BMT-222 (September 2018)

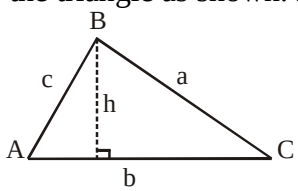
Chapter 9 The law of sines and cosines (Summary)

The Law of Sines

- The law of sines enables us to solve many oblique triangles (triangles not containing right angle).
- Solving triangles means finding the measures of all sides and angles of the triangle.

Derivation of Law of Sines

Consider the triangle as shown. Draw a perpendicular from vertex B to the opposite side.



$$\sin A = \frac{h}{c} \quad \Rightarrow h = c\sin A \qquad (1)$$

$$\sin C = \frac{h}{a} \quad \Rightarrow h = a\sin C \qquad (2)$$

From equation (1) and (2) $\quad a\sin C = c\sin A$

Dividing both side $\sin A \times \sin C$ $\quad \dfrac{a\sin C}{\sin A \times \sin C} = \dfrac{c\sin A}{\sin A \times \sin C}$

$$\Rightarrow \frac{a}{\sin A} = \frac{c}{\sin C} \qquad (3)$$

If we drop a perpendicular from C to c, we get by the same argument

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad (4)$$

Combining equations (3) and (4), we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad (5)$$

called the law of sines or simply the sine law.

To solve an oblique triangle by the law of sines, we need either:

1. Two angles and one side, or
2. Two sides and the angle opposite one of them

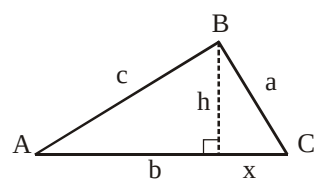
The Law of Cosines

Use the law of cosines to solve an oblique triangle given

1. Two sides and the included angle, or
2. Three side.

Derivation of Law of Cosines

Consider the triangle as shown. Draw a perpendicular from vertex B to the opposite side and let x denote the distance from C to the foot of the perpendicular.



Since, $\sin A = \dfrac{h}{c} \quad \Rightarrow h = c\sin A$, From the Pythagorean theorem: $\quad a^2 = c^2\sin^2 A + x^2 \qquad (1)$

From the above figure $\cos A = \dfrac{b-x}{c} \quad \Rightarrow b - x = c\cos A$ or $\quad x = b - c\cos A$

Substituting x in equation (1), $a^2 = c^2\sin^2 A + \left(b - c\cos A\right)^2 \Rightarrow a^2 = c^2\sin^2 A + b^2 - 2bc\cos A + c^2\cos^2 A$

$$\Rightarrow a^2 = c^2\left(\sin^2 A + \cos^2 A\right) + b^2 - 2bc\cos A \quad \Rightarrow a^2 = c^2 + b^2 - 2bc\cos A \qquad (2)$$

This formula is known as law of cosines or simply the cosine law. From dropping the perpendicular from the other vertices, we get two other forms of the cosine law, as summarized below.

$$a^2 = c^2 + b^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

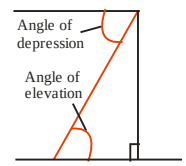
Law of cosines (verbal form): The square of any side of a triangle equals the sum of the squares of the other two sides minus twice the product of those two sides and the cosine of angle between them.

Note: If angle $A = 90^0$, then the equation (2) reduces to $a^2 = c^2 + b^2 - 2bc \times 0 \Rightarrow a^2 = c^2 + b^2$

It means that the cosine law is a generalization of the Pythagorean theorem.

Angle of elevation and depression

- If an object is located above a horizontal plane, then the angle between the horizontal and line of sight is called angle of elevation as shown.
- If an observer is looking down at an object then the angle between the horizontal and line of sight is called angle of depression as shown.
- The angle of elevation is equal to the angle of depression as shown.



Exercises / Section 9.4 (page 289-290)

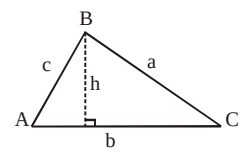


Figure 9.22

Problem # 9: Solve the triangle from the given information (Figure 9.22). $A = 29.3^o$ $a = 71.6$, $c = 136$; angle C acute.

Problem # 21: A plane maintaining an air speed of $560\,mi/hr$ is heading 10° west of north. A north wind causes the actual course to be 11° west of north. Find the velocity of the plane with respect to the ground.

Problem # 25: From a point on the ground, the angle of elevation of a balloon is 49°. From a second point $1250\ ft$ away on the opposite side of the balloon and in the same vertical plane as the balloon and the first point, the angle of elevation is 33°. Find the distance from the second point to the balloon.

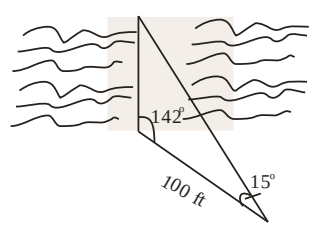
(Problems solved in class # 1, 7, 17, 23) **Home Work** (Problem # 9, 21, 25)

Problem # 1: Solve the triangle from the given information (Figure 9.22). $A = 20^o\,10', C = 50^o\,40', b = 4.00$

Problem # 7: Solve the triangle from the given information (Figure 9.22). $C = 31.5^o$, $a = 13.3$, $c = 6.82$

Problem # 17: Solve the triangle from the given information (Figure 9.22). $B = 33^o\,45^/$, $a = 1.146$, $b = 1.00$ angle C obtuse.

Problem # 23: A surveyor wants to find the width of a river from a certain point on the bank. Since no other points on the bank nearby are accessible, he takes the measurements shown in the figure. Find the width of the river.



Exercises / Section 9.5 (page 295-296)

(Problems solved in class # 5, 13, 19) **Home Work** (Problem # 1, 9, 15, 17)

Problem # 1 Solve the given triangles. $A = 46.3^o$, $b = 1.00$, $c = 2.30$

Problem # 9 Solve the given triangles. $C = 39.4^o$, $a = 126$, $b = 80.1$

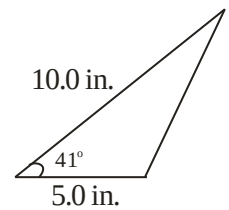
Problem # 15 A ship sails $16.0\ km$ due east, turns 20° north of east, and then continues for another $11.5\ km$. Find its distance from the starting point.

Problem # 17 A small plane is heading 5° north of east with an air speed of $151 mi/hr$. The wind is from the south at $35.3 mi/hr$. Find the actual course and the velocity with respect to the ground.

