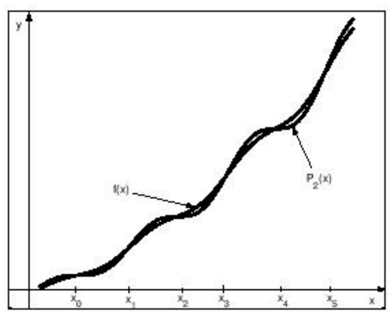
In this chapter we describe the numerical methods for the approximation of functions

We sometimes know the value of a function $f(x)$ at a set of points (say, $x_0 < x_1 < x_2 \cdots < x_n$)

but we do not have an analytic expression for $f(x)$

x	x_0	x_1	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_n)$

The task now is to estimate $f(x)$ for an arbitrary point x by, drawing a smooth curve through the data points x_i.



If the desired x is between the largest and smallest of the data point, then the problem is called *interpolation*;

if x is outside that range, it is called *extrapolation*.

Theorem 4.1 (Weierstrass Approximation Theorem)

If $f(x)$ is a continuous function in the closed interval $[a, b]$ then for every $\epsilon > 0$ there exists a polynomial $p_n(x)$, where the value of n depends on the value of ϵ, such that for all x in $[a, b]$,

$$|f(x) - p_n(x)| < \epsilon. \tag{4.2}$$

Consequently, any continuous function can be approximated to any accuracy by a polynomial of high enough degree. •

The general form of a *nth*-degree polynomial is $p_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

where n denotes the degree of the polynomial; and $a_0, a_1, \ldots, a_n$ are constants coefficients.

Polynomial Interpolation

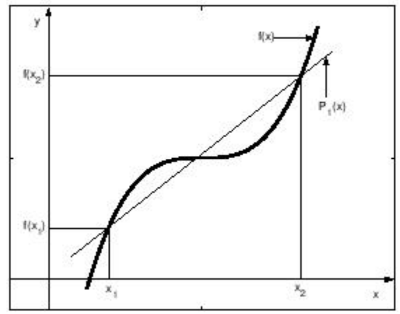
Suppose we have given a set of $(n + 1)$ data points relating a dependent variables $f(x)$ to an independent variable x as follows

x	x_0	x_1	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_n)$

Lagrange Interpolating Polynomials

It is one of the popular and well known interpolation method to approximate the functions at an arbitrary point x.

Linear Lagrange Interpolating Polynomial



Let us consider the construction of a linear polynomial $p_1(x)$ passing through two data points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

$$f(x) \approx p_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1), \qquad (4.5)$$

where

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}. \qquad (4.6)$$

Note that when $x = x_0$, then $L_0(x_0) = 1$ and $L_1(x_0) = 0$. Similarly, when $x = x_1$, then $L_0(x_1) = 0$ and $L_1(x_1) = 1$. The polynomial (4.5) is known as *linear Lagrange interpolating polynomial* and (4.6) is called the *Lagrange coefficient polynomials.*

Quadratic Lagrange Interpolating Polynomial

When $p_2(x)$ passes through three points $(x_0, f(x_0)), (x_1, f(x_1))$ and $(x_2, f(x_2))$, we have quadratic Lagrange polynomial as follows

$$f(x) \approx p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{4.7}$$

where the Lagrange coefficients are define as follows:

$$\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \\ L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}, \\ L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}. \end{aligned} \tag{4.8}$$

Example 4.5 *Consider the following table:*

x	0	3	7
$f(x)$	2	4	19

(a) Construct quadratic Lagrange polynomial $p_2(x) = ax^2 + bx + c$ *to approximate* $f(x)$.
(b) Use the polynomial in part (a) to interpolate $f(x)$ *at* $x = 4$.

Solution. *(a) Obviously, a quadratic polynomial can be determined so that it passes through the three points. Consider the quadratic Lagrange interpolating polynomial as follows:*

$$p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{4.20}$$

or

$$p_2(x) = 2L_0(x) + 4L_1(x) + 19L_2(x). \tag{4.21}$$

The Lagrange coefficients can be calculate as follows:

$$\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{1}{21}(x^2 - 10x + 21), \\ L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -\frac{1}{12}(x^2 - 7x), \\ L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{1}{28}(x^2 - 3x). \end{aligned}$$

Putting these values of the Lagrange coefficients in (4.21), we have

$$p_2(x) = \frac{1}{84}(37x^2 - 55x + 168),$$

(with $a = 37/84, b = -55/84, c = 2$) which is the required quadratic Lagrange polynomial.
(b) Now take $x = 4$ in the above polynomial, we obtain

$$p_2(4) = \frac{1}{84}\Big[37(4)^2 - 55(4) + 168\Big] = 6.4286,$$

which is the required estimate value of $f(4)$. •

Example 4.4 *Let* $f(x) = x + \frac{1}{x}$, *with points* $x_0 = 1, x_1 = 1.5, x_2 = 2.5$ *and* $x_3 = 3$. *Find the quadratic Lagrange polynomial for the approximation of* $f(2.7)$. *Also, find the relative error.*

Solution. *Consider the quadratic Lagrange interpolating polynomial as follows:*

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2).$$

Since the given interpolating point is $x = 2.7$, *therefore, the best three points for the quadratic polynomial should be as follows:*

$$x_0 = 1.5,\ f(x_0) = 2.1667, \quad x_1 = 2.5,\ f(x_1) = 2.9, \quad x_2 = 3,\ f(x_2) = 3.3333.$$

So using these values, we have

$$f(x) = p_2(x) = 2.1667L_0(x) + 2.9L_1(x) + 3.3333L_2(x),$$

where

$$L_0(x) = \frac{(x-2.5)(x-3)}{(1.5-2.5)(1.5-3)} = \frac{1}{1.5}(x^2 - 5.5x + 7.5),$$

$$L_1(x) = \frac{(x-1.5)(x-3)}{(2.5-1.5)(2.5-3)} = \frac{1}{-0.5}(x^2 - 4.5x + 4.5),$$

$$L_2(x) = \frac{(x-1.5)(x-2.5)}{(3-1.5)(3-2.5)} = \frac{1}{0.75}(x^2 - 4x + 3.75).$$

Using these Lagrange coefficients in the polynomial and after simplifying, gives

$$f(x) = p_2(x) = 0.0889x^2 + 0.3776x + 1.4003,$$

which is the required quadratic polynomial, and $f(2.7) \approx p_2(2.7) = 3.0679$. *The relative error is*

$$\frac{|f(2.7) - p_2(2.7)|}{|f(2.7)|} = \frac{|3.0704 - 3.0679|}{|3.0704|} = 0.0008.$$

Nth Degree Lagrange Interpolating Polynomial

To generalize the concept of the Lagrange interpolation, consider the construction of a polynomial $p_n(x)$ of degree at most n that passes through $(n+1)$ distinct points $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$ and satisfy the interpolation conditions $p_n(x_k) = f(x_k); \qquad k = 0, 1, 2, \ldots, n.$

Assume that there exists polynomial $L_k(x)$ $(k = 0, 1, 2, \ldots, n)$ of degree n having the property

$$L_k(x_j) = \begin{cases} 0 & \text{for} \quad k \neq j \\ 1 & \text{for} \quad k = j \end{cases} \tag{4.12}$$

and

$$\sum_{k=0}^{n} L_k(x) = 1. \tag{4.13}$$

The polynomial $p_n(x)$ is given by

$$\begin{aligned} f(x) \approx p_n(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + \cdots + L_{i-1}(x)f(x_{i-1}) \\ &+ L_i(x)f(x_i) + \cdots + L_n(x)f(x_n) = \sum_{k=0}^{n} L_k(x)f(x_k). \end{aligned} \tag{4.14}$$

