قاعدة السلسلة

مبرهنة: لتكن f دالة في متغيرين x و y. ولـ f مشتقات جزئية الأولى على D_f.

(1) إذا كان $x = x(t)$ و $y = y(t)$ حيث x و y لها مشتقات حسب دلالة t.

فإن

$$\begin{cases} F(t) = f(x(t), y(t)) \\ \boxed{F'(t) = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot y'(t)} \end{cases}$$

(2) إذا كان $x = x(u,v)$ و $y = y(u,v)$

$$w = f(x(u,v), y(u,v)) = F(u,v)$$

$$\begin{cases} \frac{\partial w}{\partial u} = \frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} \\ \frac{\partial w}{\partial v} = \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} \end{cases}$$

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مثال: لتكن: $w = f(x,y) = xy^2 + e^x + \sin y$ ، $\begin{cases} x = t^2 + 1 \\ y = t + \frac{\pi}{2} \end{cases}$

أوجد $w'(t)$ بطريقتين.

الحل: **طريقة مباشرة:** لدينا

$$w = f(x,y) = xy^2 + e^x + \sin y = (t^2+1)\left(t+\frac{\pi}{2}\right)^2 + e^{t^2+1} + \sin\left(t+\frac{\pi}{2}\right)$$
$$= (t^2+1)\left(t^2 + \pi t + \frac{\pi^2}{4}\right) + e \cdot e^{t^2} + \sin\left(t+\frac{\pi}{2}\right)$$
$$= t^4 + \pi t^3 + \left(\frac{\pi^2}{4}+1\right)t^2 + \pi t + \frac{\pi^2}{4} + e \cdot e^{t^2} + \sin\left(t+\frac{\pi}{2}\right)$$
$$w'(t) = 4t^3 + 3\pi t^2 + 2\left(\frac{\pi^2}{4}+1\right)t + \pi + 2et\, e^{t^2} + \cos\left(t+\frac{\pi}{2}\right)$$

طريقة 2: قاعدة السلسلة

لدينا قاعدة السلسلة في متغير:

$$w'(t) = \frac{\partial f}{\partial x}(x(t),y(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(x(t),y(t)) \cdot y'(t)$$

لدينا: $\frac{\partial f}{\partial x} = y^2 + e^x = \left(t+\frac{\pi}{2}\right)^2 + e^{t^2+1}$
$$= t^2 + \pi t + \frac{\pi^2}{4} + e \cdot e^{t^2}$$
$$\frac{\partial f}{\partial y} = 2xy + \cos y = 2(t^2+1)\left(t+\frac{\pi}{2}\right) + \cos\left(t+\frac{\pi}{2}\right)$$
$$= 2t^3 + \pi t^2 + 2t + \pi + \cos\left(t+\frac{\pi}{2}\right)$$

$x'(t) = 2t$

$y'(t) = 1$

وبالتالي

$$w'(t) = \left(t^2 + \pi t + \frac{\pi^2}{4} + e \cdot e^{t^2}\right) \cdot 2t + \left(2t^3 + \pi t^2 + 2t + \pi + \cos\left(t+\frac{\pi}{2}\right)\right) \cdot 1$$

$$\boxed{w'(t) = 4t^3 + 3\pi t^2 + \left(\frac{\pi^2}{2}+2\right)t + \pi + 2et\, e^{t^2} + \cos\left(t+\frac{\pi}{2}\right)}$$

ملاحظة: $\sin\left(t+\frac{\pi}{2}\right) = \cos t$ و $\cos\left(t+\frac{\pi}{2}\right) = -\sin t$

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مثال 1 ص 94: $w = x^2 + 2xy + y^2$ ، لتكن $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}$

أوجد $w'(t) = \frac{dw}{dt}$

الحل: **طريقة 1:**

لدينا

$$w = x^2 + 2xy + y^2$$
$$= (t\cos t)^2 + 2(t\cos t)(t\sin t) + (t\sin t)^2$$
$$= t^2(\cos^2 t + 2\cos t\sin t + \sin^2 t)$$
$$= t^2(1 + 2\cos t\sin t)$$
$$= t^2(1 + \sin(2t))$$
$$w'(t) = \frac{dw}{dt} = 2t(1+\sin 2t) + t^2\, 2\cos 2t$$
$$\boxed{w'(t) = 2t(1 + \sin 2t + t\cos 2t)}$$

طريقة 2: قاعدة السلسلة (في متغير)

$$w'(t) = \frac{\partial f}{\partial x} \cdot x'(t) + \frac{\partial f}{\partial y} \cdot y'(t)$$

