قاعدة السلسلة

مبرهنة: لتكن f دالة في متغيرين x و y. و f لها مشتقات جزئية الأولى على D_f.

(1) إذا كان $x = x(t)$ و $y = y(t)$ حيث x, y لها مشتقات حسب دلالة t.

فإن

$$\begin{cases} F(t) = f(x(t), y(t)) \\ \boxed{F'(t) = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot y'(t)} \end{cases}$$

(2) إذا كان $x = x(u, v)$, $y = y(u, v)$

$$w = f(x(u,v), y(u,v)) = F(u, v)$$

$$\begin{cases} \frac{\partial w}{\partial u} = \frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} \\ \frac{\partial w}{\partial v} = \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} \end{cases}$$

مثال: لتكن: $w = f(x,y) = xy^2 + e^x + \sin y$ ، $\begin{cases} x = t^2 + 1 \\ y = t + \frac{\pi}{2} \end{cases}$

أوجد $w'(t)$ بطريقتين.

الحل: طريقة مباشرة: لدينا

$$w = f(x,y) = xy^2 + e^x + \sin y = (t^2+1)\left(t+\frac{\pi}{2}\right)^2 + e^{t^2+1} + \sin\left(t+\frac{\pi}{2}\right)$$
$$= (t^2+1)\left(t^2 + \pi t + \frac{\pi^2}{4}\right) + e.e^{t^2} + \sin\left(t+\frac{\pi}{2}\right)$$
$$= t^4 + \pi t^3 + \left(\frac{\pi^2}{4}+1\right)t^2 + \pi t + \frac{\pi^2}{4} + e.e^{t^2} + \sin\left(t+\frac{\pi}{2}\right)$$
$$w'(t) = 4t^3 + 3\pi t^2 + 2\left(\frac{\pi^2}{4}+1\right)t + \pi + 2e\,t\,e^{t^2} + \cos\left(t+\frac{\pi}{2}\right)$$

طريقة 2: قاعدة السلسلة:

لدينا قاعدة السلسلة في متغير:

$$w'(t) = \frac{\partial f}{\partial x}(x(t),y(t)).x'(t) + \frac{\partial f}{\partial y}(x(t),y(t)).y'(t)$$

لدينا: $\frac{\partial f}{\partial x} = y^2 + e^x = \left(t+\frac{\pi}{2}\right)^2 + e^{t^2+1}$
$= t^2 + \pi t + \frac{\pi^2}{4} + e.e^{t^2}$

$\frac{\partial f}{\partial y} = 2xy + \cos y = 2(t^2+1)\left(t+\frac{\pi}{2}\right) + \cos\left(t+\frac{\pi}{2}\right)$
$= 2t^3 + \pi t^2 + 2t + \pi + \cos\left(t+\frac{\pi}{2}\right)$

$x'(t) = 2t$
$y'(t) = 1$

وبالتالي

$$w'(t) = \left(t^2 + \pi t + \frac{\pi^2}{4} + e.e^{t^2}\right).2t + \left(2t^3 + \pi t^2 + 2t + \pi + \cos\left(t+\frac{\pi}{2}\right)\right).1$$

$$\boxed{w'(t) = 4t^3 + 3\pi t^2 + \left(\frac{\pi^2}{2}+2\right)t + \pi + 2e\,t\,e^{t^2} + \cos\left(t+\frac{\pi}{2}\right)}$$

نلاحظ: $\sin\left(t+\frac{\pi}{2}\right) = \cos t$ ...

مثال ١ ص ٩٤: $w = x^2 + 2xy + y^2$, لات $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}$

أوجد $w'(t) = \frac{dw}{dt}$

الحل:

طريقة 1

لدينا

$w = x^2 + 2xy + y^2$

$= (t\cos t)^2 + 2(t\cos t)(t\sin t) + (t\sin t)^2$

$= t^2(\cos^2 t + 2\cos t\sin t + \sin^2 t)$

$= t^2(1 + 2\cos t\sin t)$

$= t^2(1 + \sin(2t))$

$w'(t) = \frac{dw}{dt} = 2t(1 + \sin 2t) + t^2\, 2\cos 2t$

$$\boxed{w'(t) = 2t(1 + \sin 2t + t\cos 2t)}$$

طريقة 2: قاعدة السلسلة (في متغير)