$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} = \prod_{k=0}^{n} \left(\frac{x-x_k}{x_i-x_k}\right), \quad i \neq k.$$

the Lagrange interpolation formula of degree n

$$f(x) \approx p_n(x) = \sum_{i=0}^{n} \prod_{k=0}^{n} \left(\frac{x-x_k}{x_i-x_k}\right) f(x_i), \qquad i \neq k.$$

Uniqueness of Lagrange Interpolating Polynomial

To show the *uniqueness* of the interpolating polynomial $p_n(x)$, we suppose that in addition to the polynomial $p_n(x)$ the interpolation problem has another solution $q_n(x)$ of degree $\leq n$ whose graph passes through $(x_i, y_i),\ i = 0, 1, \ldots, n$. Then define

$$r_n(x) = p_n(x) - q_n(x),$$

of the degree not greater than n. Since

$$r_n(x_i) = p_n(x_i) - q_n(x_i) = f(x_i) - f(x_i) = 0,$$

the polynomial $r_n(x)$ vanishes at $n + 1$ point. But by using the following well known result from the theory of equations: *"If a polynomial of degree n vanishes at $n + 1$ distinct points, then the polynomial is identically zero"*. Hence $r_n(x)$ vanishes identically, or equivalently, $p_n(x) = q_n(x)$.

Error Formula of Lagrange Polynomial

All can be said with certainty is that $f(x) - p_n(x) = 0$ at $x = x_0, x_1, \ldots, x_n$.

However, it is sometimes possible to obtain a bound on the error $f(x) - p_n(x)$ at an intermediate point x using the following theorem.

Theorem 4.2 (Error Formula of Nth Degree Lagrange Polynomial)

If $f(x)$ has $(n+1)$ derivatives on interval I and if it is approximated by a polynomial $p_n(x)$ passing through $(n+1)$ data points on I, then the error E_n is given by

$$E_n = f(x) - p_n(x) = \frac{f^{(n+1)}(\eta(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n), \quad \eta(x) \in I, \tag{4.30}$$

where $p_n(x)$ is Lagrange interpolating polynomial (4.14) and a unknown point $\eta(x) \in (x_0, x_n)$. •

Error Formulas of Linear, Quadratic and Cubic Lagrange Polynomials

$$E_1 = f(x) - p_1(x) = \frac{f''(\eta(x))}{2!}(x-x_0)(x-x_1), \quad \eta(x) \in I,$$

where $p_1(x)$ is the linear Lagrange polynomial (4.5) and a unknown point $\eta(x) \in (x_0, x_1)$.

$$E_2 = f(x) - p_2(x) = \frac{f'''(\eta(x))}{3!}(x-x_0)(x-x_1)(x-x_2), \quad \eta(x) \in I,$$

where $p_2(x)$ is the quadratic Lagrange polynomial (4.7) and a unknown point $\eta(x) \in (x_0, x_2)$.

$$E_3 = f(x) - p_3(x) = \frac{f^{(4)}(\eta(x))}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3), \quad \eta(x) \in I,$$

where $p_3(x)$ is the cubic Lagrange polynomial (4.9) and a unknown point $\eta(x) \in (x_0, x_3)$.

Example 4.16 *Construct the Lagrange interpolating polynomial of degree 2 for* $f(x) = x\ln x + e^{-x}$ *on the interval* $[2,4]$ *with the points* $x_0 = 2.0,\ x_1 = 3.0,\ x_2 = 4.0$. *Find a bound for the absolute error.*

Solution. *Thus, the following table is given:*

x	2.0	3.0	4.0
$f(x)$	1.5216	3.3456	5.5635

Consider a quadratic Lagrange interpolating polynomial as

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{4.30}$$

$$f(x) = p_2(x) = L_0(x)f(2) + L_1(x)f(3) + L_2(x)f(4).$$

$$f(x) = p_2(x) = 1.5216L_0(x) + 3.3456L_1(x) + 5.5635L_2(x). \tag{4.31}$$

We construct the basic Lagrange polynomials:

$$L_0(x) = \frac{(x-3)(x-4)}{(2-3)(2-4)} = \frac{(x^2-7x+12)}{2},$$

$$L_1(x) = \frac{(x-2)(x-4)}{(3-2)(3-4)} = \frac{(x^2-6x+8)}{-1},$$

$$L_2(x) = \frac{(x-2)(x-3)}{(4-2)(4-3)} = \frac{(x^2-5x+6)}{2}.$$

Putting these values of the Lagrange coefficients in (4.31), we have

$$f(x) = p_2(x) = 0.1970x^2 + 0.8392x - 0.9447,$$

which is the required quadratic polynomial. The error is given by

$$f(x) - p_2(x) = \frac{f^{(3)}(\eta(x))}{3!}(x-x_0)(x-x_1)(x-x_2).$$

$$f(x) = x \ln x + e^{-x} \qquad \textit{and} \qquad f^{(3)}(x) = -\frac{1}{x^2} - e^{-x},$$

and the maximum of the third derivative is at 2, so

$$|f^{(3)}(\eta(x))| \leq M = \max_{0 \leq x \leq 2} |f^{(3)}(x)| = \max_{0 \leq x \leq 2} |-\frac{1}{x^2} - e^{-x}| = 0.3853.$$

Next we bound $|g(x)|$, *where* $g(x) = (x-2)(x-3)(x-4)$. *To find the maximum of* $|g(x)|$, *we need to take the derivative:*

$$g(x) = x^3 - 9x^2 + 26x - 24, \quad g'(x) = 3x^2 - 18x + 26 = 0, \quad \textit{gives} \quad p_1 = 3.5774, \; p_2 = 2.4226,$$

and

$$|g(x)| = |g(3.5774)| = |-0.3849| = 0.3849 \quad \textit{and} \quad |g(x)| = |g(2.4226| = |0.3849| = 0.3849.$$

Thus, obtaining the error bound:

$$|f(x) - p_2(x)| = \frac{|f^{(3)}(\eta(x))|}{3!} |(x-x_0)(x-x_1)(x-x_2)| \leq \frac{0.3853}{6}(0.3849) = 2.47 \times 10^{-3},$$

which is desired bound for the absolute error. •

Example 4.17 *Use the quadratic Lagrange interpolating polynomial by selecting the best three points from* $\{-2, 0, 1, 2, 2.5\}$ *on the function defined by* $f(x) = (x+1)^{\frac{1}{3}}$ *to estimate the cube root of* $\frac{3}{2}$ *(that is,* $(\frac{3}{2})^{\frac{1}{3}}$*) and compute an error bound and absolute error.*

Solution. *Since the given function is a cube root of* $(x+1)$*, so by taking* $x+1 = \frac{3}{2}$*, we have* $x = \frac{1}{2}$*, therefore, the best points for the quadratic polynomial are* $x_0 = 0, x_1 = 1$*, and* $x_2 = 2$*. Consider a quadratic Lagrange interpolating polynomial as*

$$f(x) \approx p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{4.32}$$

and at $x = 0.5$*, gives*

$$f(0.5) \approx p_2(0.5) = (1)^{1/3}L_0(0.5) + (2)^{1/3}L_1(0.5) + (3)^{1/3}L_2(0.5). \tag{4.33}$$

The Lagrange coefficients can be calculate as follows:

$$\begin{aligned}
L_0(0.5) &= \frac{(0.5-1)(0.5-2)}{(0-1)(0-2)} = 0.375, \\
L_1(0.5) &= \frac{(0.5-0)(0.5-2)}{(1-0)(1-2)} = 0.75, \\
L_2(0.5) &= \frac{(0.5-0)(0.5-1)}{(2-0)(2-1)} = -0.125.
\end{aligned}$$

Putting these values of the Lagrange coefficients in (4.33), we have

$$f(0.5) \approx p_2(0.5) = (1)^{1/3}(0.375) + (2)^{1/3}(0.75) - (3)^{1/3}(0.125) = 1.1396,$$

which is the required approximation of the $\left(\frac{3}{2}\right)^{1/3}$.

