Solution of Nonlinear Equations

2.1 Introduction

I. A nonlinear equation in this chapter may be considered any one of the following types:

1. An equation may be an *algebraic equation (a polynomial equation of degree n)*

   $$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad a_n \neq 0, \quad n > 1,$$

   where $a_n, a_{n-1}, \ldots, a_1$ and a_0 are constants.

   For example, $\quad x^2 + 5x + 6 = 0; \quad x^3 = 2x + 1; \quad x^{100} + x^2 + 1 = 0.$

2. The power of the unknown variable (not a positive integer number)

   For example, $\quad x^{-1} + 2x = 1; \quad \sqrt{x} + x = 0; \quad x^{2/3} + \frac{2}{x} + 4 = 0.$

3. An equation may be a *transcendental equation*, the equation which involves the trigonometric functions, exponential functions and logarithmic functions. For example, all the following transcendental equations are nonlinear

   $$x = cos(x); \qquad e^x + x - 10 = 0; \qquad x + \ln x = 10.$$

Definition 2.1 (Root of an Nonlinear Equation)

Assume that $f(x)$ is a continuous function. An number α for which $f(\alpha) = 0$ is called a root of the equation $f(x) = 0$ or a zero of the function $f(x)$. •

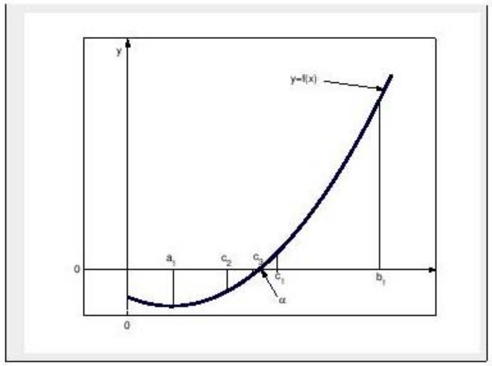
2.2 Method of Bisection

– This is one of the simplest iterative technique for determining roots of $f(x) = 0$

– we begin by supposing $f(x)$ is a continuous function defined on the interval $[a, b]$ such that

$$f(a).f(b) < 0.$$

The implication is that one of the values is negative and the other is positive.



Therefore the root must lies between a and b (by Intermediate Value Theorem)

– a new approximation to the root α be calculated as $c = \dfrac{a+b}{2}$,

– If $f(c) \approx 0$, then $c \approx \alpha$ is the desired root,

– if not, then there are two possibilities.

+ Firstly, if $f(a).f(c) < 0$, then $f(x)$ has a zero between point a and point c. The process can then be repeated on the new interval $[a, c]$.

+ Secondly, if $f(a).f(c) > 0$ it follows that $f(b).f(c) < 0$ and, $f(x)$ has zero between point c and point b and the process can be repeated with $[c, b]$

– and, in general $c_n = \dfrac{a_n + b_n}{2}, \qquad n \geq 1. \qquad (2.2)$

The iterative formula (2.2) is known as the *bisection method.*

– The process continue until the desired accuracy is achieved

Procedure 2.1 (Bisection Method)

1. *Establish an interval* $a \leq x \leq b$ *such that* $f(a)$ *and* $f(b)$ *are of opposite sign, that is,* $f(a).f(b) < 0$.
2. *Choose an error tolerance* $(\epsilon > 0)$ *value for the function.*
3. *Compute a new approximation for the root:* $c_n = \dfrac{(a_n + b_n)}{2}; \qquad n = 1, 2, 3, \ldots.$
4. *Check tolerance. If* $|f(c_n)| \leq \epsilon$, *use* $c_n,\ n \geq 1$ *for desired root; otherwise continue.*
5. *Check, if* $f(a_n).f(c_n) < 0$, *then set* $b_n = c_n$; *otherwise set* $a_n = c_n$.
6. *Go back to step 3, and repeat the process.*

Example 2.1 *Use the bisection method to find the approximation to the root of the equation*

$$x^3 = 2x + 1,$$

that is located in the interval $[1.5, 2.0]$ *accurate to within* 10^{-2}.

Solution. *Since the given function* $f(x) = x^3 - 2x - 1$ *is a polynomial function and so is continuous on* $[1.5, 2.0]$, *starting with* $a_1 = 1.5$ *and* $b_1 = 2$, *we compute:*

$$\begin{array}{lcll} a_1 & = & 1.5: & f(a_1) = -0.625 \\ b_1 & = & 2.0: & f(b_1) = 3.0, \end{array}$$

and since $f(1.5).f(2.0) < 0$, *so that a root of* $f(x) = 0$ *lies in the interval* $[1.5, 2.0]$. *Using formula (2.2) (when* $n = 1$*), we get:*

$$c_1 = \frac{a_1 + b_1}{2} = 1.75; \qquad f(c_1) = 0.859375.$$

Hence the function changes sign on $[a_1, c_1] = [1.5, 1.75]$. *To continue, we squeeze from right and set* $a_2 = a_1$ *and* $b_2 = c_1$. *Then the midpoint is:*

$$c_2 = \frac{a_2 + b_2}{2} = 1.625; \qquad f(c_2) = 0.041056.$$

Continue in this way we obtain a sequence $\{c_k\}$ *of approximation shown by Table 2.1.*

n	Left Endpoint a_n	Midpoint c_n	Right Endpoint b_n	Function Value $f(c_n)$
01	1.500000	1.750000	2.000000	0.8593750
02	1.500000	1.625000	1.750000	0.0410156
03	1.500000	1.562500	1.625000	-0.3103027
04	1.562500	1.593750	1.625000	-0.1393127
05	1.593750	1.609375	1.625000	-0.0503273
06	1.609375	1.617188	1.625000	-0.0049520

We got the desired approximation to the root of the given equation is $c_6 = 1.617188 \approx \alpha$ *after 6 iterations with accuracy* $\epsilon = 10^{-2}$. •

Theorem 2.1 (Bisection Convergence and Error Theorem)

Let $f(x)$ be continuous function defined on the given initial interval $[a_0, b_0] = [a, b]$ and suppose that $f(a)f(b) < 0$. Then bisection method (2.2) generates a sequence $\{c_n\}_{n=1}^{\infty}$ approximating $\alpha \in (a, b)$ with the property

$$|\alpha - c_n| \leq \frac{b-a}{2^n}, \quad n \geq 1. \tag{2.3}$$

Moreover, to obtain accuracy of

$$|\alpha - c_n| \leq \epsilon,$$

(for $\epsilon = 10^{-k}$) it suffices to take

$$n \geq \frac{\ln\left\{10^k(b-a)\right\}}{\ln 2}, \tag{2.4}$$

where k is nonnegative integer.

Example 2.4 *Find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-1} to the solution of $xe^x = 1$ lying in the interval $[0.5, 1]$ using the bisection method. Find an approximation to the root with this degree of accuracy.*

Solution. *Here $a = 0.5$, $b = 1$ and $k = 1$, then by using inequality (2.4), we get*

$$n \geq \frac{\ln[10^1(1-0.5)]}{\ln 2} \approx 2.3219.$$

So no more than three iterations are required to obtain an approximation accurate to within 10^{-1}. The given function $f(x) = xe^x - 1$ is continuous on $[0.5, 1.0]$, so starting with $a_1 = 0.5$ and $b_1 = 1$, we compute:

$$\begin{aligned} a_1 &= 0.5: \quad f(a_1) = -0.1756, \\ b_1 &= 1: \quad f(b_1) = 1.7183, \end{aligned}$$

since $f(0.5).f(1) < 0$, *so that a root of* $f(x) = 0$ *lies in the interval* $[0.5, 1]$. *Using formula (2.2) (when* $n = 1$*), we get:*

$$c_1 = \frac{a_1 + b_1}{2} = 0.75; \qquad f(c_1) = 0.5878.$$

Hence the function changes sign on $[a_1, c_1] = [0.5, 0.75]$. *To continue, we squeeze from right and set* $a_2 = a_1$ *and* $b_2 = c_1$. *Then the bisection formula gives*

$$c_2 = \frac{a_2 + b_2}{2} = 0.625; \qquad f(c_2) = 0.1677.$$

Finally, we have in the similar manner as

$$c_3 = \frac{a_3 + b_3}{2} = 0.5625,$$

the value of the third approximation which is accurate to within 10^{-1}. •

Example 2.5 *Use the bisection method to compute the first three approximate values for* $\sqrt[4]{18}$. *Also, compute an error bound and absolute error for your approximation.*