$w'(t) = \frac{\partial f}{\partial x}\cdot x'(t) + \frac{\partial f}{\partial y}\cdot y'(t)$

لدينا: $\frac{\partial f}{\partial x} = 2x + 2y = 2t(\cos t + \sin t)$

$\frac{\partial f}{\partial y} = 2x + 2y = 2t(\cos t + \sin t)$

$x'(t) = \cos(t) - t\sin t$

$y'(t) = \sin t + t\cos t$

بأن

$w'(t) = 2t(\cos t + \sin t)(\cos t - t\sin t) + 2t(\cos t + \sin t)(\sin t + t\cos t)$

$= 2t(\cos^2 t - t\cos t\sin t + \sin t\cos t - t\sin^2 t + \cos t\sin t + t\cos^2 t + \sin^2 t + t\sin t\cos t)$

$= 2t(1 + 2\cos t\sin t + t(\cos^2 t - \sin^2 t))$

$$\boxed{w'(t) = 2t(1 + \sin 2t + t\cos 2t)}$$

$\cos^2 t + \sin^2 t = 1$

$\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$

$\cos(a-b) = \cos a \cdot \cos b + \sin a \sin b$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\sin(a-b) = \sin a \cos b - \cos a \sin b$

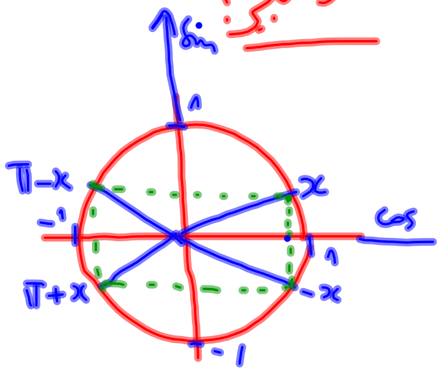
$a = b$

$\cos(2a) = \cos^2 a - \sin^2 a = 1 - 2\sin^2 a$

$= 2\cos^2 a - 1$

$\sin 2a = 2 \sin a . \cos a$

تذكير :



$\cos(-x) = \cos(x)$

$\sin(-x) = -\sin x$

$\cos(\pi - x) = \cos(\pi + x) = -\cos x$

$\sin(\pi - x) = \sin x$

$\cos(\pi + x) = -\cos x$

$\sin(\pi + x) = -\sin x$

$\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$

مثال 2 ص 95 :

$$w = f(x,y) = \ln\sqrt{x^2+y^2} \quad , \quad \begin{cases} x = re^{s} \\ y = re^{-s} \end{cases}$$

احسب كلا من

$$\frac{\partial w}{\partial r} \;,\; \frac{\partial w}{\partial s}$$

طريقة 1 :

$$w = f(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$$

$$= \frac{1}{2}\ln\left((re^{s})^2 + (re^{-s})^2\right)$$

$$= \frac{1}{2}\ln\left(r^2(e^{2s}+e^{-2s})\right)$$

$$w = \frac{1}{2}\left(\ln r^2 + \ln(e^{2s}+e^{-2s})\right)$$

$$\boxed{w = \ln r + \frac{1}{2}\ln(e^{2s}+e^{-2s})}$$

$$\boxed{\frac{\partial w}{\partial r} = \frac{1}{r} \qquad \frac{\partial w}{\partial s} = \frac{1}{2}\,\frac{2e^{2s}-2e^{-2s}}{e^{2s}+e^{-2s}} = \frac{e^{2s}-e^{-2s}}{e^{2s}+e^{-2s}}}$$

طريقة 2 : استخدام قاعدة السلسلة

لدينا

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

ولدينا

$$\frac{\partial f}{\partial x} = \frac{1}{2}\,\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$= \frac{re^{s}}{r^2(e^{2s}+e^{-2s})} = \frac{1}{r}\cdot\frac{e^{s}}{e^{2s}+e^{-2s}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\,\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2} = \frac{1}{r}\cdot\frac{e^{-s}}{e^{2s}+e^{-2s}}$$

$$\frac{\partial x}{\partial r} = e^{s} \;;\; \frac{\partial y}{\partial r} = e^{-s}$$

$$\frac{\partial x}{\partial s} = re^{s} \;;\; \frac{\partial y}{\partial s} = -re^{-s}$$

إذن :