لدينا: $\frac{\partial f}{\partial x} = 2x + 2y = 2t(\cos t + \sin t)$

$$\frac{\partial f}{\partial y} = 2x + 2y = 2t(\cos t + \sin t)$$

$x'(t) = \cos(t) - t\sin t$

$y'(t) = \sin t + t\cos t$

فإن

$$w'(t) = 2t(\cos t + \sin t)(\cos t - t\sin t) + 2t(\cos t + \sin t)(\sin t + t\cos t)$$
$$= 2t(\cos^2 t - t\cos t\sin t + \sin t\cos t - t\sin^2 t + \cos t\sin t + t\cos^2 t + \sin^2 t + t\sin t\cos t)$$
$$= 2t(1 + 2\cos t\sin t + t(\cos^2 t - \sin^2 t))$$
$$\boxed{w'(t) = 2t(1 + \sin 2t + t\cos 2t)}$$

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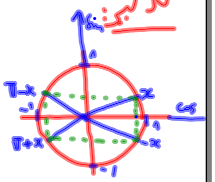
تذكير:

$$\cos^2 t + \sin^2 t = 1$$
$$\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$$
$$\cos(a-b) = \cos a \cdot \cos b + \sin a \sin b$$
$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$
$$\sin(a-b) = \sin a\cos b - \cos a\sin b$$

$a = b$

$$\cos(2a) = \cos^2 a - \sin^2 a = 1 - 2\sin^2 a$$
$$= 2\cos^2 a - 1$$
$$\sin 2a = 2\sin a \cdot \cos a$$



$$\cos(-x) = \cos(x)$$
$$\sin(-x) = -\sin x$$
$$\cos(\pi - x) = \cos(\pi + x) = -\cos x$$
$$\sin(\pi - x) = \sin x$$
$$\cos(\pi + x) = -\cos x$$
$$\sin(\pi + x) = -\sin x$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$$
$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

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مثال 2 ص 95: $w = f(x,y) = \ln\sqrt{x^2+y^2}$ ، $\begin{cases} x = re^s \\ y = re^{-s} \end{cases}$

احسب كلا من $\frac{\partial w}{\partial r}$ ، $\frac{\partial w}{\partial s}$

طريقة 1:

$$w = f(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$$
$$= \frac{1}{2}\ln((re^s)^2 + (re^{-s})^2)$$
$$= \frac{1}{2}\ln(r^2(e^{2s} + e^{-2s}))$$
$$w = \frac{1}{2}\left(\ln r^2 + \ln(e^{2s} + e^{-2s})\right)$$
$$\boxed{w = \ln r + \frac{1}{2}\ln(e^{2s} + e^{-2s})}$$
$$\boxed{\begin{aligned} \frac{\partial w}{\partial r} &= \frac{1}{r} \\ \frac{\partial w}{\partial s} &= \frac{1}{2}\,\frac{2e^{2s} - 2e^{-2s}}{e^{2s}+e^{-2s}} = \frac{e^{2s}-e^{-2s}}{e^{2s}+e^{-2s}} \end{aligned}}$$

طريقة 2: استخدام قاعدة السلسلة

لدينا

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

ولدينا

$$\frac{\partial f}{\partial x} = \frac{1}{2}\,\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2} = \frac{re^s}{r^2(e^{2s}+e^{-2s})} = \frac{1}{r}\,\frac{e^s}{e^{2s}+e^{-2s}}$$
$$\frac{\partial f}{\partial y} = \frac{1}{2}\,\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2} = \frac{1}{r}\,\frac{e^{-s}}{e^{2s}+e^{-2s}}$$
$$\frac{\partial x}{\partial r} = e^s \;;\; \frac{\partial y}{\partial r} = e^{-s}$$
$$\frac{\partial x}{\partial s} = re^s \;;\; \frac{\partial y}{\partial s} = -re^{-s}$$

فإن

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = \frac{1}{r}\,\frac{e^s}{e^{2s}+e^{-2s}} \cdot e^s + \frac{1}{r}\,\frac{e^{-s}}{e^{2s}+e^{-2s}} \cdot e^{-s} = \frac{1}{r}\,\frac{e^{2s}+e^{-2s}}{e^{2s}+e^{-2s}}$$
$$\boxed{\frac{\partial w}{\partial r} = \frac{1}{r}}$$
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{r}\,\frac{e^s}{e^{2s}+e^{-2s}} \cdot re^s + \frac{1}{r}\,\frac{e^{-s}}{e^{2s}+e^{-2s}} \cdot (-re^{-s})$$
$$= \frac{1}{r}\,\frac{r(e^{2s}-e^{-2s})}{e^{2s}+e^{-2s}} = \boxed{\frac{e^{2s}-e^{-2s}}{e^{2s}+e^{-2s}} = \frac{\partial w}{\partial s}}$$