Solution. *Consider*

$$x = \sqrt[4]{18} = (18)^{1/4}, \qquad or \qquad x^4 - 18 = 0.$$

Choose the interval $[2, 2.5]$ *on which the function* $f(x) = x^4 - 18$ *is continuous and the function* $f(x)$ *satisfies the sign property, that is*

$$f(2).f(2.5) = (-2)(21.0625) = -42.125 < 0.$$

Hence root $\alpha = \sqrt[4]{18} = 2.0598 \in [2, 2.5]$ *and we compute its first approximate value by using formula (2.2) (when* $n = 1$*) as follows:*

$$c_1 = \frac{2.0 + 2.5}{2} = 2.2500 \quad and \quad f(2.25) = 7.6289.$$

Since the function $f(x)$ changes sign on $[2.0, 2.25]$. To continue, we squeeze from right and use formula (2.2) again to get the following second approximate value of the root α as:

$$c_2 = \frac{2.0 + 2.25}{2} = 2.1250; \quad \textit{and} \quad f(2.1250) = 2.3909.$$

Then continue in the similar way, the third approximate value of the root α is $c_3 = 2.0625$ with $f(2.0625) = 0.0957$. Note that the value of the function at each new approximate value is decreasing

which shows that the approximate values are coming closer to the root α. Now to compute the error bound for the approximation we use the formula (2.3) and get

$$|\alpha - c_3| \leq \frac{2.5 - 2.0}{2^3} = 0.0625,$$

which is the possible maximum error in our approximation and

$$|E| = |2.0598 - 2.0625| = 0.0027,$$

be the absolute error in the approximation. •

2.3 Fixed-Point Method

The basic idea of this method which is also called successive approximation method or function iteration, is to rearrange the original equation

$$f(x) = 0, \tag{2.5}$$

into an equivalent expression of the form

$$x = g(x). \tag{2.6}$$

Any solution of (2.6) is called a fixed-point for the iteration function $g(x)$ and hence a root of (2.5).

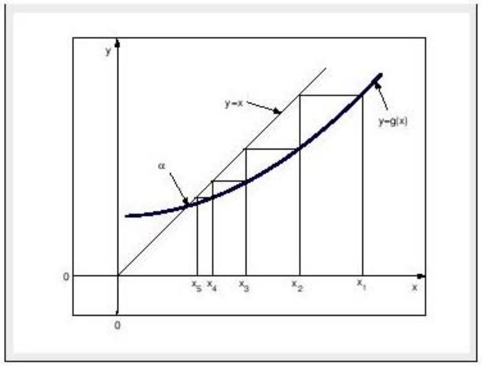


Figure 2.5: Graphical Solution of Fixed-Point Method.

Definition 2.2 (Fixed-Point of a Function)

A fixed-point of a function $g(x)$ *is a real number* α *such that* $\alpha = g(\alpha)$*. For example,* $x = 2$ *is a fixed-point of the function* $g(x) = \dfrac{x^2 - 4x + 8}{2}$ *because* $g(2) = 2$. •

Definition 2.3 (Fixed-Point Method)

The iteration defined in the following

$$x_{n+1} = g(x_n); \qquad n = 0, 1, 2, \ldots, \tag{2.7}$$

is called the fixed-point method or the fixed-point iteration. •

Procedure 2.2 (Fixed-Point Method)

1. *Choose an initial approximation* x_0 *such that* $x_0 \in [a, b]$.
2. *Choose a convergence parameter* $\epsilon > 0$.
3. *Compute new approximation* x_{new} *by using the iterative formula (2.7).*
4. *Check, if* $|x_{new} - x_0| < \epsilon$ *then* x_{new} *is the desire approximate root; otherwise set* $x_0 = x_{new}$ *and go to step 3.*

Example 2.6 *Consider the nonlinear equation $x^3 = 2x+1$ which has a root in the interval $[1.5, 2.0]$ using fixed-point method with $x_0 = 1.5$, take three different rearrangements for the equation.*

Solution. *Let us consider the three possible rearrangement of the given equation as follows:*

$$(i) \quad x_{n+1} = g_1(x_n) = \frac{(x_n^3 - 1)}{2}; \qquad n = 0, 1, 2, \ldots,$$

$$(ii) \quad x_{n+1} = g_2(x_n) = \frac{1}{(x_n^2 - 2)}; \qquad n = 0, 1, 2, \ldots,$$

$$(iii) \quad x_{n+1} = g_3(x_n) = \sqrt{\frac{(2x_n + 1)}{x_n}}; \qquad n = 0, 1, 2, \ldots,$$

then the numerical results for the corresponding iterations, starting with the initial approximation $x_0 = 1.5$ with accuracy 5×10^{-2}, are given in Table 2.3. We note that the first two considered

n	$x_{n+1} = g_1(x_n)$ $= (x_n^3 - 1)/2$	$x_{n+1} = g_2(x_n)$ $= 1/(x_n^2 - 2)$	$x_{n+1} = g_3(x_n)$ $= \sqrt{(2x_n + 1)/x_n}$
00	1.500000	1.500000	1.500000
01	1.187500	4.000000	1.632993
02	0.337280	0.071429	1.616284
03	-0.480816	-0.501279	—
04	-0.555579	-0.571847	—
05	-0.585745	-0.597731	—

sequences diverge and the last one converges. This example asks the need for a mathematical analysis of the method. The following theorem gives sufficient conditions for the convergence of the fixed-point iteration. •

Theorem 2.2 (Fixed-Point Theorem)

If g is continuously differentiable on the interval $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then

(a) g has at-least one fixed-point in the given interval $[a, b]$.

Moreover, if the derivative $g'(x)$ of the function $g(x)$ exists on an interval $[a, b]$ which contains the starting value x_0, with

$$|g'(x)| \leq k < 1; \qquad \text{for all} \qquad x \in [a, b]. \tag{2.8}$$

Then:

(b) The sequence (2.7) will converge to the attractive (unique) fixed-point α in $[a, b]$.

(c) The iteration (2.7) will converge to α for any initial approximation.

(d) We have the error estimate

$$|\alpha - x_n| \leq \frac{k^n}{1-k}|x_1 - x_0|, \qquad \text{for all} \quad n \geq 1. \tag{2.9}$$

(e) The limit holds:

$$\lim_{n \to \infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha). \tag{2.10}$$

Proof

(a) *Suppose g is continuous on $[a, b]$ and $g(x) \in [a, b]$. We need to show it has a fixed point. If $g(a) = a$ and $g(b) = b$, then the function g has a fixed-point at the endpoints. Suppose that it is not happening, that is, $g(a) \neq a$ and $g(b) \neq b$ and define a function $f(x) = g(x) - x$ which is continuous on $[a, b]$. Then $f(x)$ has a zero in $[a, b]$ if and only if $g(x)$ has a fixed point in $[a, b]$ but*

$$f(a) = g(a) - a > 0,$$

since $g(a)$ is in $[a, b]$ and hence cannot be smaller than a, and we have assumed that $g(a)$ is not equal to a. Similarly,

$$f(b) = g(b) - b < 0,$$

and so by the Intermediate Value Theorem there is a α in the interval (a, b) such that $f(\alpha) = 0$, which implies that $\alpha = g(\alpha)$. Thus the function $g(x)$ has at least one fixed-point in $[a, b]$. This proves (a).

(b) Suppose now that (2.8) holds, and α *and* β *are two fixed-points of the function* g *in* $[a,b]$*. Then we have*

$$\alpha = g(\alpha) \qquad \text{and} \quad \beta = g(\beta).$$

In addition, by the Mean Value Theorem, we have that for any two points α *and* β *in* $[a,b]$*, there exits a number* η *such that*

$$|\alpha - \beta| = |g(\alpha) - g(\beta)| = |g'(\eta)||\alpha - \beta| \leq k|\alpha - \beta|,$$

where $\eta \in (a,b)$*. Thus*

$$(1-k)|\alpha - \beta| \leq 0.$$

Since $k < 1$*, we must have* $\alpha = \beta$*; and thus, the function* g *has a unique fixed-point* α *in the interval* $[a,b]$*. This proves (b).*

(c) *For the convergence, consider the iteration*

$$x_n = g(x_{n-1}), \qquad \text{for all} \qquad n \geq 1, 2 \ldots,$$

and the definition of the fixed-point, that is

$$\alpha = g(\alpha).$$

If we subtract last two equations and take the absolute values, we get

$$|\alpha - x_n| = |g(\alpha) - g(x_{n-1})| \leq k|\alpha - x_{n-1}|.$$

The recursion can be solved readily to get

$$|\alpha - x_n| \leq k|\alpha - x_{n-1}| \leq k^2|\alpha - x - n - 2| \cdots \leq k^n|\alpha - x_0|, \tag{2.11}$$

from which it follows that

$$\text{as} \quad n \to \infty, \quad k^n \to 0, \quad (\text{since} \quad k < 1),$$

therefore, $x_n \to \alpha$. *Hence the iteration converges. This proves* (c).