To compute an error bound for the approximation of the given function in the interval $[0, 2]$, u use the following quadratic error formula

$$|f(x) - p_2(x)| = \frac{|f^{(3)}(\eta(x))|}{3!}|(x - x_0)(x - x_1)(x - x_2)|.$$

As

$$|f^{(3)}(\eta(x))| \leq M = \max_{0 \leq x \leq 2} |f^{(3)}(x)|,$$

and

$$f'(x) = \frac{1}{3}(x+1)^{-2/3}, \quad f''(x) = -\frac{2}{9}(x+1)^{-5/3}, \quad f^{(3)}(x) = \frac{10}{27}(x+1)^{-8/3},$$

so

$$M = \max_{0 \leq x \leq 2} \left| \frac{10}{27}(x+1)^{-8/3} \right| = \frac{10}{27}.$$

Hence

$$|f(0.5) - p_2(0.5)| \leq \frac{10/27}{6}|(0.5 - 0)(0.5 - 1)(0.5 - 2)|,$$

and it gives

$$|f(0.5) - p_2(0.5)| \leq \frac{10(0.375)}{162} = 0.0232,$$

which is desired error bound. Also, we have the absolute error is given as

$$|f(0.5) - p_2(0.5)| = |(1.5)^{1/3} - 1.1396| = |1.1447 - 1.1396| = 0.0051,$$

Example 4.18 *Consider* $f(x) = \sin x$ *and its values are known at five points* $\{0, 0.2, 0.4, 0.6, 0.8\}$. *If the approximation of* $\sin 0.28$ *by four degree Lagrange interpolating polynomial is* 0.2763591, *then compute the error bound and the absolute error for the approximation.*

Solution. *To compute an error bound for the approximation of the given function in the interval* $[0, 0.8]$, *we use the following error formula for Lagrange polynomial degree four*

$$|f(x) - p_4(x)| = \frac{|f^{(5)}(\eta(x))|}{5!}|(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)|,$$

or

$$|f(x) - p_4(x)| \leq \frac{M}{5!}|(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)|.$$

Since

$$|f^{(5)}(\eta(x))| \leq M = \max_{0 \leq x \leq 0.8} |f^{(5)}(x)| = \max_{0 \leq x \leq 0.8} |\cos x| = 1,$$

so

$$|f(0.28) - p_4(0.28)| \leq \frac{1}{120}|0.28(0.28-0.2)(0.28-0.4)(0.28-0.6)(0.28-0.6)(0.28-0.8)| \leq 3.7 \times 10^{-6},$$

which is desired error bound. Also, we have to compute absolute error as

$$|f(0.28) - p_4(0.28)| = |\sin 0.28 - p_4(0.28)| = |0.2763556 - 0.2763591| = 3.5 \times 10^{-6},$$

which is desired result. •

Example 4.13 *Let* $f(x) = \dfrac{1}{x}$ *be defined in the interval* $[2,4]$ *and* $x_0 = 2,\ x_1 = 2.5,\ x_2 = 4.$ *Compute the value of the unknown point* η *in the error formula of quadratic Lagrange interpolating polynomial for the approximation of* $f(3)$ *using the given points* x_0, x_1, x_2.

Solution. *Consider the quadratic Lagrange interpolating polynomial as follows:*

$$p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2),$$

At the given values of $x_0 = 2, x_1 = 2.5, x_2 = 4$, *we have,* $f(2) = 1/2, f(2.5) = 1/2.5$ *and* $f(4) = 1/4$, *so using* $x = 3$, *we have*

$$f(3) \approx p_2(3) = (1/2)L_0(3) + (1/2.5)L_1(3) + (1/5)L_2(3).$$

Then

$$L_0(3) = \frac{(3-2.5)(3-4)}{(2-2.5)(2-4)} = -\frac{1}{2},\ L_1(3) = \frac{(3-2)(3-4)}{(2.5-2)(2.5-4)} = \frac{4}{3},\ L_2(3) = \frac{(3-2)(3-2.5)}{(4-2)(4-2.5)} = \frac{1}{6}.$$

$$f(3) \approx p_2(3) = (1/2)(-1/2) + (1/2.5)(4/3) + (1/4)(1/6) = 0.325,$$

which is the required approximation of $f(3)$ *by the quadratic interpolating polynomial.*
The error is

$$f(3) - p_2(3) = 1/3 - 0.325 = 0.0083.$$

Since the error formula of the quadratic Lagrange polynomial is

$$E = f(x) - p_2(x) = \frac{f'''(\eta)}{3!}(x - x_0)(x - x_1)(x - x_2), \quad \eta \in I,$$

and the third derivative of f *is,* $f'(\eta) = -1/\eta^2, \quad f''(\eta) = 2/\eta^3, \quad f'''(\eta) = -6/\eta^4$. *Thus*

$$0.0083 = f(3) - p_2(3) = \left(\frac{(3-2)(3-2.5)(3-4)}{6} \right) \frac{(-6)}{(\eta^4)} = \frac{0.5}{\eta^4},$$

and solving for η, *we get,* $\eta^4 = 60.2410, \quad \eta^2 = 7.7615, \quad \eta = 2.7859 \in (2, 4)$, *required value of the unknown point* η. •

Theorem 4.3 (Error Bounds for Lagrange Interpolation at Equally Spaced Points)

Assume that $f(x)$ is defined on the interval $[a,b]$, which contains equally spaced points $x_k = x_0 + hk$. Additionally, assume that $f(x)$ and the derivatives of $f(x)$ up to the order $(n+1)$, are continuous and bounded on the special intervals $[x_0, x_1]$, $[x_0, x_2]$ and $[x_0, x_3]$, respectively; that is

$$|f^{(n+1)}(x)| \leq M \quad \textit{for} \quad x_0 \leq x \leq x_n,$$

for $n = 1, 2, 3$. Then error bounds for linear, quadratic and cubic polynomials are:

$$\begin{aligned} |E_1(x)| &\leq \frac{h^2}{8} M \qquad \textit{for} \qquad x_0 \leq x \leq x_1, \\ |E_2(x)| &\leq \frac{h^3}{9\sqrt{3}} M \qquad \textit{for} \qquad x_0 \leq x \leq x_2, \\ |E_3(x)| &\leq \frac{h^4}{24} M \qquad \textit{for} \qquad x_0 \leq x \leq x_3. \end{aligned}$$