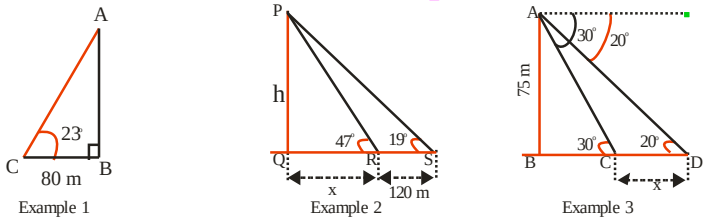
Problem # 5 Solve the given triangles. $a = 20.1$, $b = 30.3$, $c = 25.7$

Problem # 13 A civil engineer wants to find the length of a proposed tunnel. From a distant point he observes that the respective distance to the ends of the tunnel are 585 ft and 624 ft. The angle between the lines of sight is 33.4^o. Find the length of the proposed tunnel.

Problem # 19 Find the perimeter (the border or outer boundary of a two-dimensional figure) of the triangle shown below.



Solved Examples



Example # 1: An electricity pylon stands on a horizontal ground. At a point $80\,m$ from the base of the pylon, the angle of elevation of the top of the pylon is 23°. Calculate the height of the pylon to the nearest meter.

Solution: Figure show the pylon AB and the angle of elevation of A from point C is

$$\tan 23^0 = \frac{AB}{BC} \Rightarrow \tan 23^0 = \frac{AB}{80} \Rightarrow AB = 80\tan 23^0 \Rightarrow AB = 80(0.4245) \Rightarrow AB = 33.96\,m$$

Example # 2: A surveyor measures the angle of elevation of the top of a perpendicular building as 19^0. He moves $120\,m$ nearer to the building and find the angle of elevation is now 47^0. Determine the height of the building?

Solution: The building PQ and the angles of elevations are shown in the figure. In triangle PQS,

$$\tan 19^0 = \frac{h}{x+120} \Rightarrow h = \tan 19^0 (x+120) \Rightarrow h = 0.3443(x+120) \qquad (1)$$

$$\text{In triangle } PQR,\ \tan 47^0 = \frac{h}{x} \Rightarrow h = \tan 47^0 (x) \Rightarrow h = 1.0724(x) \qquad (2)$$

Equating equation (1) and (2):

$$0.3443(x+120) = 1.0724x \Rightarrow\ 1.0724x - 0.3443x = 41.316 \Rightarrow\ 0.7281x = 41.316 \Rightarrow\ x = 56.74\,m$$

From equation (2) $h = 1.0724x \Rightarrow h = 1.0724(56.74) \Rightarrow h = 60.85\,m$

Example # 3: The angle of depression of a ship viewed at a particular instant from the top of a $75\,m$ vertical cliff is 30^0. Find the distance of the ship from the base of the cliff at this instant. The ship is sailing away from the cliff at a constant speed and 1 minute later its angle of depression from the top of the cliff is 20^0. Determine the speed of the ship in km/h.

Solution: Cliff AB, the initial position of the ship is at C and the final position at D, $\angle ACB = 30^o$

$$\tan 30^0 = \frac{AB}{BC} \Rightarrow \tan 20^0 = \frac{75}{BC} \Rightarrow 0.5773 = \frac{75}{BC} \Rightarrow BC = \frac{75}{0.5773} \Rightarrow BC = 129.9\,m$$

Hence the initial position of the ship from the base of the cliff is $129.9\,m$.

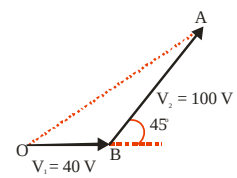
In triangle ABD,

$$\tan 20^0 = \frac{AB}{BD} \Rightarrow \tan 20^0 = \frac{75}{BC+CD} \Rightarrow \tan 20^0 = \frac{75}{129.9+x} \Rightarrow 0.3639(129.9+x) = 75 \Rightarrow x = 76.16\,m$$

The ship sails $76.16\,m$ in 1 minute (60 s),

$$speed = \frac{76.16}{60}\,m/s \Rightarrow speed = \frac{76.16\times 60\times 60}{60\times 1000}\,km/h \Rightarrow speed = 4.57\,km/h$$

Example # 4: Two voltage phasors are shown in the figure. If $V_1 = 40V$ and $V_2 = 100V$, determine the value of their resultant (that is length OA) and the angle the resultant makes with V_1.



Solution: $\angle OBA = 180^o - 45^o = 135^o$

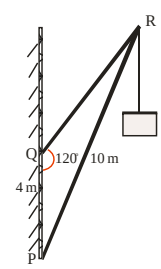
Applying law of cosines $(OA)^2 = V_1^2 + V_2^2 - 2V_1V_2\cos 135^o \Rightarrow (OA)^2 = (40)^2 + (100)^2 - 2\times 40\times 100(-0.707)$

$$\Rightarrow (OA)^2 = 1600 + 10000 + 5657 \Rightarrow (OA)^2 = 17257 \Rightarrow OA = 131.4V$$

law of sines $\dfrac{131.4}{\sin 135} = \dfrac{100}{\sin\angle AOB} \Rightarrow \sin\angle AOB = 0.5381 \Rightarrow \angle AOB = \sin^{-1}(0.5381) \Rightarrow \angle AOB = 32.55^o$

(or 147.45^o which is not possible). Hence the resultant voltage is $131.4V$ and 32.55^o to V_1.

Example # 5 In the figure PR represents the inclined jib of a crane and is $10\,m$ long. PQ is $4\,m$ long. Determine the inclination of the jib to the vertical and length of the QR.



Solution: sine rule: $\dfrac{PR}{\sin 120} = \dfrac{PQ}{\sin R} \Rightarrow \sin R = \dfrac{4\sin 120}{10} \Rightarrow \sin R = 0.3464$

$$\Rightarrow R = \sin^{-1}(0.3464) \Rightarrow R = 20.27^o \ (or\ 159.73^o\ which\ is\ not\ possible)$$

$\therefore\ P = 180^o - 120^o - 20.27^o \Rightarrow P = 39.73^o$ which is the inclination of the jib to the vertical

Again applying the sine rule, $\dfrac{10}{\sin 120} = \dfrac{QR}{\sin 39.73} = 7.38\,m$

Exercise #94

$1^\circ = 60'$

$1' = \frac{1}{60}^\circ$

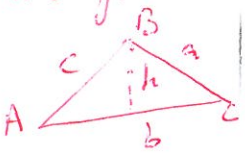
$10' = \frac{1}{6}^\circ = 0.167$

P#1

Solve the triangle from the given information

$A = 20^\circ 10'$, $b = 4.00$

$C = 50^\circ 40'$, ~~Triangle~~ ~~c~~ ~~calculate~~.