(d) Since we note that

$$\begin{aligned} |\alpha - x_0| &= |\alpha - x_1 + x_1 - x_0| \le |\alpha - x_1| + |x_1 - x_0| \\ &\le |g(\alpha) - g(x_0)| + |x_1 - x_0| \le k|\alpha - x_0| + |x_1 - x_0|, \end{aligned}$$

or

$$(1-k)|\alpha - x_0| \le |x_1 - x_0|,$$

from which it follows that

$$|\alpha - x_0| \le \frac{1}{1-k}|x_1 - x_0|.$$

From (2.11), we can write above equation as follows

$$|\alpha - x_n| \le \frac{k^n}{1-k}|x_1 - x_0|,$$

which proves (d).

(**e**) *Finally, by subtracting iteration* $x_{n+1} = g(x_n)$ *and* $\alpha = g(\alpha)$, *we have*

$$\alpha - x_{n+1} = g(\alpha) - g(x_n) = g'(\eta(x))(\alpha - x_n),$$

which implies that

$$\frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\eta(x)),$$

and by taking limits, we have

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = \lim_{n\to\infty} g'(\eta(x)) = g'(\alpha),$$

since $\eta(x) \to \alpha$ *is forced by the convergence of* x_n *to* α. *This proves* (e). •

Example 2.6

Now we come back to our previous Example 2.6 and discuss that why the first two rearrangements we considered, do not converge but on the other hand, last sequence has a fixed-point and converge. Since, we observe that $f(1.5).f(2) < 0$, then the solution we seek is in the interval $[1.5, 2]$.

(i) For $g_1(x) = \dfrac{x^3 - 1}{3}$, we have $g_1'(x) = x^2$, which is greater than unity throughout the interval $[1.5, 2]$. So by Fixed-Point Theorem 2.2 this iteration will fail to converge.

(ii) For $g_2(x) = \dfrac{1}{x^2 - 2}$, we have $g_2'(x) = \dfrac{-2x}{(x^2 - 2)^2}$, and $|g_2'(1.5)| > 1$, so from Fixed-Point Theorem 2.2 this iteration will fail to converge.

(iii) For $g_3(x) = \sqrt{\dfrac{2x + 1}{x}}$, we have $g_3'(x) = x^{-3/2}/2\sqrt{2x + 1} < 1$, for all x in the given interval $[1.5, 2]$. Also, g_3 is decreasing function of x, and $g_3(1.5) = 1.63299$ and $g_3(2) = 1.58114$ both lie in the interval $[1.5, 2]$. Thus $g_3(x) \in [1.5, 2]$, for all $x \in [1.5, 2]$, so from Fixed-Point Theorem 2.2 the iteration will converge, see Figure 2.6. •

Note 2.1 *From (2.9) Note that the rate of convergence of the fixed-point method depends on the factor* $\dfrac{k^n}{(1-k)}$*; the smaller the value of* k*, then faster the convergence. The convergence may be very slow if the value of* k *is very close to 1.* •

Note 2.2 *Assume that* $g(x)$ *and* $g'(x)$ *are continuous functions of* x *for some open interval* I*, with the fixed-point* α *contained in this interval. Moreover assume that*

$$|g'(\alpha)| < 1, \qquad \textit{for} \quad \alpha \in I,$$

then, there exists an interval $[a, b]$*, around the solution* α *for which all the conditions of Theorem 2.2 are satisfied. But if*

$$|g'(\alpha)| > 1, \qquad \textit{for} \quad \alpha \in I,$$

then the sequence (2.7) will not converge to α*. In this case* α *is called a repulsive fixed-point. If*

$$|g'(\alpha)| = 0, \qquad \textit{for} \quad \alpha \in I,$$

then the sequence (2.7) converges very fast to the root α *while if*

$$|g'(\alpha)| > 1, \qquad \textit{for} \quad \alpha \in I,$$

then the convergence the sequence (2.7)is not guaranteed and if the convergence happened, it would be very slow. Thus to get the faster convergence, the value of $|g'(\alpha)|$ *should be equal to zero or very close to zero.* •

Example 2.8 *Find an interval* $[a, b]$ *on which fixed-point problem* $x = \dfrac{2 - e^x + x^2}{3}$ *will converges. Estimate the number of iterations* n *within accuracy* 10^{-5}.

Solution. *Since* $x = \dfrac{2 - e^x + x^2}{3}$ *can be written as*

$$e^x - x^2 + 3x - 2 = 0,$$

and we observe that $f(0)f(1) = (-1)(e^1) < 0$, *then the solution we seek is in the interval* $[0, 1]$. *For* $g(x) = \dfrac{2 - e^x + x^2}{3}$, *we have* $g'(x) = \dfrac{2x - e^x}{3} < 1$, *for all* x *in the given interval* $[0, 1]$. *Also,*

g *is decreasing function of* x *and* $g(0) = 0.3333$ *and* $g(1) = \dfrac{3 - e}{3} = 0.0939$ *both lie in the interval* $[0, 1]$. *Thus* $g(x) \in [0, 1]$, *for all* $x \in [0, 1]$, *so from fixed-point theorem the* $g(x)$ *has a unique fixed-point in* $[0, 1]$. *Taking* $x_0 = 0.5$, *we have*

$$x_1 = g(x_0) = \frac{2 - e^{x_0} + x_0^2}{3} = 0.2004.$$

Also, we have

$$k_1 = |g'(0)| = 0.3333 \quad \text{and} \quad k_2 = |g'(1)| = 0.2394,$$

which give $k = \max\{k_1, k_2\} = 0.3333$. *Thus the error estimate (2.9) within the accuracy* 10^{-5} *is*

$$|\alpha - x_n| \leq 10^{-5}, \qquad \text{gives} \qquad \frac{(0.3333)^n}{1 - 0.3333}(0.2996) \leq 10^{-5},$$

and by solving this inequality, we obtain $n \geq 9.7507$. *So we need ten approximations to get the desired accuracy for the given problem.* •

Example 2.12 *Show that the fixed point form of the equation* $x = N^{1/3}$ *can be written as* $x = Nx^{-2}$ *and the associated iterative scheme*

$$x_{n+1} = Nx_n^{-2}, \qquad n \geq 0,$$

will not successful (diverge) in finding the 3rd root of the positive number N*.*

Solution. *Given* $x = N^{1/3}$ *and it can be written as*

$$x^3 - N = 0 \quad or \quad x = \frac{N}{x^2} = Nx^{-2}.$$

It gives the iterative scheme

$$x_{n+1} = Nx_n^{-2} = g(x_n), \qquad n \geq 0.$$

From this, we have

$$g(x) = Nx^{-2} \qquad and \qquad g'(x) = -2Nx^{-3}.$$

Since $\alpha = x = N^{1/3}$*, therefore*

$$g'(\alpha) = -2N\alpha^{-3} \quad and \quad g'(N^{1/3}) = -2N(N^{1/3})^{-3} = -2NN^{-1} = -2.$$

Thus

$$|g'(N^{1/3})| = |-2| = 2 > 1,$$

which shows the divergence. •

Example 2.16 *One of the possible rearrangement of the nonlinear equation* $e^x = x + 2$, *which has root in* $[1, 2]$ *is*

$$x_{n+1} = g(x_n) = \ln(x_n + 2); \qquad n = 0, 1, \ldots.$$

(a) *Show that* $g(x)$ *has a unique fixed-point in* $[1, 2]$.

(b) *Use fixed-point iteration formula (2.7) to compute approximation* x_3, *using* $x_0 = 1.5$.