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مثال 4 ص 96: برهن على أن الدالة $z = f\left(\frac{1}{2}bx^2 - \frac{1}{3}ay^3\right)$ تحقق المعادلة $ay^2\frac{\partial z}{\partial x} + bx\frac{\partial z}{\partial y} = 0$ حيث أن $a, b \in \mathbb{R}$

الحل: نفرض أن $u = \frac{1}{2}bx^2 - \frac{1}{3}ay^3$

ولدينا:

$$\frac{\partial u}{\partial x} = bx$$
$$\frac{\partial u}{\partial y} = -ay^2$$

لدينا:

$$\frac{\partial z}{\partial x} = f'(u) \cdot \frac{\partial u}{\partial x} = \frac{df}{du}\,\frac{\partial u}{\partial x} = bx\,\frac{df}{du}$$
$$\frac{\partial z}{\partial y} = f'(u) \cdot \frac{\partial u}{\partial y} = \frac{df}{du} \cdot (-ay^2) = -ay^2\,\frac{df}{du}$$

وبالتالي:

$$ay^2\frac{\partial z}{\partial x} + bx\frac{\partial z}{\partial y} = ay^2 \cdot bx\frac{df}{du} + bx\left(-ay^2\frac{df}{du}\right)$$
$$= abxy^2\frac{df}{du} - abxy^2\frac{df}{du} = 0$$

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مثال (ص 96,97): برهن على أن الدالة $z = f(s^2-t^2, t^2-s^2)$ تحقق العلاقة التالية:

$$t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = 0$$

الحل: لدينا: $z = f(s^2-t^2, t^2-s^2)$ ونفرض أن

$$\begin{cases} x = s^2-t^2 \\ y = t^2-s^2 \end{cases}$$

حسب قاعدة السلسلة لدينا:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{\partial z}{\partial x}2s + \frac{\partial z}{\partial y}\cdot(-2s)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{\partial z}{\partial x}(-2t) + \frac{\partial z}{\partial y}(2t)$$

فإن:

$$t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = t\left(2s\frac{\partial z}{\partial x} - 2s\frac{\partial z}{\partial y}\right) + s\left(-2t\frac{\partial z}{\partial x} + 2t\frac{\partial z}{\partial y}\right)$$

$$= 2ts\left(\cancel{\frac{\partial z}{\partial x}} - \cancel{\frac{\partial z}{\partial y}} - \cancel{\frac{\partial z}{\partial x}} + \cancel{\frac{\partial z}{\partial y}}\right) = 0$$

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مثال (ص 97) $w = f(x,y)$ دالة في متغيرين مشتقاتها من الرتبة الثانية متصلة ونفرض أن $x = u+v$, $y = u-v$ برهن أن:

$$\frac{\partial^2 w}{\partial u\partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}$$

الحل: حسب قاعدة السلسلة لدينا

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$

فإن وحسب قاعدة السلسلة (للمرة الثانية)

$$\frac{\partial^2 w}{\partial u\partial v} = \frac{\partial\left(\frac{\partial w}{\partial u}\right)}{\partial v} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\right)\frac{\partial y}{\partial v}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\cdot 1 + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\cdot(-1)$$

$$= \frac{\partial^2 w}{\partial x^2} + \underbrace{\frac{\partial^2 w}{\partial y\partial x} - \frac{\partial^2 w}{\partial x\partial y}} - \frac{\partial^2 w}{\partial y^2}$$

بما أن $w = f$ دالة في متغيرين لها مشتقات من الرتبة الثانية متصلة فإن $\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial^2 w}{\partial y\partial x}$ وبالتالي

$$\boxed{\frac{\partial^2 w}{\partial u\partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}}$$

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ملاحظة:

$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial u}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\underbrace{\frac{\partial x}{\partial u}}_{1} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right)\underbrace{\frac{\partial y}{\partial u}}_{1}$$

$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y\partial x} + \frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 w}{\partial y^2}$$

لأن $\left(\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial^2 w}{\partial y\partial x}\right)$

$$\boxed{\frac{\partial^2 w}{\partial u^2} = \frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 w}{\partial y^2}}$$