Using law of Sine

$\angle A = 20^\circ 10' = 20.167^\circ$

$\angle C = 50^\circ 40' = 50.67^\circ$

$\angle B = 180^\circ - \angle A - \angle C$

$\Rightarrow \angle B = 109.163^\circ$

$\Rightarrow \boxed{\angle B = 109^\circ 10'}$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow a = \frac{\sin A}{\sin B} \times b$

$\Rightarrow a = \frac{\sin 20.167}{\sin 109.163} \times 4 = \frac{0.345}{0.945} \times 4$

$\Rightarrow a = 1.46$

$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow c = \frac{\sin C \times a}{\sin A} = \frac{1.46 \sin 50.67}{0.345}$

$\Rightarrow c = \frac{1.46 \times 0.774}{0.345} \Rightarrow c = 3.25$

P#7 $C = 31.5^\circ$, $a = 13.3$, $c = 6.82$

$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \sin A = \frac{a \sin C}{c} = \frac{13.3 \times \sin 31.5}{6.82}$

$\Rightarrow \sin A = \frac{13.3 \times 0.522}{6.82}$

$\Rightarrow \sin A = 1.02$

No Solution Sin max. value is 1

P#17 $\angle B = 33^\circ 45'$, $a = 1.146$, $b = 1$, "C" is obtuse

$= 33.75^\circ$

$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \sin A = \frac{a \sin B}{b} = \frac{1.146 \sin 33.75}{1}$

$\sin A = \frac{0.556 \times 1.146}{1} = 0.637$

$\Rightarrow A = \sin^{-1}(0.637) = 39.57^\circ \simeq 39^\circ 34'$

$\angle A + \angle B + \angle C = 180^\circ$

$\angle C = 180^\circ - 33.75 - 39.57 = 106.68^\circ$

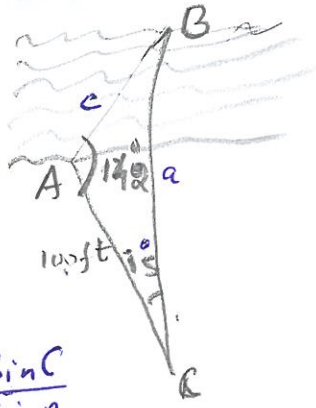
$\angle C \simeq 106^\circ 41'$

$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow c = \frac{b \sin C}{\sin B}$

$\Rightarrow c = \frac{\sin 106.68}{\sin 33.75} = \frac{0.958}{0.556}$

$c = 1.72$

P#23 A surveyor wants to find the width of a river from a certain point on the bank. Since no other points on the bank nearby are accessible, he ~~takes~~ takes measurement shown in figure. Find the width of the river.



$\angle A + \angle B + \angle C = 180^\circ$

$\angle B = 180^\circ - \angle A - \angle C$

$= 180 - 142 - 15$

$\angle B = 23^\circ$

$\frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow c = \frac{b \sin C}{\sin B}$

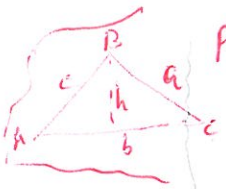
$= \frac{100 \sin 15}{\sin 23} = \frac{25.882}{0.391}$

$= 66.19$

$\therefore c = 66.2\,ft$

Exercise #9.5

P#5 Solve the given triangle



$a = 20.1,\ b = 30.3,\ c = 25.7$

For Angle A

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Rightarrow 2bc\cos A = b^2 + c^2 - a^2$$

$$\Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{(30.3)^2 + (25.7)^2 - (20.1)^2}{2 \times 30.3 \times 25.7}$$

$$\Rightarrow \cos A = \frac{918.09 + 660.49 - 404.01}{1557.42}$$

$$\cos A = \frac{1174.57}{1557.42} = 0.7542$$

$$\Rightarrow A = \cos^{-1}(0.7542)$$

$$\Rightarrow A = 41.04^\circ$$

For B using Sine law

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\sin B = \frac{b \sin A}{a} = \frac{30.3 \times \sin 41.04}{20.1} = \frac{30.3 \times 0.6566}{20.1}$$

$$\Rightarrow \sin B = 0.9898 \Rightarrow B = \sin^{-1}(0.9898)$$

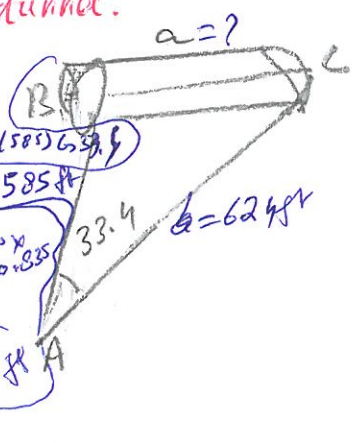
$$\Rightarrow B = 81.87^\circ$$

$$\angle A + B + \angle C = 180$$

$$\angle C = 180 - 41.04 - 81.87$$

$$\angle C = 57.15^\circ$$

P#13 A civil engineer wants to find the length of a proposed tunnel. From a distant point he observes that the respective distances to the ends of the tunnel are 585 ft and 624 ft. The angle between lines of sight is 33.4°. Find the length of the proposed tunnel.



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$= (624)^2 + (585)^2 - 2(624)(585)\cos 33.4$$

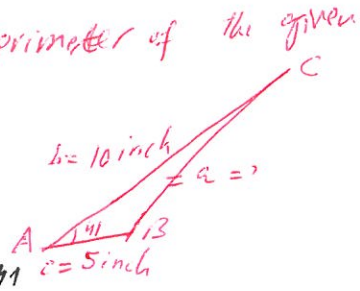
$$\Rightarrow a^2 = 389376 + 342225 - 730080 \times 0.835$$

$$a^2 = 731601 - 609616.8$$

$$= 121984.2$$

$$a = 349.3\text{ ft}$$

P#19 Find the perimeter of the given triangle



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$= 100 + 25 - 2(10)(5)\cos 41$$

$$a^2 = 125 - 75.5$$

$$a^2 = 49.5$$

$$\Rightarrow a = 7.04\text{ inch}$$

$$\text{Perimeter} = a + b + c$$

$$= 7.04 + 10 + 5$$

$$\text{Perimeter} = 22.04\text{ inch}$$

284

17. Find the force required to keep a 3,125-lb car parked on a hill that makes an angle of 11°31′ with the horizontal.

18. What is the force required to keep a 295-lb cart on a ramp inclined at an angle of 15.0° to the horizontal?

19. A force of 473 lb is required to pull a boat up a ramp inclined at 19.0° to the horizontal. Find the weight of the boat.