Continue in the similar manner for the interval $[x_0, x_n]$, for $n = 1, 2, \ldots, n$, we have

$$|E_n(x)| \leq \frac{M}{4(n+1)} \left(\frac{b-a}{n}\right)^{n+1} = \frac{M}{4(n+1)} h^{n+1}, \quad \textit{for} \quad x_0 \leq x \leq x_n,$$

the general error bound formula. •

Example 4.20 *Find an error bound if $f(x) = \sin x$ is approximated by an interpolation polynomial with ten equally spaced data points in $[0, 1.6875]$.*

Solution. *Given $n = 9$ and $a = 0, b = 1.6875$,*

Note that $f^{(n)}(x) = \pm \sin x$ for even n and $f^{(n)}(x) = \pm \cos x$ for odd n, so we have a uniform bound on $f^{(n)}(x)$ for all n. That is $|f^{(n)}(x)| \leq 1$ for all x and for all n.

$$M = \max_{0 \leq x \leq 1.6875} |f^{(10)}(x)| = \max_{0 \leq x \leq 1.6875} |-\sin x| \leq 1, \quad \forall\ x \in [0, 1.6875].$$

Hence, the interpolation error (use Theorem 4.36) can be bounded by

$$|E_9(x)| = |\sin x - p_9(x)| \leq \frac{1}{40}\left(\frac{1.6875}{9}\right)^{10} \approx 1.34 \times 10^{-9},$$

for all $x \in [0, 1.6875]$. •

Example 4.25 *(a) Let $f(x) = (x+1)\ln(x+1)$ be the function defined over the interval $[1,2]$. Find the approximations of $(2.9\ln 2.9)$ using linear, quadratic and cubic Lagrange interpolating polynomials for equally spaced data points defined over the interval $[1,2]$. Compute the error bounds for linear, quadratic and cubic Lagrange interpolating polynomials for equally spaced data points. Also, compute absolute error for each case.*
(b) Determine the step size h and the number of points to be used in the tabulation of the given function $f(x) = (x+1)\ln(x+1)$ in $[1,2]$ so that linear, quadratic and cubic interpolations will be correct to 5×10^{-6}

Solution. *(a) For linear Lagrange polynomial, we have $h = 2-1 = 1$, so using $x_0 = 1, x_1 = 2$ and $x = 1.9$, in the linear Lagrange formula, we have*

$$f(1.9) \approx p_1(1.9) = L_0(1.9)f(1) + L_1(1.9)f(2) = 1.3863L_0(1.9) + 3.2958L_1(1.9). \quad (4.38)$$

The Lagrange coefficients can be calculate as follows:

$$L_0(1.9) = \frac{(1.9-2)}{(1-2)} = 0.1 \quad \text{and} \quad L_1(1.9) = \frac{(1.9-1)}{(2-1)} = 0.9.$$

Putting these values of the Lagrange coefficients in (4.38), we have

$$f(1.9) \approx p_1(1.9) = 1.3863(0.1) + 3.2958(0.9) = 3.1049.$$

Now for quadratic Lagrange polynomial, take $h = \dfrac{2-1}{2} = 0.5$, *using* $x_0 = 1, x_1 = 1.5, x_2 = 2$ *and* $x = 1.9$, *in the quadratic Lagrange formula, we have*

$$\begin{aligned} f(1.9) \approx p_2(1.9) &= L_0(1.9)f(1) + L_1(1.9)f(1.5) + L_2(1.9)f(2) \\ &= 1.3863L_0(1.9) + 2.2907L_1(1.9) + 3.2958L_2(1.9). \end{aligned} \tag{4.39}$$

The Lagrange coefficients can be calculate as follows:

$$\begin{aligned} L_0(1.9) &= \frac{(1.9-1.5)(1.9-2)}{(1-1.5)(1-2)} = -0.08, \\ L_1(1.9) &= \frac{(1.9-1)(1.9-2)}{(1.5-1)(1.5-2)} = 0.36, \\ L_2(1.9) &= \frac{(1.9-1)(1.9-1.5)}{(2-1)(2-1.5)} = 0.72. \end{aligned}$$

Putting these values of the Lagrange coefficients in (4.39), we have

$$f(1.9) \approx p_2(1.9) = 1.3863(-0.08) + 2.2907(0.36) + 3.2958(0.72) = 3.0867.$$

For cubic Lagrange polynomial, we have $h = \frac{2-1}{3} = 1/3$, *so using* $x_0 = 1, x_1 = 4/3, x_2 = 5/3$, $x_3 = 2$ *and* $x = 1.9$, *in the cubic Lagrange formula, we have*

$$\begin{aligned} f(1.9) \approx p_3(1.9) &= L_0(1.9)f(1) + L_1(1.9)f(4/3) + L_2(1.9)f(5/3) + L_3(1.9)f(2) \\ &= 1.3863L_0(1.9) + 1.9770L_1(1.9) + 2.6156L_2(1.9) + 3.2958L_3(1.9). \quad (4.40) \end{aligned}$$

The Lagrange coefficients can be calculate as follows:

$$L_0(1.9) = \frac{(1.9-4/3)(1.9-5/3)(1.9-2)}{(1-4/3)(1-5/3)(1-2)} = 0.0595,$$

$$L_1(1.9) = \frac{(1.9-1)(1.9-5/3)(1.9-2)}{(4/3-1)(4/3-5/3)(4/3-2)} = -0.2835,$$

$$L_2(1.9) = \frac{(1.9-1)(1.9-4/3)(1.9-2)}{(5/3-1)(5/3-4/3)(5/3-2)} = 0.6885,$$

$$L_3(1.9) = \frac{(1.9-1)(1.9-4/3)(1.9-5/3)}{(2-1)(2-4/3)(2-5/3)} = 0.5355.$$

Putting these values of the Lagrange coefficients in (4.40), we have

$$f(1.9) \approx p_3(1.9) = 1.386(0.0595) + 1.977(-0.2835) + 2.616(0.6885) + 3.296(0.5355) = 3.0881.$$

The derivatives of the given function $f(x) = (x+1)\ln(x+1)$ *are as follows:*

$$f'(x) = 1 + \ln(x+1), \quad f''(x) = \frac{1}{x+1}, \quad f'''(x) = -\frac{1}{(x+1)^2}, \quad f^{(4)}(x) = \frac{2}{(x+1)^3}.$$

Now for error bound of linear Lagrange polynomial, we use the formula

$$|E_1| \leq \frac{Mh^2}{8},$$

where $h = 2 - 1 = 1$ *and* $M = \max_{1 \leq x \leq 2} |f''(x)| = \max_{1 \leq x \leq 2} \left| \frac{1}{x+1} \right| = \frac{1}{2}$. *So*

$$|E_1| \leq \frac{(1/2)(1)^2}{8} = \frac{1}{16} = 0.0625,$$

the error bound for the linear Lagrange polynomial.