(c) *Compute an error estimate* $|\alpha - x_3|$ *for your approximation.*

(d) *Determine the number of iterations needed to achieve an approximation with accuracy* 10^{-2} *to the solution of* $g(x) = \ln(x + 2)$ *lying in the interval* $[1, 2]$ *by using the fixed-point iteration method.*

Solution. *Since, we observe that* $f(1).f(2) < 0$, *then the solution we seek is in the interval* $[1, 2]$.

(a) *For* $g(x) = \ln(x+2)$, *we have* $g'(x) = 1/(x+2) < 1$, *for all* x *in the given interval* $[1, 2]$. *Also,* g *is increasing function of* x, *and* $g(1) = \ln(3) = 1.0986123$ *and* $g(2) = \ln(4) = 1.3862944$ *both lie in the interval* $[1, 2]$. *Thus* $g(x) \in [1, 2]$, *for all* $x \in [1, 2]$, *so from fixed-point theorem the* $g(x)$ *has a unique fixed-point, see Figure 2.8.*

(b) *using the given initial approximation* $x_0 = 1.5$, *we have the other approximations as*

$$x_1 = g(x_0) = 1.252763, \quad x_2 = g(x_1) = 1.179505, \quad x_3 = g(x_2) = 1.156725.$$

(c) *Since* $a = 1$ *and* $b = 2$, *then the value of* k *can be found as follows*

$$k_1 = |g'(1)| = |1/3| = 0.333 \quad \textit{and} \quad k_2 = |g'(2)| = |1/4| = 0.25,$$

which give $k = \max\{k_1, k_2\} = 0.333$. *Thus using the error formula (2.9), gives*

$$|\alpha - x_3| \leq \frac{(0.333)^3}{1 - 0.333}|1.252763 - 1.5| = 0.013687.$$

(d) From the error bound formula (2.9), we have

$$\frac{k^n}{1-k}|x_1 - x_0| \leq 10^{-2}.$$

By using above parts (b) and (c), we have

$$\frac{(0.333)^n}{1-0.333}|1.252763 - 1.5| \leq 10^{-2}.$$

Solving this inequality, we obtain

$$n \ln(0.333) \leq \ln(0.02698), \quad \textit{gives} \quad n \geq 3.28539.$$

So we need four approximations to get the desired accuracy for the given problem. •

2.4 Newton's Method

This is one of the most popular and powerful iterative method for finding roots of the nonlinear equatioı $f(x) = 0$ This method is also called the *Newton-Raphson method.*

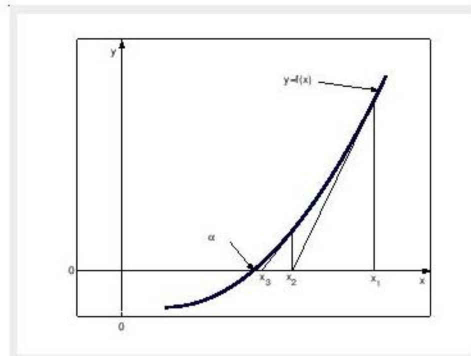


Figure 2.9: Graphical Solution of Newton's Method.

The Newton's method consists geometrically of expanding the tangent line at a current point x_i until it crosses zero, then setting the next guess x_{i+1} to the abscissa of that zero crossing, see Figure 2.9.

description of the Newton's method.

Let $f \in C^2[a,b]$ and let x_n be the nth approximation to the root α such that $f'(x_n) \neq 0$ and $|\alpha - x_n|$ is small. Consider the first Taylor polynomial for $f(x)$ expanded about x_n, so we have

$$f(x) = f(x_n) + (x - x_n)f'(x_n) + \frac{(x - x_n)^2}{2} f''(\eta(x)), \tag{2.12}$$

where $\eta(x)$ lies between x and x_n. Since $f(\alpha) = 0$, then (2.12), with $x = \alpha$, gives

$$f(\alpha) = 0 = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\eta(\alpha)).$$

Since $|\alpha - x_n|$ is small, then we neglect the term involving $(\alpha - x_n)^2$ and so

$$0 \approx f(x_n) + (\alpha - x_n)f'(x_n).$$

Solving for α, we get

$$\alpha \approx x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2.13}$$

which should be better approximation to α than is x_n. We call this approximation as x_{n+1}, then we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \qquad \text{for all} \quad n \geq 0. \tag{2.14}$$

The iterative method (2.14) is called the Newton's method.

Procedure 2.3 (Newton's Method)

1. *Find the initial approximation* x_0 *for the root by sketching the graph of the function.*

2. *Evaluate function* $f(x)$ *and the derivative* $f'(x)$ *at initial approximation.*

   Check: if $f(x_0) = 0$ *then* x_0 *is the desire approximation to a root. But if* $f'(x_0) = 0$*, then go back to step 1 to choose new approximation.*

3. *Establish Tolerance* $(\epsilon > 0)$ *value for the function.*

4. *Compute new approximation for the root by using the iterative formula (2.14).*

5. *Check Tolerance. If* $|f(x_n)| \leq \epsilon$*, for* $n \geq 0$*, then end; otherwise, go back to step 4, and repeat the process.*

Example 2.17 *Use the Newton's method to find the root of* $x^3 = 2x + 1$ *that is located in the interval* $[1.5, 2.0]$ *accurate to* 10^{-2}*, take an initial approximation* $x_0 = 1.5$.

Solution. *Given* $f(x) = x^3 - 2x - 1$ *and so* $f'(x) = 3x^2 - 2$. *Now evaluating* $f(x)$ *and* $f'(x)$ *at the give approximation* $x_0 = 1.5$*, gives*

$$x_0 = 1.5, \qquad f(1.5) = -0.625, \qquad f'(1.5) = 4.750.$$

Using the Newton's iterative formula (2.14), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{(-0.625)}{4.75} = 1.631579.$$

Now evaluating $f(x)$ *and* $f'(x)$ *at the new approximation* x_1*, gives*

$$x_1 = 1.631579, \qquad f(1.631579) = 0.0801869, \qquad f'(1.631579) = 5.9861501.$$

Using the iterative formula (2.14) again to get other new approximation. The successive iterates were shown in the Table 2.4. Just after the third iterations the required root is approximated to

Table 2.4: Solution of $x^3 = 2x + 1$ by Newton's method

n	x_n	$f(x_n)$	$f'(x_n)$	Error $x - x_n$
00	1.500000	-0.625000	4.750000	0.1180339
01	1.631579	0.0801869	5.9861501	-0.0135451
02	1.618184	0.000878	5.855558	-0.0001501
03	1.618034	0.00000007	5.854102	-0.0000001

be $x_3 = 1.618034$ *and the functional value is reduced to* 6.57×10^{-8}. *Since the exact solution is* 1.6180339*, so the actual error is* 1×10^{-7}. *We see that the convergence is quite faster than the methods considered previously.* •

Example 2.21 *Successive approximations* x_n *to the desired root are generated by the scheme*

$$x_{n+1} = \frac{1+3x_n^2}{4+x_n^3}, \qquad n \geq 0.$$

Find $f(x_n)$ *and* $f'(x_n)$ *and then use the Newton's method to find the approximation of the root accurate to* 10^{-2}*, starting with* $x_0 = 0.5$.

Solution. *Given*

$$x = \frac{1+3x^2}{4+x^3} = g(x),$$

and

$$x - g(x) = x - \frac{1+3x^2}{4+x^3} = \frac{x^4 - 3x^2 + 4x - 1}{4+x^3}.$$

Since

$$f(x) = x - g(x) = 0,$$

therefore, we have

$$f(x_n) = x_n^4 - 3x_n^2 + 4x_n - 1 \quad \textit{and} \quad f'(x_n) = 4x_n^3 - 6x_n + 4.$$

Using these functions values in the Newton's iterative formula (2.14), we have (see Figure 2.10),

$$x_{n+1} = x_n - \frac{x_n^4 - 3x_n^2 + 4x_n - 1}{4x_n^3 - 6x_n + 4}.$$

Finding the first approximation of the root using the initial approximation $x_0 = 0.5$*, we get*

$$x_1 = x_0 - \frac{x_0^4 - 3x_0^2 + 4x_0 - 1}{4x_0^3 - 6x_0 + 4} = 0.5 - \frac{0.3125}{1.5} = 0.2917.$$

Similarly, the other approximations can be obtained as

$$x_2 = 0.2917 - \frac{(-0.0813)}{2.3491} = 0.3263; \quad x_3 = 0.3263 - \frac{(-0.0029}{2.1812} = 0.3276.$$

Notice that $|x_3 - x_2| = 0.0013.$ •

Lemma 2.1 *Assume that* $f \in C^2[a,b]$ *and there exists a number* $\alpha \in [a,b]$*, where* $f(\alpha) = 0$*. If* $f'(\alpha) \neq 0$*, then there exists a number* $\delta > 0$ *such that the sequence* $\{x_n\}_{n=0}^{\infty}$ *defined by the iteration*

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \textit{for} \quad n = 0, 1, \ldots, \tag{2.17}$$

will converges to α *for any initial approximation* $x_0 \in [\alpha - \delta, \alpha - \delta]$. •

The Newton's method uses the iteration function

$$g(x) = x - \frac{f(x)}{f'(x)}, \tag{2.18}$$

is called the *Newton's iteration function.* Since $f(\alpha) = 0$, it is easy to see that $g(\alpha) = \alpha$. Thus the Newton's iteration for finding the root of the equation $f(x) = 0$ is accomplished by finding a fixed-point of the equation $g(x) = x$.