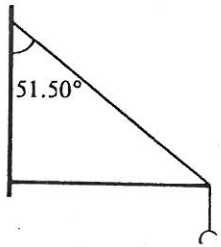
20. A woman is dragging a crate across the floor by means of a rope. She is able to pull with a force of 79 lb at an angle of 41° with the ground. What force parallel to the floor would have the same effect?

21. A 98-lb weight hanging on a rope is pulled sideways by a force of 25 lb. Assuming the system to be in equilibrium, what is the tension on the rope and the angle of the rope with the vertical?

22. A 31-lb weight hanging on a rope is to be pulled sideways so that the rope makes an angle of 17° with the vertical. What is the force required to accomplish this?

23. Determine the force against the horizontal support in Figure 9.21.

CHAPTER #9



Basic Technical Mathematics with Calculus
Peter K. F. Kuhfitting

24. A 51-lb weight hanging on a rope is pulled sideways by a force of 15 lb. If the system is in equilibrium, determine the tension on the rope and the angle that the rope makes with the vertical.

9.4 The Law of Sines

So far all of our applications have involved right triangles. We will now consider oblique triangles (triangles not containing a right angle) and thereby greatly increase the usefulness of our previous methods. We shall begin by studying the *law of sines* in this section and then the *law of cosines* in the next.

To derive the law of sines, consider the triangles in Figure 9.22. Now pick an arbitrary vertex such as B and drop a perpendicular to the side opposite. (The triangles in the figure illustrate the different cases.) Making use of the resulting right triangles, we get from Figure 9.22(a)

$$\sin A = \frac{h}{c} \quad \text{and} \quad \sin C = \frac{h}{a}$$

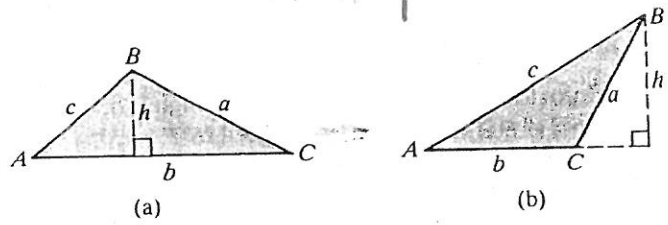


Figure 9.22

or $h = c \sin A$ and $h = a \sin C$. From Figure 9.22(b),

$$h = c \sin A \quad \text{and} \quad h = a \sin(180° - C) = a \sin C$$

In both cases

$$c \sin A = a \sin C$$

and

$$\frac{c \cancel{\sin A}}{\cancel{\sin A} \sin C} = \frac{a \cancel{\sin C}}{\sin A \cancel{\sin C}} \quad \text{dividing by } \sin A \sin C$$

$$\frac{a}{\sin A} = \frac{c}{\sin C} \tag{9.5}$$

If we drop a perpendicular from C to c, we get by the same argument

$$\frac{a}{\sin A} = \frac{b}{\sin B} \tag{9.6}$$

Combining equations (9.5) and (9.6), we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{9.7}$$

called the **law of sines** or simply the **sine law.**

Law of sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The law of sines enables us to solve many oblique triangles. **Solving a triangle** means finding the measures of all sides and angles of the triangle.

Note: Examples 1 and 2 refer to Figure 9.22.

Example # 1

Example 1 If $A = 45°41'$, $C = 130°26'$, and $b = 5.000$, find B, a, and c. 286

Solution. Since the sum of the angles of a triangle is 180°, $B = 180° - (45°41' + 130°26') = 179°60' - 176°7' = 3°53'$. From the law of sines

$$\frac{a}{\sin 45°41'} = \frac{5.000}{\sin 3°53'} \quad \text{or} \quad a = \frac{5.000 \sin 45°41'}{\sin 3°53'} = 52.82$$

$$\frac{c}{\sin 130°26'} = \frac{5.000}{\sin 3°53'} \quad \text{or} \quad c = \frac{5.000 \sin 130°26'}{\sin 3°53'} = 56.19$$

Thus $B = 3°53'$, $a = 52.82$, and $c = 56.19$.

Example # 2

Example 2 Given that $B = 54°10'$, $b = 15.1$, and $c = 10.0$, find the remaining parts (Figure 9.23).

Solution. Since only one angle is known, one of the ratios must involve this angle. We now find angle C:

$$\frac{\sin C}{10.0} = \frac{\sin 54°10'}{15.1}$$

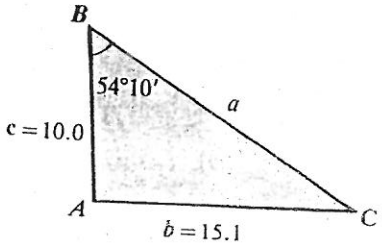


Figure 9.23

Thus

$$\sin C = \frac{10.0 \sin 54°10'}{15.1} = 0.5369$$

and $C = 32°30'$ (to the nearest 10′). It follows that $A = 180° - (32°30' + 54°10') = 179°60' - (86°40') = 93°20'$.

Using angle A, we now find a:

$$\frac{a}{\sin 93°20'} = \frac{15.1}{\sin 54°10'} \quad \text{or} \quad a = \frac{15.1 \sin 93°20'}{\sin 54°10'} = 18.6$$

Thus $A = 93°20'$, $C = 32°30'$, and $a = 18.6$.

If three sides of a triangle are known, then the sine law is of no help: All the ratios in the sine law involve an angle, at least one of which has to be known. We run into a similar problem if $a = 5$, $b = 6$, and $C = 20°$, even though one angle is known:

$$\frac{c}{\sin 20°} = \frac{6}{\sin B} \quad \text{and} \quad \frac{c}{\sin 20°} = \frac{5}{\sin A}$$

Both equations involve two unknowns. These cases will be studied in the next section. The cases to which the sine law does apply are summarized below. 287

> To solve an oblique triangle by the **law of sines**, we need either:
>
> 1. Two angles and one side, or
> 2. Two sides and the angle opposite one of them.

The sine law can be applied to certain problems involving vectors, as shown in the next example.

Example # 3

A plane has an air speed of 525 mi/hr and wants to fly on a course 20.0° east of north. If the wind is out of the north at 40.1 mi/hr, determine the direction in which the plane must head to stay on the proper course. What is the resultant velocity relative to the ground?