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{1}{r}\,\frac{e^{s}}{e^{2s}+e^{-2s}}\cdot e^{s} + \frac{1}{r}\cdot\frac{e^{-s}}{e^{2s}+e^{-2s}}\cdot e^{-s} = \frac{1}{r}\,\frac{e^{2s}+e^{-2s}}{e^{2s}+e^{-2s}}$$

$$\boxed{\frac{\partial w}{\partial r} = \frac{1}{r}}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{1}{r}\cdot\frac{e^{s}}{e^{2s}+e^{-2s}}\cdot re^{s} + \frac{1}{r}\,\frac{e^{-s}}{e^{2s}+e^{-2s}}\cdot(-re^{-s})$$

$$= \frac{1}{r}\,\frac{r(e^{2s}-e^{-2s})}{e^{2s}+e^{-2s}} = \boxed{\frac{e^{2s}-e^{-2s}}{e^{2s}+e^{-2s}} = \frac{\partial w}{\partial s}}$$

مثال 4 ص 96: برهن على أن الدالة: $z = f\left(\frac{1}{2}bx^2 - \frac{1}{3}ay^3\right)$

تحقق المعادلة: $ay^2\frac{\partial z}{\partial x} + bx\frac{\partial z}{\partial y} = 0$ حيث أن $a, b \in \mathbb{R}$

الحل: نفرض أن $u = \frac{1}{2}bx^2 - \frac{1}{3}ay^3$

ولدينا:

$$\frac{\partial u}{\partial x} = bx$$

$$\frac{\partial u}{\partial y} = -ay^2$$

لدينا:

$$\frac{\partial z}{\partial x} = f'(u)\cdot\frac{\partial u}{\partial x} = \frac{df}{du}\cdot\frac{\partial u}{\partial x} = bx\,\frac{df}{du}$$

$$\frac{\partial z}{\partial y} = f'(u)\cdot\frac{\partial u}{\partial y} = \frac{df}{du}\cdot(-ay^2) = -ay^2\,\frac{df}{du}$$

وبالتالي:

$$ay^2\frac{\partial z}{\partial x} + bx\frac{\partial z}{\partial y} = ay^2\cdot bx\frac{df}{du} + bx\left(-ay^2\frac{df}{du}\right)$$

$$= abxy^2\frac{df}{du} - abxy^2\frac{df}{du} = 0$$

مثال (ص 96, 97):

برهن على أن الدالة: $z = f(s^2-t^2, t^2-s^2)$

تحقق العلاقة التالية: $t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = 0$

الحل: لدينا: $z = f(s^2-t^2, t^2-s^2) \Rightarrow$ نفرض أن

$$\begin{cases} x = s^2-t^2 \\ y = t^2-s^2 \end{cases}$$

حسب قاعدة السلسلة لدينا:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{\partial z}{\partial x}\,2s + \frac{\partial z}{\partial y}\cdot(-2s)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{\partial z}{\partial x}(-2t) + \frac{\partial z}{\partial y}(2t)$$

بأن:

$$t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = t\left(2s\frac{\partial z}{\partial x} - 2s\frac{\partial z}{\partial y}\right) + s\left(-2t\frac{\partial z}{\partial x} + 2t\frac{\partial z}{\partial y}\right)$$

$$= 2ts\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} - \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 0$$

مثال (ص 97)

$w=f(x,y)$ دالة في متغيرين مشتقاتها من الرتبة الثانية متصلة ونفرض أن $x=u+v$, $y=u-v$ بيّن أن: $\frac{\partial^2 w}{\partial u \partial v}=\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}$

الحل: حسب قاعدة السلسلة لدينا

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}$$

فإن وحسب قاعدة السلسلة (للمرة الثانية)

$$\frac{\partial^2 w}{\partial u \partial v}=\frac{\partial\left(\frac{\partial w}{\partial u}\right)}{\partial v}=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right)\frac{\partial x}{\partial v}+\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\right)\frac{\partial y}{\partial v}$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\cdot 1+\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)\cdot(-1)$$

$$=\frac{\partial^2 w}{\partial x^2}+\underbrace{\frac{\partial^2 w}{\partial y \partial x}-\frac{\partial^2 w}{\partial x \partial y}}-\frac{\partial^2 w}{\partial y^2}$$

بما أن $w=f$ دالة في متغيرين لها مشتقات من الرتبة الثانية متصلة

فإن $\frac{\partial^2 w}{\partial x \partial y}=\frac{\partial^2 w}{\partial y \partial x}$