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مثال (ص 98): u دالة في متغيرين x, y , $x = e^s$, $y = e^t$

برهن أن: المعادلة:

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

يمكن كتابتها على الصيغة:

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = 0$$

الحل: لدينا حسب قاعدة السلسلة:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} = e^s\frac{\partial u}{\partial x} + 0\cdot\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial s} = x\frac{\partial u}{\partial x}}$$

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s}\left(x\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial s}$$

$$= \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)\cdot\underbrace{e^s}_{x} + \frac{\partial}{\partial y}(\quad)\cdot 0 = \frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2}$$

$$\boxed{\frac{\partial^2 u}{\partial s^2} = x\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right)} = x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\boxed{\frac{\partial u}{\partial t} = e^t\frac{\partial u}{\partial y} = y\frac{\partial u}{\partial y}}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(y\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(\quad)\underbrace{\frac{\partial x}{\partial t}}_{0} + \frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)\underbrace{\frac{\partial y}{\partial t}}_{e^t}$$

$$= \left(\frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2}\right)y$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2}}$$

وبالتالي

$$\boxed{\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y}}$$

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الدوال المتجانسة

تعريف: لتكن f دالة في متغيرين x و y وحيث D هو مجال الدالة f.

f دالة متجانسة من الدرجة k $(k\in\mathbb{R})$ إذا كان:

$$\begin{cases} \text{لكل } t>0 \text{ ولكل } (x,y)\in D \text{ فإن } (tx,ty) = t(x,y)\in D \\ \text{لكل } t>0 \text{ , } (x,y)\in D \text{ لدينا: } f(tx,ty) = t^k f(x,y) \end{cases}$$

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مثال:

$$f(x,y) = \frac{xy}{x^2+y^2}$$

لدينا $D_f = \mathbb{R}^2 \setminus \{(0,0)\}$

ليكن $t>0$, $(x,y)\in D_f$ فإن $(x,y)\neq(0,0)$, وبما أن $t>0$ وبالتالي $(tx,ty) = t(x,y)\neq(0,0)$

فإن $(tx,ty)\in D_f$

ليكن $t>0$, $(x,y)\in D_f$ لدينا:

$$f(tx,ty) = \frac{tx\cdot ty}{(tx)^2+(ty)^2} = \frac{t^2xy}{t^2(x^2+y^2)} = \frac{xy}{x^2+y^2} = f(x,y) = t^0 f(x,y)$$

فإن f دالة متجانسة من الدرجة 0.

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مثال 2: لتكن $f(x,y)=\sqrt{x+2y}$

لدينا لكل $t>0$, $(x,y)\in D_f$ $(x+2y\geq 0)$

بأن $(tx,ty)=t(x,y)$

و $(tx)+2(ty)=t(x+2y)$ بما أن $t>0$, $x+2y\geq 0$

بأن $tx+2ty\geq 0$ بأن $(tx,ty)\in D_f$

ولدينا: $f(tx,ty)=\sqrt{(tx)+2(ty)}=\sqrt{t(x+2y)}=\sqrt{t}\sqrt{x+2y}$

$=t^{1/2}f(x,y)$

وبالتالي f دالة متجانسة من الدرجة $\frac{1}{2}$

Jun 25-2:03 PM

مثال 3: $f(x,y)=\dfrac{x^2}{x^4+y^4}$

Jun 25-2:07 PM

مثال 4 $f(x,y)=\dfrac{x+y-3}{x^2+y^2}$

ليكن $t>0$, $(x,y)\in D_f$ بأن $x^2+y^2\neq 0$

ولدينا $(tx)^2+(ty)^2=t^2(x^2+y^2)\neq 0$ بأن $(tx,ty)\in D_f$

لدينا $f(1,3)=\dfrac{1+3-3}{1^2+2^2}=\dfrac{1}{5}$, لدينا $f(t\cdot 1,t\cdot 3)=\dfrac{t+3t-3}{t^2+(2t)^2}$

$=\dfrac{4t-3}{t^2(5)}$

نفرض أن $\dfrac{4t-3}{5t^2}=t^k\cdot\dfrac{1}{5}$

بأن $4t-3=t^{k+2} \Leftrightarrow \begin{cases} t^{k+2}-4t-3=0 \\ t>0 \text{ لكل} \end{cases}$

إذا كان $\underline{t=1}$ بأن $1^{k+2}-4-3\neq 0$

وبالتالي f دالة غير متجانسة.

Jun 25-2:07 PM