Solution. Denote the desired velocity vector by **v**. We can see from the diagram in Figure 9.24 that the angle x opposite **v** has to be known in order to

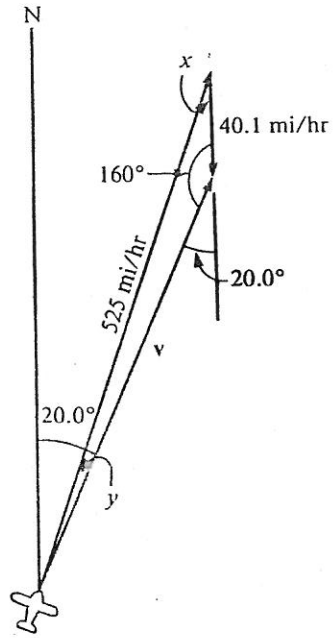


Figure 9.24

determine $|\mathbf{v}| = v$. To this end we first find angle y:

$$\frac{\sin y}{40.1} = \frac{\sin 160°}{525} \quad \text{or} \quad \sin y = \frac{40.1 \sin 160°}{525} = 0.0261$$

Thus $y = 1.5°$. It follows that $x = 180° - 161.5 = 18.5°$. Finally,

$$\frac{v}{\sin 18.5°} = \frac{525}{\sin 160°} \quad \text{or} \quad v = \frac{525 \sin 18.5°}{\sin 160°}$$

The calculator sequence is

525 [×] 18.5 [SIN] [÷] 160 [SIN] [=]

Display: 487.06179

If this is rounded off to three significant figures, we get 487.

Thus the plane must head 18.5° east of north, and it will fly at the rate of 487 mi/hr relative to the ground.

Caution. Suppose we want to find the obtuse angle C (greater than 90°) in Figure 9.25. By the sine law

$$\frac{\sin C}{4} = \frac{\sin 35°}{3}$$

or $\sin C = 0.7648$. If we now use a calculator to find C, we might conclude that $C = 49.9°$. However, since angle C is obtuse, the correct value is $180° - 49.9° = 130.1°$. Consequently, *we have to know in advance whether an angle to be found by the sine law is obtuse or acute*. (Recall that an angle in a triangle is *obtuse* if its measure is more than 90° and *acute* if its measure is less than 90°.)

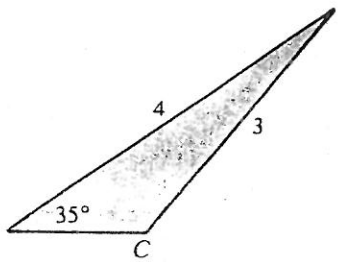
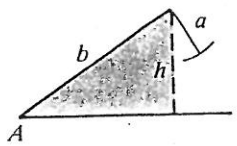
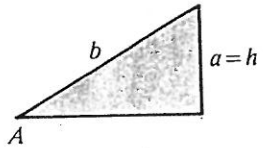
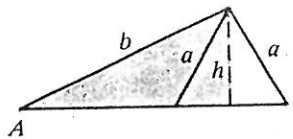


Figure 9.25

The case just described is called the *ambiguous case*. If for some reason the relevant information is withheld, then there may exist two solutions or no solution at all. The different cases are summarized in the chart on the next page.

Number of possible triangles	Sketch	Condition
0	 Figure 9.26	$a < h$ or $a < b \sin A$
1	 Figure 9.27	$a = h$ or $a = b \sin A$
2	 Figure 9.28	$b > a > h$ or $b > a > b \sin A$

Exercises / Section 9.4

In Exercises 1–20, solve the triangles from the given information. (Refer to Figure 9.22 on page 285.)

1. $A = 20°10'$, $C = 50°40'$, $b = 4.00$
2. $A = 40.3°$, $B = 80.5°$, $a = 5.32$
3. $B = 25°50'$, $C = 130°20'$, $c = 15.1$
4. $A = 112.1°$, $C = 10.5°$, $c = 36.0$
5. $A = 19°50'$, $a = 102$, $b = 46.5$
6. $C = 47°38'$, $a = 0.7980$, $c = 1.320$
7. $C = 31.5°$, $a = 13.3$, $c = 6.82$
8. $A = 56°$, $a = 8.1$, $c = 10$
9. $A = 29.3°$, $a = 71.6$, $c = 136$; angle C acute
10. Same as Exercise 9 with angle C obtuse
11. $B = 46.0°$, $C = 54.0°$, $a = 236$
12. $A = 10.1°$, $C = 22.7°$, $c = 0.450$

13. $C = 63.6°$, $a = 12.4$, $c = 11.6$; angle A acute
14. Same as Exercise 13 with angle A obtuse
15. $B = 33°45'$, $a = 1.146$, $b = 2.805$
16. $B = 33°45'$, $a = 1.146$, $b = 0.6200$
17. $B = 33°45'$, $a = 1.146$, $b = 1.000$; angle C obtuse
18. $A = 72.3°$, $a = 86.0$, $c = 73.0$
19. $A = 126.50°$, $C = 10.40°$, $a = 136.4$
20. $B = 28.0°$, $C = 16.7°$, $c = 1.03$
21. A plane maintaining an air speed of 560 mi/hr is heading 10° west of north. A north wind causes the actual course to be 11° west of north. Find the velocity of the plane with respect to the ground.
22. Two forces act on the same object in directions that are 40.0° apart. If one force is 48.0 lb, what must the other force be so that the combined effect is equivalent to a force of 65.0 lb?
23. A surveyor wants to find the width of a river from a certain point on the bank. Since no other points on the bank nearby are accessible, he takes the measurements shown in Figure 9.29. Find the width of the river.

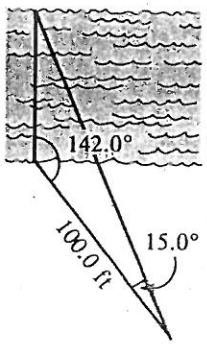


Figure 9.29

24. A building 81 ft tall stands on top of a hill. From a point at the foot of the hill, the angles of elevation of the top and bottom of the building are 39°10′ and 37°40′, respectively. What is the distance from the point to the bottom of the building?
25. From a point on the ground, the angle of elevation of a balloon is 49.0°. From a second point 1,250 ft away on the opposite side of the balloon and in the same vertical plane as the balloon and the first point, the angle of elevation is 33.0°. Find the distance from the second point to the balloon.
26. Find the area of the triangle in Figure 9.30.

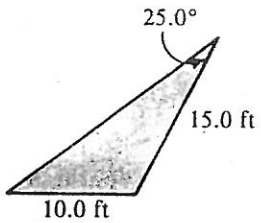


Figure 9.30

9.5

The Law of Cosines

291

We saw in the previous section that not all triangles can be solved by the law of sines. In this section we shall consider the solution of these triangles by means of the *law of cosines*.

> Use the law of cosines to solve an oblique triangle given:
> 1. Two sides and the included angle, or
> 2. Three sides.