2.5 Secant Method

Since we known the main obstacle to using the Newton's method is that it may be difficult or impossible to differentiate the function $f(x)$. The calculation of $f'(x_n)$ may be avoided by approximating the slope of the tangent at $x = x_n$ by that of the chord joining the two points $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$, see Figure 2.14.

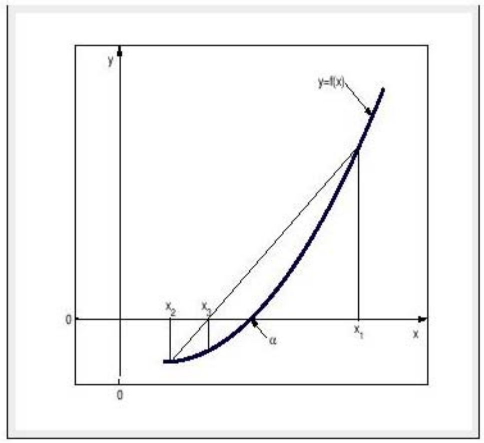


Figure 2.14: Graphical Solution of Secant Method.

The slope of the chord (or secant) is

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}. \tag{2.19}$$

Then by using this approximation of the derivative of the function in the Newton's iterative formula (2.14), we get

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}, \quad n \geq 1. \tag{2.20}$$

Procedure 2.4 (Secant Method)

1. *Choose the two initial approximation* x_0 *and* x_1.
2. *Check, if* $f(x_0) = f(x_1)$*, go to step 1 otherwise, continue.*
3. *Establish Tolerance* $(\epsilon > 0)$ *value for the function.*
4. *Compute new approximation for the root by using the iterative formula (2.20).*
5. *Check tolerance. If* $|x_n - x_{n-1}| \leq \epsilon$*, for* $n \geq 1$*, then end; otherwise, go back to step 4, and repeat the process.*

Example 2.27 *Use the secant method to find the approximate root of the following equation within the accuracy* 10^{-2} *take* $x_0 = 1.5$ *and* $x_1 = 2.0$ *as starting values*

$$x^3 = 2x + 1.$$

Solution. *Since* $f(x) = x^3 - 2x - 1$ *and*

$$\begin{aligned} x_0 &= 1.5, \qquad f(x_0) = -0.625, \\ x_1 &= 2.0, \qquad f(x_1) = 3.0, \end{aligned}$$

therefore, we see that $f(x_0) \neq f(x_1)$. *Hence, one can use the iterative formula (2.20), to get new approximation:*

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.5)(3.0) - (2.0)(-0.625)}{3.0 - (-0.625)}.$$

Hence $x_2 = 1.586207$ *and* $f(x_2) = -0.18434$. *Similar way, we can find the other possible approximation of the root. A summary of the calculations is given in Table 2.5.* •

Table 2.5: Solution of $x^3 = 2x + 1$ by secant method

n	x_{n-1}	x_n	x_{n+1}	$f(x_{n+1})$
01	1.500000	2.000000	1.586207	-0.1814342
02	2.000000	1.586207	1.609805	-0.0478446
03	1.586207	1.609805	1.618257	0.0013040

Example 2.26 *Show that the secant method for finding approximation of the square root of a positive number N is*

$$x_{n+1} = \frac{x_n x_{n-1} + N}{x_n - x_{n-1}}, \quad n \geq 1. \tag{2.22}$$

Carry out the first three approximations for the square root of 9, using $x_0 = 2, x_1 = 2.5$ and also compute absolute error.

Solution. *We shall compute $x = N^{1/2}$ by finding a positive root for the nonlinear equation*

$$x^2 - N = 0,$$

where $N > 0$ is the number whose root is to be found. If $f(x) = 0$, then $x = \alpha = N^{1/2}$ is the exact zero of the function

$$f(x) = x^2 - N.$$

Since the secant formula is

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}, \qquad n \geq 1.$$

Hence, assuming the initial estimates to the root, say, $x = x_0, x = x_1$, and by using the secant iterative formula, we have

$$x_2 = x_1 - \frac{(x_1 - x_0)(x_1^2 - N)}{(x_1^2 - N) - (x_0^2 - N)} = \frac{x_1 x_0 + N}{(x_1 + x_0)}.$$

In general, we have

$$x_{n+1} = \frac{x_n x_{n-1} + N}{x_n + x_{n-1}}, \qquad n = 1, 2, \ldots,$$

the secant formula for approximation of the square root of number N. Now using this formula for approximation of the square root of $N = 9$, taking $x_0 = 2$ and $x_1 = 2.5$, we have

$$x_2 = 3.1111, \qquad x_3 = 2.9901, \qquad x_4 = 2.9998.$$

Hence

$$\textit{Absolute Error} = |9^{1/2} - x_4| = |3 - 2.9998| = 0.0002,$$

is the possible absolute error •

2.6 Multiplicity of a Root

So far we discussed about the function which has simple root. Now we will discuss about the function which has multiple roots. A root is called a *simple root* if it is distinct, otherwise roots that are of the same order of magnitude are called *multiple*.

Definition 2.4 *(***Order of a Root***)*

The equation $f(x) = 0$ *has a root* α *of order* m*, if there exists a continuous function* $h(x)$*, and* $f(x)$ *can be expressed as the product*

$$f(x) = (x-\alpha)^m h(x), \qquad \textit{where} \quad h(\alpha) \neq 0. \tag{2.23}$$

So $h(x)$ *can be used to obtain the remaining roots of* $f(x) = 0$*. It is called polynomial deflation.*

A root of order $m = 1$ is called a *simple root* and if $m > 1$ it is called *multiple root*. In particular, a root of order $m = 2$ is sometimes called a *double root*, and so on.

Lemma 2.2 *Assume that* $f(x)$ *and its derivatives* $f'(x), f''(x), \cdots, f^{(m)}(x)$ *are defined and continuous on an interval about* $x = \alpha$*. Then* $f(x) = 0$ *has a root* α *of order* m *if and only if*

$$f(\alpha) = f'(\alpha) = f''(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0, \qquad f^{(m)}(\alpha) \neq 0. \tag{2.24}$$

Example

For example, consider the function $f(x) = x^3 - x^2 - 21x + 45$, which has two roots; a simple root at $\alpha = -5$ and a double root at $\alpha = 3$. This can be verified by considering the derivatives of the function as follows:

$$f'(x) = 3x^2 - 2x - 21, \qquad f''(x) = 6x - 2.$$

At the value $\alpha = -5$, we have $f(5) = 0$ and $f'(5) = 64 \neq 0$, so by (2.23), we see that $m = 1$. Hence $\alpha = -5$ is a simple root of the function. For the value $\alpha = 3$, we have

$$f(3) = 0, \qquad f'(3) = 0, \qquad f''(3) = 16 \neq 0,$$

so that $m = 2$ by (2.24), hence $\alpha = 3$ is a double root of the function. Note that this function $f(x)$ has the factorization and can be written in the form of (2.23) as (see Figure 2.18),

$$f(x) = (x-3)^2(x+5).$$

Example 2.29 *Find the multiplicity of the root $\alpha = 1$ of the equation $x \ln x = \ln x$.*

Solution. *From the given equation, we have*

$$\begin{array}{rclcrl} f(x) &=& x\ln x - \ln x & \textit{and} & f(1) = & 0, \\ f'(x) &=& \ln x + 1 - \dfrac{1}{x} & \textit{and} & f'(1) = & 0, \\ f''(x) &=& \dfrac{1}{x} + \dfrac{1}{x^2} & \textit{and} & f''(1) \neq & 0. \end{array}$$

Thus the multiplicity of the root $\alpha = 1$ of the given equation is 2. •

Example 2.30 *Consider the following two nonlinear equations*

$$(1) \qquad xe^x = 0 \qquad\qquad (2) \qquad x^2 e^x = 0.$$

(a) Find the Newton's method for the solutions of the given equations.
(b) Explain why one of the sequences converges much faster than the other to the root $\alpha = 0$.