وبالتالي

$$\boxed{\frac{\partial^2 w}{\partial u \partial v}=\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}}$$

ملحوظة :

$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial u}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\underbrace{\frac{\partial x}{\partial u}}_{1} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\underbrace{\frac{\partial y}{\partial u}}_{1}$$

$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}$$

لأن $\left(\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}\right)$

$$\boxed{\frac{\partial^2 w}{\partial u^2} = \frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}}$$

مثال 7 ص 98 : u دالة في متغيرين x , y ، $x=e^s$, $y=e^t$

برهن أن: المعادلة:

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

يمكن كتابتها على الصيغة:

$$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=0$$

الحل: لدينا حسب قاعدة السلسلة:

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}=e^s\frac{\partial u}{\partial x}+0\cdot\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial s}=x\frac{\partial u}{\partial x}}$$

$$\frac{\partial^2 u}{\partial s^2}=\frac{\partial}{\partial s}\left(x\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial s}+\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial s}$$

$$=\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)\cdot\underbrace{e^s}_{x}+\frac{\partial}{\partial y}\left(\quad\right)\cdot 0=\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}$$

$$\boxed{\frac{\partial^2 u}{\partial s^2}=x\left(x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\right)}\boxed{=x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}}$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\boxed{\frac{\partial u}{\partial t}=e^t\frac{\partial u}{\partial y}=y\frac{\partial u}{\partial y}}$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)=\frac{\partial}{\partial t}\left(y\frac{\partial u}{\partial y}\right)$$

$$=\frac{\partial}{\partial x}\left(\quad\right)\underbrace{\frac{\partial x}{\partial t}}_{0}+\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)\underbrace{\frac{\partial y}{\partial t}}_{e^t}$$

$$=\left(\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}\right)y$$

$$\boxed{\frac{\partial^2 u}{\partial t^2}=y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}}$$

وبالتالي

$$\boxed{\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}}$$

الدوال المتجانسة

تعريف: لتكن f دالة في متغيرين x و y وحيث D هو مجال الدالة f.

f دالة متجانسة من الدرجة k $(k\in\mathbb{R})$ إذا كان:

$$\begin{cases} \text{لكل } t>0,\ (x,y)\in D \text{ فإن } (tx,ty)=t(x,y)\in D \\ \text{لكل } t>0,\ (x,y)\in D \text{ لدينا: } f(tx,ty)=t^k f(x,y) \end{cases}$$

مثال :

$$f(x,y)=\frac{xy}{x^2+y^2}$$

لدينا $D_f=\mathbb{R}^2\setminus\{(0,0)\}$

ليكن $t>0$, $(x,y)\in D_f$ بأن $(x,y)\neq(0,0)$ وبما أن $t>0$

وبالتالي $(tx,ty)=t(x,y)\neq(0,0)$

بأن $(tx,ty)\in D_f$

ليكن $t>0$, $(x,y)\in D_f$ لدينا :

$$f(tx,ty)=\frac{tx\cdot ty}{(tx)^2+(ty)^2}=\frac{t^2xy}{t^2(x^2+y^2)}=\frac{xy}{x^2+y^2}=f(x,y)=t^0f(x,y)$$

بأن f دالة متجانسة من الدرجة 0 .

مثال 2: لتكن $f(x,y)=\sqrt{x+2y}$

لدينا لكل $t>0$, $(x,y)\in D_f$ $\quad (x+2y\geq 0)$

بأن $(tx,ty)=t(x,y)$

و $(tx)+2(ty) = t(x+2y)$ بما أن $t>0$, $x+2y\geq 0$

بأن $tx+2ty\geq 0$ بأن $(tx,ty)\in D_f$

ولدينا: $f(tx,ty)=\sqrt{(tx)+2(ty)} = \sqrt{t(x+2y)}=\sqrt{t}\sqrt{x+2y}$

$= t^{1/2} f(x,y)$

وبالتالي f دالة متجانسة من الدرجة $\frac{1}{2}$

مثال 3:

$$f(x,y)=\frac{x^2}{x^4+y^4}$$

لدينا $D_f=\mathbb{R}^2\setminus\{(0,0)\}$

← ليكن $t>0$, $(x,y)\in D_f$ فإن $(x,y)\neq(0,0)$

فإن $t(x,y)\neq(tx,ty)\neq(0,0)$

وبالتالي $(tx,ty)\in D_f$

← ليكن $t>0$, $(x,y)\in D_f$ لدينا

$$f(tx,ty)=\frac{(tx)^2}{(tx)^4+(ty)^4}=\frac{t^2x^2}{t^4(x^4+y^4)}=t^{-2}f(x,y)$$