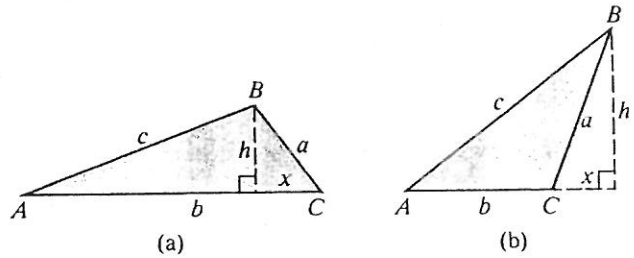


Figure 9.31

To derive the law of cosines, consider the triangles in Figure 9.31. Drop a perpendicular from B to the opposite side and let x denote the distance from C to the foot of the perpendicular. Since $h = c \sin A$, we get from the Pythagorean theorem

$$a^2 = c^2 \sin^2 A + x^2 \tag{9.8}$$

In Figure 9.31(a), $b - x = c \cos A$, or $x = b - c \cos A$. In Figure 9.31(b), $x + b = c \cos A$, or $x = c \cos A - b$. Substituting the respective expressions in equation (9.8), we get

$$a^2 = c^2 \sin^2 A + (b - c \cos A)^2$$

and

$$a^2 = c^2 \sin^2 A + (c \cos A - b)^2$$

If we multiply the expressions on the right, we find that in both cases

$$a^2 = c^2 \sin^2 A + c^2 \cos^2 A - 2bc \cos A + b^2$$

or

$$a^2 = c^2(\sin^2 A + \cos^2 A) + b^2 - 2bc \cos A \tag{9.9}$$

Now recall that for an angle θ in standard position, $\sin \theta = y/r$ and $\cos \theta = x/r$. Hence

$$\sin^2 \theta + \cos^2 \theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

292

Thus $\sin^2 A + \cos^2 A = 1$ for any angle A, so that equation (9.9) becomes

$$a^2 = b^2 + c^2 - 2bc \cos A \tag{9.10}$$

This formula is known as the **law of cosines** or simply the **cosine law.**

By dropping the perpendicular from the other vertices, we get two other forms of the cosine law, as summarized below.

Law of cosines:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

The law of cosines can also be stated verbally.

Law of cosines (verbal form): The square of any side of a triangle equals the sum of the squares of the other two sides minus twice the product of those two sides and the cosine of the angle between them.

Note especially that if $A = 90°$, then formula (9.10) reduces to

$$a^2 = b^2 + c^2 - 2bc \cdot 0 = b^2 + c^2$$

It follows that the cosine law is a generalization of the Pythagorean theorem.

The first two examples illustrate how the law of cosines is used to solve oblique triangles.

Example # 1

Given that $C = 20.4°$, $a = 7.60$, and $b = 10.0$, find the remaining parts (Figure 9.32).

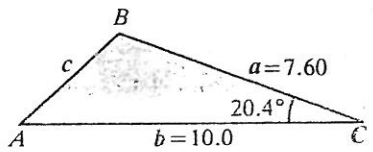


Figure 9.32

Solution. By the law of cosines

$$c^2 = (7.60)^2 + (10.0)^2 - 2(7.60)(10.0) \cos 20.4°$$

Using a calculator we find that $c = 3.91$.

293

Having obtained c, we can now find A. Suppose we use the cosine law again. Then

$$(7.60)^2 = (3.91)^2 + (10.0)^2 - 2(3.91)(10.0) \cos A$$
$$(7.60)^2 - (3.91)^2 - (10.0)^2 = -2(3.91)(10.0) \cos A$$
$$\cos A = \frac{(7.60)^2 - (3.91)^2 - (10.0)^2}{-2(3.91)(10.0)} = 0.7357$$

and $A = 42.6°$. Finally, $B = 117.0°$.

The calculator sequence is

OR 7.6 [x^2] [−] 3.91 [x^2] [−] 10 [x^2] [=] [÷] 2 [+/−] [÷] 3.91 [÷] 10
NT [=] [INV] [COS]

Display: 42.6 (to the nearest tenth of a degree)

Note that once we have obtained c in Example 1, we can find angle A by the sine law:

$$\frac{\sin A}{7.60} = \frac{\sin 20.4°}{3.91} \quad \text{or} \quad \sin A = \frac{7.60 \sin 20.4°}{3.91}$$

which yields $A = 42.7°$. The difference in the values is due to round-off errors.

Besides disagreements due to round-off errors, we are actually faced with a much more serious problem. As noted in the previous section, whenever the sine law is used, we need to know in advance whether an angle is acute or obtuse. The next example demonstrates the problem that can arise.

Example #2

le 2 Given that $a = 4.3$, $b = 5.2$, and $c = 8.2$, find the three angles (Figure 9.33).

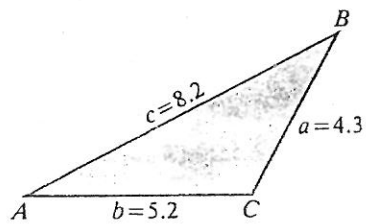


Figure 9.33

Solution. Suppose we first find angle B:

$$(5.2)^2 = (4.3)^2 + (8.2)^2 - 2(4.3)(8.2) \cos B$$

or

$$\cos B = \frac{(5.2)^2 - (4.3)^2 - (8.2)^2}{-2(4.3)(8.2)} = 0.8322$$

and $B = 34°$ (to the nearest degree).

Let us now find angle C by the cosine law:

$$\cos C = \frac{(8.2)^2 - (5.2)^2 - (4.3)^2}{-2(5.2)(4.3)} = -0.4855.$$

It follows that $C = 119°$ and $A = 180° - (119° + 34°) = 27°$.

However, suppose without examining the situation more closely, we had decided to use the sine law to get C. We now step into

$$\frac{\sin C}{8.2} = \frac{\sin 34°}{5.2}$$

so that $C = 62°$. Yet the correct value is $180° - 62° = 118°$, since C is obtuse. (Notice again the difference due to round-off errors.)

One way to avoid this problem is to find the second angle by the cosine law, especially if a calculator is used: Since $\cos C$ is negative, the calculator yields the correct value automatically. A good alternative is given next.

> First use the cosine law to find the angle opposite the longest side. Then use the sine law to find the remaining angles.

The law of cosines can also be used to solve problems involving vectors, as shown in the remaining examples.

Example #3

Example 3 Two forces of 55.0 lb and 37.0 lb, respectively, are acting on the same object. If the angle between their directions is 23.4°, what single force would produce the same effect?