Similarly, for error bound of quadratic Lagrange polynomial, we use the formula

$$|E_2| \leq \frac{Mh^3}{9\sqrt{3}},$$

where $h = (2-1)/2 = 1/2$ *and* $M = \max_{1 \leq x \leq 2} |f'''(x)| = \max_{1 \leq x \leq 2} \left| -\frac{1}{(x+1)^2} \right| = \frac{1}{4}$. *So*

$$|E_2| \leq \frac{(1/4)(1/2)^3}{9\sqrt{3}} = \frac{(1/32)}{9\sqrt{3}} = 0.0020,$$

the error bound for the quadratic Lagrange polynomial.
Finally, for error bound of cubic Lagrange polynomial, we use the formula

$$|E_3| \leq \frac{Mh^4}{24},$$

where $h = (2-1)/3 = 1/3$ *and* $M = \max_{1 \leq x \leq 2} |f^{(4)}(x)| = \max_{1 \leq x \leq 2} \left| \frac{2}{(x+1)^3} \right| = \frac{1}{4}$. *Thus*

$$|E_3| \leq \frac{(1/4)(1/3)^4}{24} = \frac{1}{7776} = 0.0001,$$

the error bound for the cubic Lagrange polynomial. Finally,

$$|f(1.9) - p_1(1.9)| = |3.0877 - 3.1049| = 0.0172,$$

$$|f(1.9) - p_2(1.9)| = |3.0877 - 3.0867| = 0.0010,$$

$$|f(1.9) - p_3(1.9)| = |3.0877 - 3.0881| = 0.0004,$$

are respectively, the absolute error for linear, quadratic and cubic polynomials.
(b) Since we know that the upper bound of error in linear polynomial is

$$|E_1| \leq \frac{Mh^2}{8} \quad and \quad M = \frac{1}{2},$$

therefore,

$$\frac{h^2}{16} \leq 5 \times 10^{-6}, \; h \leq 0.0089, \quad n = 113.$$

As the upper bound of error in quadratic polynomial is

$$|E_2| \leq \frac{Mh^3}{9\sqrt{3}} \quad and \quad M = \frac{1}{4},$$

therefore,

$$\frac{h^3}{36\sqrt{3}} \leq 5 \times 10^{-6}, \quad or \quad h^3 \leq 311.7691 \times 10^{-6}, \ h \leq 0.0678 \ and \ n = 14.7476 \approx 15.$$

Finally, as the upper bound of error in cubic polynomial is

$$|E_3| \leq \frac{Mh^4}{24} \ and \ M = \frac{1}{4}, \ \frac{h^4}{96} \leq 5 \times 10^{-6}, \ or \ h^4 \leq 480 \times 10^{-6}.$$

This gives, [$h \leq 0.1480$ *and* $n = 6.7560 \approx 7$. *Thus we need, respectively,* 114 *points,* 16 *points and* 8 *points for the linear, quadratic and cubic interpolations.* •

Newton's General Interpolating Formula

Since we noted in the previous section that for a small number of data point one can easily use the Lagrange formula of the interpolating polynomial. However, for a large number of data points there will be many multiplication and more significantly, whenever a new data point is added to an existing set, the interpolating polynomial has to be completely recalculated. Here, we describe an efficient way of organizing the calculations so as to overcome these disadvantages.
Let us consider the *nth*-degree polynomial $p_n(x)$ that agrees with the function $f(x)$ at the distinct numbers $x_0, x_1, \ldots, x_n$. The divided differences of $f(x)$ with respect to $x_0, x_1, \ldots, x_n$ are derived to express $p_n(x)$ in the form

$$\begin{aligned} p_n(x) = a_0 &+ a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \\ &+ a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}), \end{aligned} \tag{4.41}$$

for appropriate constants $a_0, a_1, \ldots, a_n$.
Now to determine the constants, firstly, by evaluating $p_n(x)$ at x_0, we have

$$p_n(x_0) = a_0 = f(x_0) \tag{4.42}$$

Similarly, when $p_n(x)$ is evaluated at x_1, then

$$p_n(x_1) = a_0 + a_1(x_1 - x_0) = f(x_1),$$

which implies that

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \tag{4.43}$$

Divided Differences

Firstly, we define the *Zeroth divided difference* at the point x_i by

$$f[x_i] = f(x_i), \tag{4.44}$$

which is simply the value of the function $f(x)$ at x_i.

The *first-order* or *first divided difference* at the points x_i and x_{i+1} can be defined by

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}. \tag{4.45}$$

In general, the *nth divided difference* $f[x_i, x_{i+1}, \ldots, x_{i+n}]$ is defined by

$$f[x_i, x_{i+1}, \ldots, x_{i+n}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+n}] - f[x_i, x_{i+1}, \ldots, x_{i+n-1}]}{x_{i+n} - x_i}. \tag{4.46}$$

By using this definition, (4.42) and (4.43) can be written as

$$a_0 = f[x_0]; \qquad a_1 = f[x_0, x_1],$$

respectively. Similarly, one can have the values of other constants involving in (4.41) such as

$$\begin{aligned} a_2 &= f[x_0, x_1, x_2], \\ a_3 &= f[x_0, x_1, x_2, x_3], \\ \cdots &= \quad \cdots \\ \cdots &= \quad \cdots \\ a_n &= f[x_0, x_1, \ldots, x_n]. \end{aligned}$$

Table 4.1: Divided difference table for a function $y = f(x)$

k	x_k	Zero Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	x_0	$f[x_0]$			
1	x_1	$f[x_1]$	$f[x_0, x_1]$		
2	x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
3	x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$

Linear Newton's Interpolating Polynomial

The linear Newton's interpolating polynomial passing through two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ can be written as

$$f(x) \approx p_1(x) = f[x_0] + (x - x_0)f[x_0, x_1].$$

Quadratic Newton's Interpolating Polynomial

The quadratic Newton's interpolating polynomial passing through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ can be written in terms of divided differences as

$$p_2(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2].$$

This polynomial can also be written as

$$f(x) \approx p_2(x) = p_1(x) + (x - x_0)(x - x_1)f[x_0, x_1, x_2],$$

Nth Degree Newton's Interpolating Polynomial

Repeating this entire process again, $p_3(x), p_4(x)$ and higher degree interpolating polynomials can be consecutively obtained in the same way. In general, the interpolating polynomial $p_n(x)$ passing through the points $(x_i, f(x_i))(i = 0, 1, \ldots, n)$, can be written in terms of divided differences as

$$f(x) \approx \begin{aligned} p_n(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ \cdots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}), \end{aligned} \tag{4.49}$$

Theorem 4.4 (Newton's Interpolating Polynomial)

Suppose that $x_0, x_1, \ldots, x_n$ are $(n+1)$ distinct points in the interval $[a, b]$. There exists a unique polynomial $p_n(x)$ of degree at most n with the property that

$$f(x_i) = p_n(x_i), \quad \textit{for} \quad i = 0, 1, \ldots, n.$$

The Newton's form of this polynomial is

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}),$$

where

$$a_k = f[x_0, x_1, x_2, \cdots, x_k], \quad \textit{for} \quad k = 0, 1, 2, \ldots, n.$$

Example 4.29 *Consider the following table of date points*

x	3	1	5	6
$f(x)$	1	-3	2	4

Find the third divided difference $f[3,1,5,6]$ and use it to find the Newton's form of the interpolating polynomial. Find approximation of $f(2)$.