Solution.

(a) $$x_{n+1} = g_1(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{x_n^2}{(1+x_n)}, \qquad n \geq 0,$$

$$x_{n+1} = g_2(x_n) = x_n - \frac{x_n^2 e^{x_n}}{(2x_n + x_n^2)e^{x_n}} = \frac{x_n + x_n^2}{(2+x_n)}, \qquad n \geq 0,$$

(b) From the first sequence, we have $g_1(x) = \dfrac{x^2}{(1+x)}$ *and* $g_1'(x) = \dfrac{x^2+2x}{(1+x)^2}$.

Thus $\quad |g_1'(\alpha)| = |g_1'(0)| = \left|\dfrac{0}{1}\right| = 0,$

which shows that the first sequence converges to zero. Similarly, from the second sequence, we have

$$g_2(x) = \frac{x_+ x^2}{(2+x)} \quad \text{and} \quad g_2'(x) = \frac{x^2+4x+2}{(2+x)^2}.$$

Thus $\quad |g_2'(0)| = \left|\dfrac{2}{4}\right| = \dfrac{1}{2} < 1,$

which shows that the second sequence is also converges to zero. Since the value of $|g_1'(0)|$ *is smaller than* $|g_2'(0)|$, *therefore, the first sequence converges faster than the second one.* •

Remark

Note that in the Example 2.30 the root $\alpha = 0$ is the simple root for the first equation because

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 1 \neq 0,$$

and for the second equation it is a multiple root because

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 0.$$

Therefore, the Newton's method converges very fast for the first equation and converges very slow for the second equation. However, in some cases simple modifications can be made to the methods to maintain the rate of convergence. Two such modified methods are considered here, called the Newton modified methods.

First Newton's Modified Method

If we wish to determine a root of known multiplicity m for the equation $f(x) = 0$, then the *first Newton's modified method* (also called the *Schroeder's method*) may be used. It has the form

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots \tag{2.25}$$

Second Newton's Modified Method

$$x_{n+1} = x_n - \frac{q(x_n)}{q'(x_n)}, \qquad q(x) = \frac{f(x)}{f'(x)}. \qquad f(x) = (x-\alpha)^m h(x),$$

which gives $$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - [f(x_n)][f''(x_n)]}, \quad n = 0, 1, 2, \ldots$$

Example 2.33 *Show that the function* $f(x) = e^x - \frac{x^2}{2} - x - 1$ *has zero of multiplicity 3 at* $\alpha = 0$ *and then, find the approximate solution of the zero of the function with the help of the Newton's method, first and second modified Newton's methods, by taking initial approximation* $x_0 = 1.5$ *within an accuracy of* 10^{-4}.

Solution. *Since* $\alpha = 0$ *is a root of* $f(x)$,

$$\begin{array}{rclrcl} f(x) & = & e^x - \frac{x^2}{2} - x - 1, & f(0) & = & 0, \\ f'(x) & = & e^x - x - 1, & f'(0) & = & 0, \\ f''(x) & = & e^x - 1, & f''(0) & = & 0, \\ f'''(x) & = & e^x, & f'''(0) & = & 1 \neq 0, \end{array}$$

the function has zero of multiplicity 3. In Table 2.6 we showed the comparison of three methods. •

n	Newton's Method x_n	1st. M.N. Method x_n	2nd. M.N. Method x_n
00	1.500000	1.500000	1.500000
01	1.067698	0.2030926	-0.297704
02	0.745468	3.482923e-03	-6.757677e-03
03	0.513126	1.010951e-06	-3.798399e-06
..	....		
25	7.331582e-05		

2.7 Convergence of Iterative Methods

Definition 2.5 (Order of Convergence)

Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ *converges to* α, *and let* $e_n = \alpha - x_n$ *define the error of the nth iterate. If two positive constants* $\beta \neq 0$ *and* $R > 0$ *exist, and*

$$\lim_{n \to \infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^R} = \lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^R} = \beta, \tag{2.27}$$

then the sequence is said to converge to α *with order of convergence* R. *The number* β *is called the asymptotic error constant. The cases* $R = 1, 2$ *are given special consideration.*
If $R = 1$, *the convergence of the sequence* $\{x_n\}_{n=0}^{\infty}$ *is called linear.*
If $R = 2$, *the convergence of the sequence* $\{x_n\}_{n=0}^{\infty}$ *is called quadratic.*

If R is large, the sequence $\{x_n\}$ converges rapidly to α; that is, (2.27) implies that for large values of n we have the approximation $|e_{n+1}| \approx \beta |e_n|^R$. For example, suppose that $R = 2$ and $|e_n| \approx 10^{-3}$; then we could expect that $|e_{n+1}| \approx \beta \times 10^{-6}$. •

Example 2.34 *Show that the following sequence*

$$x_{n+1} = \frac{1}{2}x_n\left(1 + \frac{N}{x_n^2}\right), \qquad n \geq 0,$$

will converge quadratically to $\sqrt{N}$.

Solution. *Since the sequence is given as*

$$x_{n+1} = \frac{1}{2}x_n\left(1 + \frac{N}{x_n^2}\right),$$

and $\alpha = \sqrt{N}$, *then we have*

$$\begin{aligned} x_{n+1} - \sqrt{N} &= \frac{1}{2}x_n\left(1 + \frac{N}{x_n^2}\right) - \sqrt{N} = \frac{1}{2}\left(x_n + \frac{N}{x_n} - 2\sqrt{N}\right) \\ &= \frac{1}{2}\left(\sqrt{x_n} - \frac{\sqrt{N}}{\sqrt{x_n}}\right)^2 = \frac{1}{2x_n}(x_n - \sqrt{N})^2. \end{aligned}$$

Thus

$$e_{n+1} = \frac{1}{2x_n}e_n^2, \quad \text{or} \quad e_{n+1} \propto e_n^2,$$

which shows the quadratic convergence.

Lemma 2.3 (Linear Convergence)

Let g is continuously differentiable on the interval $[a, b]$ and suppose that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose that $g'(x)$ is continuous on (a, b) with

$$|g'(x)| \leq k < 1; \qquad \textit{for all} \quad x \in (a, b).$$

If $g'(\alpha) \neq 0$, then for any $x_0 \in [a, b]$, the sequence $x_{n+1} = g(x_n)$, for $n \geq 0$, converges only linearly to the unique fixed-point α in $[a, b]$. •

Example 2.36 *Consider an iterative scheme*

$$x_{n+1} = 0.4 + x_n - 0.1x_n^2, \qquad n \geq 0.$$

Will this scheme converge to the fixed-point $\alpha = 2$? If yes, find its rate of convergence.

Solution. *Since*

$$g(x) = 0.4 + x - 0.1x^2 \quad \textit{and} g(2) = 0.4 + 2 - 0.1(2)^2 = 2,$$

which shows that the scheme converges to $\alpha = 2$. Also

$$g'(x) = 1 - 0.2x, \quad \textit{gives} \quad g'(2) = 1 - 0.4 = 0.6 \neq 0.$$

Therefore, the scheme converges linearly. •

Lemma 2.4 (Quadratic Convergence)

Let α be a solution of the equation $x = g(x)$. Suppose that $g'(\alpha) = 0$ and g'' is continuous on an open interval (a, b) containing α. Then there exists a $\delta > 0$ such that, for $x_0 \in [\alpha - \delta, \alpha + \delta]$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the iteration $x_{n+1} = g(x_n)$, for $n \geq 0$, converges at least quadratically to α. •

Example 2.37 *The iterative scheme*

$$x_{n+1} = 2 - (1 + a)x_n + ax_n^2, \qquad n \geq 0,$$

converges to $\alpha = 1$ for some values of a. Find the value of a for which the convergence is at least quadratic.