Solution. Denote the resultant force by **F** and consider the diagram in Figure 9.34. By the law of cosines

$$|\mathbf{F}|^2 = (55.0)^2 + (37.0)^2 - 2(55.0)(37.0)\cos 156.6°$$

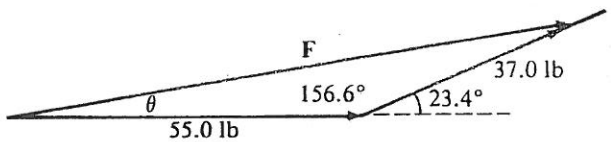


Figure 9.34

Using a calculator we get $|\mathbf{F}| = 90.2$ lb. Since the angle opposite the longest side is already known, we can safely apply the sine law to find angle θ, the direction of **F** relative to the 55.0-lb force. By the sine law, $\theta = 9.4°$. (If the cosine law is used, the calculated value turns out to be 9.3°.)

Example 4 A ship sailing at the rate of 21 knots in still water is heading 16° west of south. It runs into a strong current of 5.3 knots from the south. Find the resultant velocity vector.

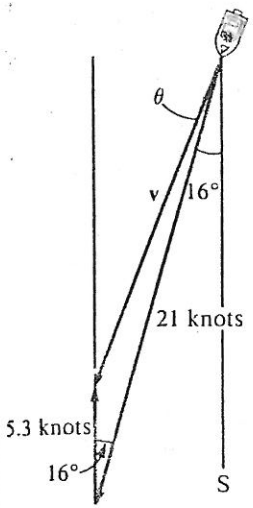


Figure 9.35

Solution. In Figure 9.35, denote the desired velocity vector by **v**. Then

$$|\mathbf{v}|^2 = (5.3)^2 + (21)^2 - 2(5.3)(21)\cos 16°$$

Using a calculator, we find that $|\mathbf{v}| = 16$ knots. To obtain the direction, we need to find θ in Figure 9.35. By the cosine law:

$$(5.3)^2 = (21)^2 + (16)^2 - 2(21)(16)\cos\theta$$

from which $\theta = 5°$. (The angle θ can also be found by the sine law.) So the direction is $16° + 5° = 21°$ west of south.

Exercises / Section 9.5

In Exercises 1–12, solve the given triangles. (See Examples 1 and 2.)

1. $A = 46.3°$, $b = 1.00$, $c = 2.30$
2. $B = 62.7°$, $a = 7.00$, $c = 10.0$
3. $C = 125°10'$, $a = 178$, $b = 137$
4. $A = 100.0°$, $b = 2.36$, $c = 1.97$
5. $a = 20.1$, $b = 30.3$, $c = 25.7$
6. $a = 2.46$, $b = 1.97$, $c = 4.10$
7. $A = 14°40'$, $b = 11.7$, $c = 7.80$
8. $a = 0.471$, $b = 0.846$, $c = 0.239$
9. $C = 39.4°$, $a = 126$, $b = 80.1$
10. $A = 63.0°$, $b = 35.1$, $c = 86.1$
11. $a = 12.85$, $b = 21.46$, $c = 9.179$
12. $a = 20$, $b = 23$, $c = 18$
13. A civil engineer wants to find the length of a proposed tunnel. From a distant point he observes that the respective distances to the ends of the tunnel are 585 ft and 624 ft. The angle between the lines of sight is 33.4°. Find the length of the proposed tunnel.
14. Two forces of 150.0 lb and 80.0 lb produce a resultant force of 209 lb. Find the angle between the forces.
15. A ship sails 16.0 km due east, turns 20.0° north of east, and then continues for another 11.5 km. Find its distance from the starting point.

16. A car travels 85 km due west of point A and then northwest for another 50 km. Find its dis
from point A.

17. A small plane is heading 5.0° north of east with an air speed of 151 mi/hr. The wind is from the sc
at 35.3 mi/hr. Find the actual course and the velocity with respect to the ground.

18. In Figure 9.36 the point P is moving around the circle. Find d as a function of θ.

19. Find the perimeter of the triangle in Figure 9.37.

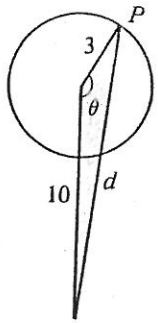


Figure 9.36

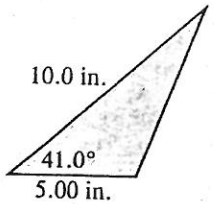


Figure 9.37

20. A river is flowing east at 4.10 mi/hr. A boat crosses the river in the direction 29.0° ea
if the speedometer reads 10.0 mi/hr, what is the boat's actual speed and direction?

Review Exercises / Chapter 9

In Exercises 1–8, find the magnitude and direction of each vector.

1. $-\mathbf{i} + \mathbf{j}$
2. $-\mathbf{i} - \sqrt{3}\mathbf{j}$
3. $\mathbf{i} + 2\mathbf{j}$
4. $\mathbf{i} - 3\mathbf{j}$
5. $\sqrt{15}\mathbf{i} + \mathbf{j}$
6. $-3\mathbf{i} + \sqrt{7}\mathbf{j}$
7. $-\sqrt{2}\mathbf{i} - 2\mathbf{j}$
8. $\sqrt{2}\mathbf{i} + 3\mathbf{j}$

In Exercises 9–14, resolve each vector into its components. (Give answers to two decimals.)

9. $|\mathbf{A}| = 5,\ \theta = 75°$
10. $|\mathbf{A}| = 3,\ \theta = 115°$
11. $|\mathbf{A}| = \sqrt{7},\ \theta = 220°$
12. $|\mathbf{A}| = \sqrt{3},\ \theta = 328°$
13. $|\mathbf{A}| = 6,\ \theta = 216°$
14. $|\mathbf{A}| = \sqrt{10},\ \theta = 153°$

In Exercises 15–24, solve each triangle from the given information.

15. $A = 46.3°,\ C = 53.7°,\ b = 5.26$
16. $A = 29.4°,\ B = 115.2°,\ c = 63.5$
17. $B = 10°40',\ C = 130°20',\ c = 236$
18. $A = 41°,\ B = 52°,\ b = 22$
19. $A = 26°,\ a = 25,\ c = 37$ (angle C is obtuse)
20. $C = 31.6°,\ a = 38.4,\ b = 62.0$
21. $A = 19°23',\ b = 11.23,\ c = 30.04$
22. $B = 51°,\ a = 1.9,\ c = 1.4$
23. $a = 3.74,\ b = 5.86,\ c = 5.50$
24. $a = 321.7,\ b = 276.4,\ c = 248.2$

25. A railroad car weighing 9.8 tons is resting on a track inclined 7° with the horizontal. What is the force needed to keep it from rolling downhill?

26. A 320-lb cart is resting on a ramp inclined 12.0° to the horizontal. How much of the weight does the ramp support?

27. A surveyor wants to determine the width of a river. He can reach one point on the bank, but no other point nearby is accessible. Instead, he takes the measurements shown in Figure 9.38. Find the width of the river.