Solution. *The third divided differences for the given data points are listed in Table 4.5. The cubic*

Table 4.5: Divided difference table for a function $y = f(x)$

k	x_k	Zero Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	$x_0 = 3$	$f[x_0] = 1$			
1	$x_1 = 1$	$f[x_1] = -3$	$f[x_0, x_1] = 2$		
2	$x_2 = 5$	$f[x_2] = 2$	$f[x_1, x_2] = 5/4$	$f[x_0, x_1, x_2] = -3/8$	
3	$x_3 = 6$	$f[x_3] = 4$	$f[x_2, x_3] = 2$	$f[x_1, x_2, x_3] = 3/20$	$f[x_0, x_1, x_2, x_3] = 7/40$

Newton's interpolating polynomial passing through the given can be written as

$$p_3(x) = f[x_0]+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]+(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3],$$

so using Table 4.5, we have

$$p_3(x) = 1 + 2(x - x_0) - \frac{3}{8}(x - x_0)(x - x_1) + \frac{7}{40}(x - x_0)(x - x_1)(x - x_2),$$

or

$$p_3(x) = \frac{1}{40}[7x^3 - 78x^2 + 301x - 350].$$

Thus at $x = 2$, we get

$$f(2) \approx p_3(2) = \frac{1}{40}[7(2)^3 - 78(2)^2 + 301(2) - 350] = -\frac{1}{10},$$

Example 4.28 *Find the Lagrange and the Newton forms of the interpolating polynomial for the following data*

x	0	1	3
$f(x)$	1	2	3

Write both polynomials in the form $a + bx + cx^2$ *to verify that they are identical as functions.*

Solution. *With* $x_0 = 0, x_1 = 1$ *and* $x_2 = 3$, *we obtain the quadratic Lagrange interpolating polynomial*

$$f(x) = p_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

$$= \frac{(x-1)(x-3)}{(0-1)(0-3)}(1) + \frac{(x-0)(x-3)}{(1-0)(1-3)}(2) + \frac{(x-0)(x-1)}{(3-0)(3-1)}(3).$$

$$f(x) = p_2(x) = 1 + \frac{7}{6}x - \frac{1}{6}x^2.$$

The result of the divided difference is listed in Table 4.3.
Since the Newton's interpolating polynomial of degree 2 is defined as

Table 4.3: Divided differences table for the Example 4.28.

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference
0	0	**1**		
1	1	2	**1**	
2	3	3	$\frac{1}{2}$	$-\frac{\mathbf{1}}{\mathbf{6}}$

$$f(x) = p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1).$$

By using Table 4.3, we have Newton's polynomial

$$f(x) = p_2(x) = 1 + (1)(x - 0) + \left(-\frac{1}{6}\right)(x - 0)(x - 1) = 1 + \frac{7}{6}x - \frac{1}{6}x^2,$$

which show that both polynomials are identical as functions. •

Example 4.33 *(a) Construct the divided difference table for the function* $f(x) = \ln(x+2)$ *in the interval* $0 \le x \le 3$ *for the stepsize* $h = 1$.

(b) Use Newton divided difference interpolation formula to construct the interpolating polynomials of degree 2 and degree 3 to approximate $\ln(3.5)$.

(c) Compute error bounds for the approximations in part (b).

Solution. *(a) The results of the divided differences are listed in Table 4.9.*

(b) Firstly, we construct the second degree polynomial $p_2(x)$ *by using the quadratic Newton interpolation formula as follows*

$$p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1),$$

then with the help of the divided differences Table 4.9, we get

$$p_2(x) = 0.6932 + 0.4055(x - 0) - 0.0589(x - 0)(x - 1),$$

which implies that $p_2(x) = -0.0568x^2 + 0.4644x + 0.6932$ *and* $p_2(1.5) = 1.2620$,

with possible actual error $f(1.5) - p_2(1.5) = 1.2528 - 1.2620 = -0.0072.$

Now to construct the cubic interpolatory polynomial $p_3(x)$ *that fits at all four points. We only have to add one more term to the polynomial* $p_2(x)$:

$$p_3(x) = p_2(x) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2),$$

and this gives

$$p_3(x) = p_2(x) + 0.0089(x^3 - 3x^2 + 2x) \quad \text{and} \quad p_3(1.5) = 1.2620 - 0.0033 = 1.2587,$$

Table 4.9: Divide differences table for the Example 4.33

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	0	0.6932			
1	1	1.0986	0.4055		
2	2	1.3863	0.2877	- 0.0589	
3	3	1.6094	0.2232	- 0.0323	0.0089

with possible actual error $f(1.5) - p_3(1.5) = 1.2528 - 1.2587 = -0.0059.$

(c) Now to compute the error bounds for the approximations in part (b), we use the error formula (4.30). For the polynomial $p_2(x)$*, we have*

$$|f(x) - p_2(x)| = \frac{|f'''(\eta(x))|}{3!}|(x-x_0)(x-x_1)(x-x_2)|.$$

The third derivative of the given function is given as

$$f'''(x) = \frac{2}{(x+2)^3} \quad \text{and} \quad |f'''(\eta(x))| = \left|\frac{2}{(\eta(x)+2)^3}\right|, \quad \text{for} \quad \eta(x) \in (0,2).$$

Then $M = \max_{0\le x\le 2}\left|\frac{2}{(x+2)^3}\right| = 0.25,$ *and* $|f(1.5) - p_2(1.5)| \le (0.375)(0.25)/6 = 0.0156,$

the error bound for the cubic polynomial $p_3(x)$ *is* $|f(x) - p_3(x)| = \frac{|f^{(4)}(\eta(x))|}{4!}|(x-x_0)(x-x_1)(x-x_2)(x-x_3)|$

$$f^{(4)}(x) = \frac{-6}{(x+2)^4} \quad \text{and} \quad |f^{(4)}(\eta(x))| = \left|\frac{-6}{(\eta(x)+2)^4}\right|, \quad \text{for} \quad \eta(x) \in (0,3).$$

Since $|f^{(4)}(0)| = 0.375$ *and* $|f^{(4)}(3)| = 0.0096,$

so $|f^{(4)}(\eta(x))| \le \max_{0\le x\le 3}\left|\frac{-6}{(x+2)^4}\right| = 0.375$ *and* $|f(1.5) - p_3(1.5)| \le (0.5625)(0.375)/24 = 0.0088,$

Example 4.37 *Consider the points* $x_0 = 0, x_1 = 0.4, x_2 = 0.7$ *and for a function* $f(x)$, *the divided differences are* $f[x_2] = 6, f[x_1, x_2] = 10, f[x_0, x_1, x_2] = 50/7$. *Use linear Newton's polynomial* $p_1(x)$ *to find quadratic Newton's polynomial* $p_2(x)$ *for the approximation of* $f(0.3)$.

Solution. *First we construct the complete divided differences table for the given data points. Since we know that the second divided difference is defined as*

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}, \qquad \frac{50}{7} = \frac{10 - f[x_0, x_1]}{0.7 - 0}.$$

Solving for $f[x_0, x_1]$, *we have,* $f[x_0, x_1] = 5$. *We need to find the values of the zeroth order divided differences* $f[x_0]$ *and* $f[x_1]$ *which can be obtained by using the first-order divided differences* $f[x_0, x_1]$ *and* $f[x_1, x_2]$. *Firstly, we find the value of* $f[x_1]$ *as follows*

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}, \qquad 10 = \frac{6 - f[x_1]}{0.7 - 0.4}, \qquad f[x_1] = 6 - 10(0.3) = 3.$$

The other zeroth divided difference $f[x_0]$ *can be computed as follows*

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}, \qquad 5 = \frac{3 - f[x_0]}{0.4 - 0}, \qquad f[x_0] = 3 - 5(0.4) = 1.$$

The completed divided differences table is shown by Table 4.11. We first find the linear Newton's

Table 4.11: Divided differences table for the Example 4.37.