Solution. *Given*

$$g(x) = 2 - (1 + a)x + ax^2 \quad \text{and} \quad g(1) = 2 - (1 + a) + a = 1.$$

Thus, the given iterative scheme converges to 1. Also

$$g'(x) = -(1 + a) + 2ax,$$

and so

$$g'(1) = 0 = -(1 + a) + 2a, \quad \text{gives} \quad a = 1.$$

Thus, the convergence of the given iterative scheme is at least quadratic for the value of $a = 1$. •

Note 2.3 *The sequence* $\{x_n\}_{n=0}^{\infty}$ *defined by the iteration*

$$x_{n+1} = g(x_n), \quad \textit{for} \qquad n \geq 0,$$

converges only quadratically to α *if*

$$g'(\alpha) = 0 \quad \textit{but} \quad g''(\alpha) \neq 0.$$

and cubically (order three) to α *if*

$$g''(\alpha) = 0 \quad \textit{but} \quad g'''(\alpha) \neq 0.$$

In the similar manner the higher order of convergence can be achieved.

Example 2.38 *What is the order of convergence of the iteration*

$$x_{n+1} = \frac{x_n(x_n^2 + 3k)}{3x_n^2 + k}, \qquad k > 0, \qquad \textit{as it converges to the fixed-point } \alpha = \sqrt{k}.$$

Solution.

$$x_{n+1} = \frac{x_n(x_n^2 + 3k)}{3x_n^2 + k} = g(x_n), \quad \textit{which gives ,} \quad g(x) = \frac{x(x^2 + 3k)}{3x^2 + k}. \quad g'(x) = \frac{3(x^2 - k)^2}{(3x^2 + k)^2}.$$

$$g'(\sqrt{k}) = \frac{3((\sqrt{k})^2 - k)^2}{(3(\sqrt{k})^2 + k)^2} = 0. \qquad g''(x) = \frac{48xk(x^2 - k)}{(3x^2 + k)^3}, \quad \textit{gives,} \quad g''(\sqrt{k}) = 0, \quad \textit{but} \quad g'''(\sqrt{k}) \neq 0.$$

Hence, the order of convergence for the given iteration is exactly cubic.

Case 2.1 (Bisection Method)

Example 2.39 *If α is the fixed-point of the equation $x = g(x)$ in $[x, b]$. Then show that rate of convergence of the bisection method is linear.*

Solution. *Since the bisection iteration function is define in the interval $[x, b]$, so by using the bisection formula (2.2), we have*

$$g(x) = \frac{x+b}{2} \quad \text{and} \quad g'(x) = \frac{1}{2}.$$

So at $x = \alpha$, we have

$$g'(\alpha) = \frac{1}{2} \neq 0,$$

therefore, by the Lemma 2.3, the convergence is linear.

Case 2.2 (Fixed-point Method)

The convergence rate of the fixed-point iteration can be analyzed as follows. The general procedure is given by

$$x_{n+1} = g(x_n), \qquad n = 0, 1, 2, \ldots \tag{2.28}$$

Let $x = \alpha$ denote the solution to $f(x) = 0$, so $f(\alpha) = 0$ and $\alpha = g(\alpha)$. Then

$$x_{n+1} - \alpha = e_{n+1} = g(x_n) - g(\alpha), \tag{2.29}$$

where e_{n+1} denote the error of the (n+1)th iterate. Expressing $g(\alpha)$ in the Taylor series about x_n gives:

$$g(\alpha) = g(x_n) + g'(\eta)(\alpha - x_n), \qquad x_n \leq \eta \leq \alpha. \tag{2.30}$$

Solving (2.30) for $g(x_n) - g(\alpha)$ and substituting into (2.29), we get

$$e_{n+1} = g'(\eta)e_n, \tag{2.31}$$

or

$$|e_{n+1}| = |g'(\eta)||e_n|. \tag{2.32}$$

Now suppose that $|g'(x)| \leq k < 1$ for all values of x in an interval. If x_1 is choose in this interval, x_2 will also be in the interval and the fixed-point iteration method will converge, since

$$\left|\frac{e_{n+1}}{e_n}\right| = |g'(\eta)| < 1. \tag{2.33}$$

Convergence is linear since e_{n+1} is linearly dependent on e_n. If $|g'(\eta)| > 1$, the procedure diverges. If $|g'(\eta)| < 1$, but close to one, convergence is quite slow. •

Example 2.40 *(a) Show that $\alpha = 1$ is a unique fixed-point of*

$$g(x) = \frac{x^2 - 4x + 7}{4}.$$

(b) Find the rate of convergence of the sequence

$$x_n = \frac{x_{n-1}^2 - 4x_{n-1} + 7}{4}.$$

Solution. *(a) Firstly, we show that $\alpha = 1$ is a fixed-point of $g(x)$ by showing that $g(1) = 1$ and it happened because*

$$g(1) = \frac{1 - 4 + 7}{4} = 1.$$

It is unique also because

$$g'(x) = \frac{2x - 4}{4}, \quad \text{and} \quad |g'(1)| = 0.5 < 1.$$

(b) To find the rate of convergence of the given sequence, we have

$$g(x) = \frac{x^2 - 4x + 7}{4} \quad \text{and} \quad g'(x) = \frac{2x - 4}{4}.$$

Taking $x = \alpha = 1$, gives

$$g'(1) = \frac{2 - 4}{4} = -\frac{1}{2} \neq 0.$$

Hence the rate of the convergence of the given sequence is linear. •

Case 2.3 (Newton's Method)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = g(x_n). \qquad g(x) = x - \frac{f(x)}{f'(x)},$$

then

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}.$$

Now we show that the Newton's method is quadratically convergent for the simple root. Let $x = \alpha$ denote the solution to $f(x) = 0$, so $f(\alpha) = 0$ and $\alpha = g(\alpha)$. Since $x_{n+1} = g(x_n)$, we can write

$$x_{n+1} - \alpha = e_{n+1} = g(x_n) - g(\alpha), \tag{2.35}$$

where e_n denote the error of the nth iterate. Let us expand $g(x_n)$ as a Taylor series in terms of $(x_n - \alpha)$ with the second derivative term as the remainder:

$$g(x_n) = g(\alpha) + g'(\alpha)(x_n - \alpha) + \frac{g''(\eta)}{2}(x_n - \alpha)^2, \quad x_n \le \eta \le \alpha.$$

Since

$$g'(\alpha) = \frac{f(\alpha)f''(\alpha)}{[f'(\alpha)]^2} = 0, \quad \text{because } f(\alpha) = 0, \text{ we have}$$

$$g(x_n) = g(\alpha) + \frac{g''(\eta)}{2}(x_n - \alpha)^2. \qquad \text{we get} \quad e_{n+1} = g(\alpha) - g(x_n) = -\frac{g''(\eta)}{2}(e_n)^2.$$

This implies that each error is (in the limit) proportional to the square of the previous error, that is, the Newton's method is quadratically convergent.

Example 2.42 *If $x = \alpha$ is a simple root of $f(x) = 0$, then show that the rate of convergence of the Newton's method is at least quadratic.*

Solution. *Consider the Newton's iteration function which is define as follows:*

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Since α is a simple root of nonlinear equation $f(x) = 0$, so

$$f(\alpha) = 0 \qquad \text{and} \qquad f'(\alpha) \neq 0.$$

Thus taking derivative of $g(x)$, we get

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}.$$

At $x = \alpha$, we know that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, so at $x = \alpha$, we get

$$g'(\alpha) = 1 - \frac{f'(\alpha)f'(\alpha) - f(\alpha)f''(\alpha)}{[f'(\alpha)]^2} = \frac{f(\alpha)f''(\alpha)}{[f'(\alpha)]^2} = 0.$$

Thus from Lemma 2.4, the rate of convergence of Newton's method is at least quadratic.