28. Find the perimeter of the triangle in Figure 9.39.

29. A 12.0-lb weight hanging from a rope is pushed sideways so that the rope makes a 15.0° angle with the vertical. If the system is in equilibrium, find the tension on the rope.

30. A 463-lb weight is supported as shown in Figure 9.40. Find the compression force on the angled bracket PQ.

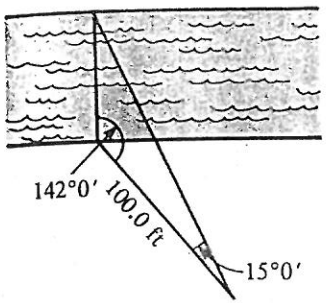


Figure 9.38

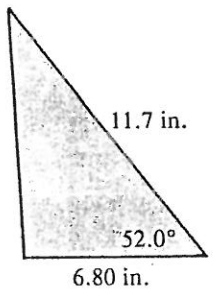


Figure 9.39

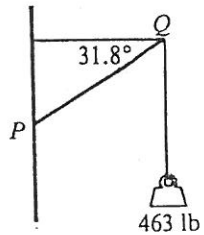


Figure 9.40

31. A force of 25.1 lb and a force of 39.4 lb produce a resultant force of 59.9 lb. Find the angle between the two forces.

32. From a point on the ground, the angle of elevation of the top of a tower is 26.3°. From a second point 48.3 ft closer to the tower, the angle of elevation of the top is 37.4°. Find the height of the tower.

33. A racing car is traveling at 90.0 mi/hr along a straight road inclined at 7.0° to the horizontal. Determine the rate of ascent.

34. A plane is heading 20°0' south of west at 311 mi/hr. If the wind is from the east at 20.0 mi/hr, determine its velocity with respect to the ground and the resulting direction.

13. 401 km/hr, 4° east of north 15. 223 km/hr, 7°40′ east of south 17. 623.9 lb 19. 1,450 lb
21. 101 lb, 14° 23. 188.6 lb

Section 9.4 (page 289)

1. $B = 109°10'$, $a = 1.46$, $c = 3.28$ 3. $A = 23°50'$, $a = 8.00$, $b = 8.63$
5. $B = 8°50'$, $C = 151°20'$, $c = 144$ 7. no solution 9. $C = 68.4°$, $B = 82.3°$, $b = 145$
11. $A = 80.0°$, $b = 172$, $c = 194$ 13. $A = 73.2°$, $B = 43.2°$, $b = 8.87$
15. $A = 13°7'$, $C = 133°8'$, $c = 3.684$ 17. $A = 39°33'$, $C = 106°42'$, $c = 1.724$
19. $B = 43.10°$, $b = 115.9$, $c = 30.63$ 21. 510 mi/hr 23. 66.2 ft 25. 953 ft

Section 9.5 (page 295)

1. $a = 1.76$; by the cosine law: $C = 109.8°$, $B = 23.9°$ (by the sine law: $B = 24.3°$)
3. $c = 280$, $A = 31°20'$, $B = 23°30'$
5. by the cosine law: $B = 81.9°$, $A = 41.0°$, $C = 57.1°$ (by the sine law: $A = 41.1°$)
7. $a = 4.60$, $B = 139°50'$, $C = 25°30'$
9. $c = 81.8$, $A = 102.2°$ (by the cosine law), $B = 38.4°$
11. $B = 153°31'$, $A = 15°29'$, $C = 11°0'$ 13. 349 ft 15. 27.1 km
17. 17.9° north of east, 158 mi/hr 19. 22.0 in.

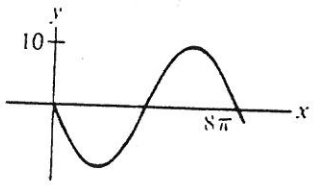
Review Exercises for Chapter 9 (page 296)

1. $\sqrt{2}$, 135° 3. $\sqrt{5}$, 63°26′ 5. 4, 14°29′ 7. $\sqrt{6}$, 234°44′ 9. $1.29\mathbf{i} + 4.83\mathbf{j}$
11. $-2.03\mathbf{i} - 1.70\mathbf{j}$ 13. $-4.85\mathbf{i} - 3.53\mathbf{j}$ 15. $B = 80.0°$, $a = 3.86$, $c = 4.30$
17. $A = 39°0'$, $a = 195$, $b = 57.3$ 19. $B = 14°$, $C = 140°$, $b = 14$
21. $a = 19.80$, $C = 149°46'$, $B = 10°51'$ 23. $B = 76.1°$, $A = 38.3°$, $C = 65.6°$ 25. 1.2 tons
27. 66.24 ft 29. 12.4 lb 31. 44.7° 33. 11.0 mi/hr

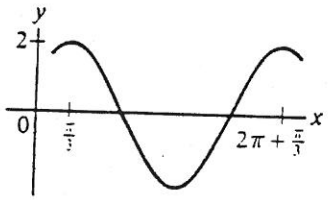
Cumulative Review Exercises for Chapters 7–9 (page 298)

1. $\frac{\sqrt{5}}{2}$ 2. a. $-\frac{2\sqrt{3}}{3}$; b. $-\frac{\sqrt{3}}{3}$ 3. a. 45°, 315°; b. 150°, 330° 4. 1.388
5. 228°13′, 311°47′ 6. $\frac{28\pi}{45}$ 7. 500° 8. 1.11 cm 9. 98.0 in.^{2} 10. 85.0 m/sec

11. 19 m/sec 12.



13.



14. 4, 150° 15. 22.5 lb 16. 146 mi/hr, 8.0° north of west 17. 549 lb
18. 24 knots, 18° east of south

Chapter 10

Section 10.1 (page 301)

1. 72 3. $\frac{27}{64}$ 5. $6x^5y^9$ 7. $2a^4b^3$ 9. R^4S^8 11. $-512C_1{}^9C_2{}^{12}$ 13. $\frac{3sv}{7t}$ 15. $\frac{1}{3}ns$
17. $3V^3$ 19. $\frac{x^2}{2y^3z^6}$ 21. $3abc$ 23. $\frac{7v_0}{s_2{}^5}$ 25. $\frac{8x^3}{27y^3}$ 27. $\frac{16q^{12}}{81p^8r^4}$ 29. $5.625x^8y^{16}z^{10}$
31. $\frac{3bxy^4}{2a^4}$ 33. $\frac{R^2V^2}{32C}$ 35. x 37. y 39. xy^a 41. a 43. x^{5b} 45. x^{6b}