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference
0	0	1		
1	0.4	3	5	
2	0.7	6	10	$\frac{50}{7}$

polynomial to approximate $f(0.3)$ using Table 4.11 as follows:

$$f(0.3) \approx p_1(0.3) = 1 + (5)(0.3 - 0.0) = 1 + 1.5 = 2.5,$$

and then use it to find quadratic Newton's polynomial using Table 4.11, we have

$$f(0.3) \approx p_2(0.3) = p_1(0.3) + 5(0.3 - 0) + \frac{50}{7}(0.3 - 0)(0.3 - 0.4) = 2.5 + 1.5 - 0.2143 = 3.7857,$$

the approximation of $f(0.3)$ using quadratic Newton's polynomial. •

Theorem 4.5 *Let $p_n(x)$ be the polynomial of degree at most n that interpolates a function $f(x)$ at a set of $n+1$ distinct points $x_0, x_1, \ldots, x_n$. If x is a point different from the points $x_0, x_1, \ldots, x_n$, then*

$$f(x) - p_n(x) = f[x_0, x_1, \ldots, x_n, x] \prod_{j=0}^{n} (x - x_j). \tag{4.55}$$

Theorem 4.6 (Divided Differences and Derivatives)

Suppose that $f \in C^n[a,b]$ and $x_0, x_1, \ldots, x_n$ are distinct number in $[a,b]$. Then for some point $\eta(x)$ in the interval (a,b) spanned by $x_0, \ldots, x_n$ exists with

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\eta(x))}{n!}. \tag{4.56}$$

Example 4.37 *Let $f(x) = x \ln x$, and the points $x_0 = 1.1, x_1 = 1.2, x_2 = 1.3$. Compute the best approximate value for unknown point $\eta(x)$ by using the relation (4.56).*

Solution. *Given $f(x) = x \ln x$, then*

$$\begin{aligned} f(1.1) &= 1.1 \ln(1.1) = 0.1048, \\ f(1.2) &= 1.2 \ln(1.2) = 0.2188, \\ f(1.3) &= 1.3 \ln(1.3) = 0.3411. \end{aligned}$$

Since the relation (4.56) for the given data points is

$$f[x_0, x_1, x_2] = \frac{f''(\eta(x))}{2!}. \tag{4.57}$$

To compute the value of the left-hand side of the relation (4.57), we have to find the values of the first-order divided differences

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{0.2188 - 0.1048}{1.2 - 1.1} = 1.1400,$$

and

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0.3411 - 0.2188}{1.3 - 1.2} = 1.2230.$$

Using these values, we can compute the second-order divided difference as

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1.2230 - 1.1400}{1.3 - 1.1} = 0.4150.$$

Now we calculate the right-hand side of the relation (4.57) for the given points and which gives us

$$\frac{f''(x_0)}{2} = \frac{1}{2x_0} = 0.4546, \quad \frac{f''(x_1)}{2} = \frac{1}{2x_1} = 0.4167, \quad \frac{f''(x_2)}{2} = \frac{1}{2x_2} = 0.3846.$$

We note that the left-hand side of (4.57) is nearly equal to the right-hand side when $x_1 = 1.2$*. Hence the best approximate value of* $\eta(x)$ *is* 1.2. •

Properties of Divided Differences

1. Divided difference of a constant is zero. Let $f(x) = a$, then

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{a - a}{x_1 - x_0} = 0.$$

2. Divided difference of $h(x) = af(x)$, a is constant, is the divided difference of $f(x)$ multiplied by a. Let $h(x) = af(x)$, then

$$h[x_0, x_1] = \frac{h(x_1) - h(x_0)}{x_1 - x_0} = \frac{af(x_1) - af(x_0)}{x_1 - x_0} = a\frac{f(x_1) - f(x_0)}{x_1 - x_0} = af[x_0, x_0].$$

3. Divided difference obeys linear property.
Let $F(x) = af_1(x) + bf_2(x)$, then $\quad F[x_0, x_1] \;=\; af_1[x_0, x_1] + bf_2[x_0, x_1].$

4. If $p_n(x)$ is a polynomial of degree n, then the divided differences of order n is always constant and $(n+1), (n+2), \ldots$ are identically zero.

5. The divided difference is a symmetric function of its arguments. Thus if $(t_0, t_1, \ldots, t_n)$ is a permutation of $(x_0, x_1, \ldots, x_n)$, then

$$f[t_0, t_1, \ldots, t_n] = f[x_0, x_1, \ldots, x_n],$$

6. The interpolating polynomial of degree n can be obtained by adding a single term to the polynomial of degree $(n-1)$ expressed in the Newton form.

$$p_n(x) = p_{n-1}(x) + f[x_0, \ldots, x_n] \prod_{j=0}^{n-1} (x - x_j).$$

7. The divided difference $f[x_0, \ldots, x_{n-1}]$ is the coefficient of x^{n-1} in the polynomial that interpolates $(x_0, f_0), (x_1, f_1), \ldots, (x_{n-1}, f_{n-1})$.

8. A sequence of divided differences may be constructed recursively from the formula

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0},$$

and the zeroth-order divided difference is defined by

$$f[x_i] = f(x_i), \quad i = 0, 1, \ldots, n.$$

9. $$f[x_0, x_0] = f'(x_0). \qquad f[x_0, x_0, x_0] = \frac{f''(x_0)}{2}. \qquad f[x_0, x_0, \ldots, x_0] = \frac{f^{(n)}(x_0)}{n!},$$

Example 4.40 *Let* $f(x) = e^{-x}$ *and let* $x_0 = 0, x_1 = 1$. *Using (4.56) and the above divide difference property 9, calculate* $f[x_0, x_1, x_0]$, $f[x_0, x_0, x_1, x_1]$ *and* $f[x_0, x_1, x_1, x_1]$.

Solution. *By using (4.56), we have* $f[x_0, x_0] = \frac{1}{1!} f'(x_0) = f'(x_0)$. *Therefore*

$$f[x_0, x_1, x_0] = f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0} = \frac{f[x_0, x_1] - f'(x_0)}{x_1 - x_0}.$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}, \quad \textit{gives} \quad f[0, 1] = \frac{0.368 - 1}{1 - 0} = -0.632.$$

Using $f'(x_0) = -e^{-x_0}$, $f'(0) = -1$, *we obtain,* $f[0, 1, 0] = f[0, 0, 1] = \frac{-0.632 + 1}{1 - 0} = 0.368$. *Now to find the value of the third divided difference which is defined as*

$$f[x_0, x_0, x_1, x_1] = \frac{f[x_0, x_1, x_1] - f[x_0, x_0, x_1]}{x_1 - x_0},$$

$$f[x_0, x_0, x_1, x_1] = \frac{f'(x_1) - 2f[x_0, x_1] + f'(x_0)}{(x_1 - x_0)^2}.$$

$$f[0, 0, 1, 1] = \frac{-0.368 - 2(-0.632) - 1}{(1 - 0)^2} = -0.014.$$

$$f[x_0,x_1,x_1,x_1] = \frac{f[x_1,x_1,x_1] - f[x_0,x_1,x_1]}{x_1 - x_0},$$

$$f[x_0,x_1,x_1,x_1] = \frac{f''(x_1)/2! - (f'(x_1) - f[x_0,x_1])/(x_1 - x_0)}{x_1 - x_0}.$$

$$f[x_0,x_0,x_1,x_1] = \frac{(x_1 - x_0)f''(x_1) - 2f'(x_1) + 2f[x_0,x_1]}{2(x_1 - x_0)^2}.$$

$$As\ f''(1) = e^{-1} = 0.368,\ \ f[0,1,1,1] = \frac{(1-0)(0.368) - 2(-1) + 2(-0.632)}{2(1-0)^2} = 1.104.$$