وبالتالي فإن f هي دالة متجانسة من الرتبة (-2)

<u>مثال 4</u>

$$f(x,y)=\frac{x+y-3}{x^2+y^2}$$

لیکن $t>0$, $(x,y)\in D_f$ فإن $x^2+y^2\neq 0$

ولدینا $(tx)^2+(ty)^2=t^2(x^2+y^2)\neq 0$ فإن $(tx,ty)\in D_f$

لدینا $f(1,3)=\frac{1+3-3}{1^2+2^2}=\frac{1}{5}$, لدینا $f(t\cdot 1, t\cdot 3)=\frac{t+3t-3}{t^2+(2t)^2}$

$$=\frac{4t-3}{t^2(5)}$$

نفرض أن $\frac{4t-3}{5t^2}=t^k\cdot\frac{1}{5}$

فإن $4t-3=t^{k+2} \iff \begin{cases} t^{k+2}-4t-3=0 \\ t>0 \end{cases}$ لـ

إذا كان $\underline{t=1}$ فإن $1^{k+2}-4-3\neq 0$

وبالتالي f دالة غير متجانسة .

مبرهنة: لتكن f دالة متجانسة من الرتبة k.

ولتكن f لها مشتقات جزئية أولى متصلة.

فان: (1) $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ كل منهما دالة متجانسة من الرتبة $(k-1)$

(2) لكل $(x,y)\in D_f$ لدينا:

$$\boxed{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=k\cdot f}$$

البرهان: لدينا f دالة متجانسة من الرتبة k.

بأن لكل $t>0$, $(x,y)\in D_f$, $(tx,ty)\in D_f$,

$f(tx,ty)=t^k f(x,y)$

وبالتالي:

$$\frac{\partial f}{\partial x}(tx,ty)\cdot\frac{\partial (tx)}{\partial x}=t^k\cdot\frac{\partial f}{\partial x}(x,y) \quad (1)$$

$$\Leftrightarrow t\cdot\frac{\partial f}{\partial x}(tx,ty)=t^k\frac{\partial f}{\partial x}(x,y) \Leftrightarrow \boxed{\frac{\partial f}{\partial x}(tx,ty)=t^{k-1}\frac{\partial f}{\partial x}(x,y)}$$

وبنفس الطريقة نبرهن:

$$\boxed{\frac{\partial f}{\partial y}(tx,ty)=t^{k-1}\frac{\partial f}{\partial y}(x,y)}$$

(٢) لدينا $\forall\, t>0$, $\forall\, (x,y)\in D_f$, $(tx,ty)\in D_f$

و $f(tx,ty)=t^k f(x,y)$

حسب قاعدة السلسلة لدينا :

$$\frac{\partial}{\partial t}\big(f(tx,ty)\big)=\frac{\partial}{\partial t}\big(t^k f(x,y)\big)$$

فإن :

$$\frac{\partial f}{\partial x}(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial f}{\partial y}(tx,ty)\cdot\frac{\partial(ty)}{\partial t}=k\,t^{k-1}f(x,y)$$

$\forall\, t>0$ $\forall\, (x,y)\in D_f$

$$(*)\quad \boxed{x\cdot\frac{\partial f}{\partial x}(tx,ty)+y\,\frac{\partial f}{\partial y}(tx,ty)=k\cdot t^{k-1}f(x,y)}$$

$\boxed{t=1}$

$$\boxed{x\cdot\frac{\partial f}{\partial x}(x,y)+y\,\frac{\partial f}{\partial y}(x,y)=k\,f(x,y)}$$

ملاحظة: في النتيجة (*) $t>0$

وبالتالي يمكن أن نأخذ $t=\sqrt{2}$ فإن القاعدة:

$$x\frac{\partial f}{\partial x}(\sqrt{2}x,\sqrt{2}\cdot y)+y\frac{\partial f}{\partial y}(\sqrt{2}x,\sqrt{2}y)=k\cdot(\sqrt{2})^{k-1}f(x,y).$$