Example 2.44 *If $x = \alpha$ is a root of multiplicity m of $f(x) = 0$, then show that the rate of convergence of the Newton's method is linear.*

Solution. *Consider the Newton's iteration function which is define as follows:*

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Since the function $f(x)$ has multiple root, so

$$f(x) = (x-\alpha)^m h(x),$$

and its derivative is

$$f'(x) = m(x-\alpha)^{m-1}h(x) + (x-\alpha)^m h'(x).$$

Substituting the values of the $f(x)$ and $f'(x)$ in the above equation, we get

$$g(x) = x - \frac{(x-\alpha)^m h(x)}{(m(x-\alpha)^{m-1}h(x) + (x-\alpha)^m h'(x))},$$

or

$$g(x) = x - \frac{(x-\alpha)h(x)}{(mh(x) + (x-\alpha)h'(x))}.$$

Then

$$\begin{aligned} g'(x) = 1 \quad - \quad & \{([mh(x) + (x-\alpha)][h(x) + (x-\alpha)h'(x)] - [(x-\alpha)h(x)] \\ & [mh'(x) + h'(x) + (x-\alpha)h''(x)])\}/([mh(x) + (x-\alpha)h'(x)]^2). \end{aligned}$$

At $x = \alpha$, and since $f(\alpha) = 0$, we have

$$g'(\alpha) = 1 - \frac{[mh(\alpha)][h(\alpha)]}{[mh(\alpha)]^2} = 1 - \frac{1}{m} \neq 0, \qquad (m > 1).$$

Therefore, the Newton's method converges to a multiple zero from any sufficiently close approximation and the convergence is linear (by the Lemma 2.3), with ration $(1 - \frac{1}{m})$. In particular for a double root, the ration is $\frac{1}{2}$, which is comparable with the convergence of the bisection method. •

Case 2.4 (Secant Method)

The convergence rate of the secant method can be analyzed as follows. The general procedure is

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \tag{2.37}$$

As before, let $x_{n-1} = \alpha - e_{n-1}, x_n = \alpha - e_n$, *and* $x_{n+1} = \alpha - e_{n+1}$. *Then*

$$e_{n+1} = e_n - \frac{f(\alpha - e_n)(e_n - e_{n-1})}{f(\alpha - e_n) - f(\alpha - e_{n-1})}$$

$$|e_n| \approx \beta |e_{n-1}|^{1.61803}.$$

Thus the error of the secant method is of order 1.61803, *which is in-between* 1 *and* 2. *This shows that the order of the secant method is better than the bisection method and fixed-point method but less than the Newton's method.*
Remember, however, that the secant method does not require the derivative of $f(x)$ *to be evaluated at each step, so that in many ways the secant method is a very attractive alternative to the standard Newton's method.* •

Example 2.45 *If $x = \alpha$ is a root of multiplicity m of $f(x) = 0$, then show that the rate of convergence of the modified Newton's method is at least quadratic.*

Solution. *The first modified Newton's iteration function is define as follows:*

$$g(x) = x - m\frac{f(x)}{f'(x)}. \tag{2.39}$$

Since the function $f(x)$ has multiple root, so

$$f(x) = (x-\alpha)^m h(x),$$

and its derivative is

$$f'(x) = m(x-\alpha)^{m-1}h(x) + (x-\alpha)^m h'(x).$$

Substituting the values of the $f(x)$ and $f'(x)$ in (2.39), we get

$$g(x) = x - \frac{m(x-\alpha)^m h(x)}{(m(x-\alpha)^{m-1}h(x) + (x-\alpha)^m h'(x))},$$

or

$$g(x) = x - \frac{m(x-\alpha)h(x)}{(mh(x) + (x-\alpha)h'(x))}.$$

Then

$$\begin{aligned} g'(x) = 1 \quad - \quad & m\{([mh(x) + (x-\alpha)][h(x) + (x-\alpha)h'(x)] - [(x-\alpha)h(x)] \\ & [mh'(x) + h'(x) + (x-\alpha)h''(x)])\}/([mh(x) + (x-\alpha)h'(x)]^2). \end{aligned}$$

At $x = \alpha$, and since $f(\alpha) = 0$, we have

$$g'(\alpha) = 1 - \frac{[m^2h^2(\alpha)]}{[mh(\alpha)]^2},$$

it gives

$$g'(\alpha) = 0.$$

Therefore, the modified Newton's method converges to a multiple root α and the convergence is at least quadratically (by the Lemma 2.4).
Similarly, if $x = \alpha$ is a root of multiplicity m of $f(x) = 0$, then by using the Example 2.45 one can easily show that the rate of convergence of the Newton's method is linear. As the Newton iteration function is defined by

$$g(x) = x - m\frac{f(x)}{f'(x)},$$

and proceeding in the same way as we did in the Example 2.45, one can get

$$g'(\alpha) = 1 - \frac{1}{m} \neq 0, \quad \text{because} \quad m > 1.$$

Hence the Newton's method converges to a multiple root α from any sufficiently close approximation and the convergence is linear (by the Lemma 2.3) with ration $(1 - \frac{1}{m})$. In particular for a double root, the ration is $\frac{1}{2}$, which is comparable with the convergence of the bisection method. •

2.8 Systems of Nonlinear Equations

$$f_1(x,y)=0, \quad \text{and} \quad f_2(x,y)=0.$$

Given the continuous functions $f_1(x,y)$ and $f_2(x,y)$, find the values $x=\alpha$ and $y=\beta$ such that

$$f_1(\alpha,\beta)=0 \quad \text{and} \quad f_2(\alpha,\beta)=0. \tag{2.42}$$

Newton's Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

We call the following matrix J a *Jacobian matrix*

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}.$$

Example 2.46 *For the following system of two equations*

$$\begin{aligned} x^3 + 3y^2 &= 21 \\ x^2 + 2y &= -2 \end{aligned}$$

Find the Jacobian matrix and its inverse using initial approximation $(1, -1)$*, then find the first approximation by using the Newton's method.*

Solution. *Given*

$$\begin{aligned} f_1(x,y) &= x^3 + 3y^2 - 21, \quad f_{1x} = 3x^2, \quad f_{1y} = 6y \\ f_2(x,y) &= x^2 + 2y + 2, \quad f_{2x} = 2x, \quad f_{2y} = 2. \end{aligned}$$

At the given initial approximation $x_0 = 1$ *and* $y_0 = -1$*, we have*

$$\begin{aligned} f_1(1,-1) &= -17, \quad \frac{\partial f_1}{\partial x} = f_{1x} = 3, \quad \frac{\partial f_1}{\partial y} = f_{1y} = -6 \\ f_2(1,-1) &= 1, \quad \frac{\partial f_1}{\partial x} = f_{2x} = 2, \quad \frac{\partial f_2}{\partial y} = f_{2y} = 2 \end{aligned}$$

The Jacobian matrix J *at the given initial approximation can be calculated as*

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad J^{-1} = \frac{1}{18}\begin{pmatrix} 2 & 6 \\ -2 & 3 \end{pmatrix},$$

is the inverse of the Jacobian matrix.Now to find the first approximation we have to solve the following equation

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{18}\begin{pmatrix} 2 & 6 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -17 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.5556 \\ -3.0556 \end{pmatrix},$$

the required first approximation. •

Similarly, for a large system of equations it is convenient to use vector notation. Consider the system

$$\mathbf{f}(\mathbf{x}) = \mathbf{0},$$

where $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$. Denoting the nth iterate by $\mathbf{x}^{[n]} = (x_1^{[n]}, x_2^{[n]}, x_3^{[n]}, \ldots, x_n^{[n]})^T$, then the Newton's method is defined by

$$\mathbf{x}^{[n+1]} = \mathbf{x}^{[n]} - \left[J(\mathbf{x}^{[n]})\right]^{-1} \mathbf{f}(\mathbf{x}^{[n]}), \tag{2.49}$$

where the Jacobian matrix J is defined as

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

Since the iterative formula (2.49) involves the inverse of Jacobian J, in practice we do not attempt to find this explicitly. In stead of using the form of (2.49) we use the following form

$$J(\mathbf{x}^{[\mathbf{n}]})\mathbf{Z}^{[\mathbf{n}]} = -\mathbf{f}(\mathbf{x}^{[n]}), \tag{2.50}$$

where $\mathbf{Z}^{[n]} = \mathbf{x}^{[n+1]} - \mathbf{x}^{[n]}$.

This represents a system of linear equations for $\mathbf{Z}^{[n]}$ and can be solved by any methods described in the next Chapter 3. Once $\mathbf{Z}^{[n]}$ has been found, the next iterate is calculated from

$$\mathbf{x}^{[n+1]} = \mathbf{Z}^{[n]} + \mathbf{x}^{[n]}. \tag{2.51}$$