-

Example 4.41 *Let* $f(x) = \ln(x+2)$ *and* $x_0 = 0, x_1 = 0, x_2 = 1, x_3 = 1$, *find the best approximation of* $\ln(2.5)$ *by using the Newton's polynomial.*

Solution. *Using* $f(x) = \ln(x+2)$ *and* $x_0 = 0, x_1 = 0, x_2 = 1, x_3 = 1$, *the cubic Newton's interpolating polynomial has the following form*

$$p_3(x) = f[x_0]+(x-x_0)f[x_0,x_0]+(x-x_0)(x-x_0)f[x_0,x_0,x_1]+(x-x_0)(x-x_0)(x-x_1)f[x_0,x_0,x_1,x_1].$$

Now we find the second and third-order divided differences as follows:

$$f[0,0,1] = \frac{f[0,1]-f'(0)}{1-0} = f(1)-f(0)-f'(0) = 1.0986-0.6932-0.5 = -0.0946.$$

$$f[0,1,1] = \frac{f[1,1]-f[0,1]}{1-0} = f'(1)-f(1)+f(0) = 0.3333-1.0986+0.6932 = -0.0721,$$

$$f[0,0,1,1] = \frac{f[0,1,1]-f[0,0,1]}{1-0} = -0.0721+0.0946 = 0.0225.$$

$p_3(0.5) = f(0)+(0.5-0)f'(0)+(0.5-0)(0.5-0)f[0,0,1]+(0.5-0)(0.5-0)(0.5-1)f[0,0,1,1],$

$\ln(2.5) \approx p_3(0.5) = \ln(2)+(0.5)(0.5)+(0.25)(-0.0946)+(-0.1250)(0.0225) = 0.6932+0.25-0.0237-0.0028 = 0.916$

the required approximation of $\ln(2.5)$ *and*

$$|f(0.5)-p_3(0.5)| = |\ln(2.5)-p_3(0.5)| = |0.9163-0.9167| = 4.0\times 10^4,$$

the possible absolute error in the approximation. •

Interpolation with Spline Functions

Definition 4.1 (Spline Function)

Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$. A function $s : [a, b] \to \mathbf{R}$ is a spline or spline function of degree m with points $x_0, x_1, \ldots, x_n$ if
1. A function s is a piecewise polynomial such that, on each subinterval $[x_k, x_{k+1}]$, s has degree at most m.
2. A function s is $m - 1$ times differentiable everywhere. •

Piecewise Linear Interpolation

It is the one of the simplest piecewise polynomial interpolation for the approximation of the function,

Consider the set of seven data points (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5) and (x_6, y_6)

If we use a straight line on each subinterval (see Figure 4.4) then we can interpolate the data with a piecewise linear function, where

$$s_k(x) = p_k(x) = \frac{(x - x_{k+1})}{(x_k - x_{k+1})} y_k + \frac{(x - x_k)}{(x_{k+1} - x_k)} y_{k+1},$$

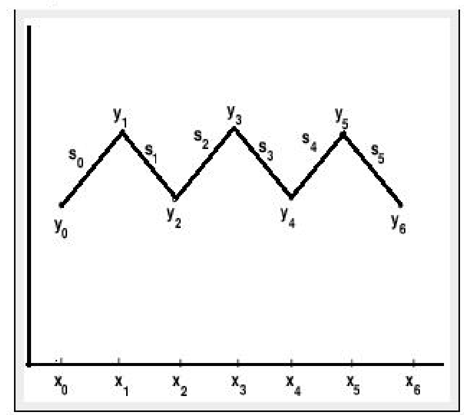


Figure 4.4: Linear spline.

or

$$s_k(x) = y_k + \frac{(y_{k+1} - y_k)}{(x_{k+1} - x_k)}(x - x_k).$$

It gives us

$$s_k(x) = A_k + B_k(x - x_k), \tag{4.59}$$

where the values of the coefficients A_k and B_k are given as

$$A_k = y_k \quad \text{and} \quad B_k = \frac{(y_{k+1} - y_k)}{(x_{k+1} - x_k)}. \tag{4.60}$$

Note that the linear spline must be continuous at given points $x_0, x_1, \ldots, x_n$ and

$$s(x_k) = f(x_k), \qquad \text{for} \quad k = 0, 1, \ldots, n.$$

Example 4.43 *Find the linear splines which interpolates the following data*

x	1	2	3	4
$f(x)$	1.0	0.67	0.50	0.40

Find the approximation of the function $f(x) = \dfrac{2}{x+1}$ *at* $x = 2.9$*. Compute absolute error.*

Solution. *Given* $x_0 = 1.0, x_1 = 2.0, x_2 = 3.0, x_3 = 4.0,$ *then using (4.60), we have*

$$A_0 = y_0 = 1.0,\ A_1 = y_1 = 0.67,\ A_2 = y_2 = 0.50,\ A_3 = y_3 = 0.4,$$

and

$$B_0 = \frac{(y_1 - y_0)}{(x_1 - x_0)} = \frac{(0.67 - 1.0)}{(2.0 - 1.0)} = -0.33,$$

$$B_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(0.50 - 0.67)}{(3.0 - 2.0)} = -0.17,$$

$$B_2 = \frac{(y_3 - y_2)}{(x_3 - x_2)} = \frac{(0.40 - 0.50)}{(4.0 - 3.0)} = -0.10.$$

Now using (4.59), the linear splines for three subintervals are define as

$$s(x) = \begin{cases} s_0(x) = 1.0 - 0.33(x - 1.0) = 1.33 - 0.33x, & 1 \le x \le 2, \\ s_1(x) = 0.67 - 0.17(x - 2.0) = 1.01 - 0.17x, & 2 \le x \le 3, \\ s_2(x) = 0.50 - 0.10(x - 3.0) = 0.80 - 0.10x, & 3 \le x \le 4. \end{cases}$$

The value $x = 2.9$ *lies in the interval* $[2, 3]$, *so*

$$f(2.9) \approx s_1(2.9) = 1.01 - 0.17(2.9) = 0.517.$$

Also,

$$|f(2.9) - s_1(2.9)| = |0.513 - 0.517| = 0.004,$$

Example 4.44 *Find the values of unknown coefficients a and b so that the following function is a linear spline.*

$$s(x) = \begin{cases} a - x, & 0 \leq x \leq 1, \\ 3x - b, & 1 \leq x \leq 2, \\ 2x + 1, & 2 \leq x \leq 3. \end{cases}$$

Solution. *Since the given function is a linear spline, so s must be continuous at the internal points 1 and 2. Continuity at $x = 1$ implies that*

$$\begin{aligned} \lim_{x \to 1^-} s(x) &= \lim_{x \to 1^+} s(x), \\ \lim_{x \to 1^-} a - x &= \lim_{x \to 1^+} 3x - b, \\ a - 1 &= 3 - b, \end{aligned}$$

and it gives an equation of the from

$$a + b = 4.$$

Now continuity at $x = 2$ implies that

$$\begin{aligned} \lim_{x \to 2^-} s(x) &= \lim_{x \to 2^+} s(x), \\ \lim_{x \to 2^-} 3x - b &= \lim_{x \to 2^+} 2x + 1, \\ 6 - b &= 5, \end{aligned}$$

and it gives $b = 1$. Using this value of b, we get $a = 3$, and so

$$s(x) = \begin{cases} 3 - x, & 0 \leq x \leq 1, \\ 3x - 1, & 1 \leq x \leq 2, \\ 2x + 1, & 2 \leq x \leq 3, \end{cases}$$

is the linear spline function. •