I Vector spaces (Linear spaces)

Definition

we say that a set V is a vector space over the field F if we have:

(I) $(V, +)$ satisfies the following conditions:

(1) closed : $a, b \in V \Rightarrow a+b \in V$

(2) associetive : $(a+b)+c = a+(b+c)$ $\forall a, b, c \in V$.

(3) identity $0 \in V$: $a+0 = 0+a = a$ $\forall a \in V$

(4) inverse : $\forall a \in V$ there is $(-a) \in V$ where
$a + (-a) = (-a) + a = 0$

(5) Commutative : $a+b = b+a$

(II) the product $\boxed{\lambda a \in V}$ where $\lambda \in F$ and $a \in V$
which satisfies the following

(1) $1 \cdot a = a$ $\forall a \in V$

(2) $\lambda(a+b) = \lambda a + \lambda b$ $\forall \lambda \in F$ and $a, b \in V$

(3) $\lambda_1(\lambda_2 a) = (\lambda_1 \lambda_2) a$ $\forall \lambda_1, \lambda_2 \in F$ and $a \in V$

(4) $(\lambda_1 + \lambda_2) a = \lambda_1 a + \lambda_2 a$ $\forall \lambda_1, \lambda_2 \in F$ and $a \in V$.

and then, we write $\boxed{V(F)}$

Remark

For any $V(F)$, we have to know the definition of

① $v_1 + v_2 \in V$ $\forall v_1, v_2 \in V$

② $\lambda v \in V$ $\forall \lambda \in F$ and $v \in V$

Remark

In this course, F is either $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers)

Example $\mathbb{Z}(\mathbb{R})$ where $(\mathbb{Z},+)$ is the regular addition and $\lambda \cdot a$ is regular product where $\lambda \in \mathbb{R}$ and $a \in \mathbb{Z}$, is not Linear space

Let $\lambda = 1/2$ and $v = 3$ then

$\lambda v = \frac{3}{2} \notin \mathbb{Z}$

Example $\mathbb{R}(\mathbb{R})$ is linear space.

Example $\mathbb{R}^2(\mathbb{R})$ is linear space where

(i) $(a,b) + (c,d) = (a+c, b+d) \in \mathbb{R}^2$

(ii) $\lambda(a,b) = (\lambda a, \lambda b) \in \mathbb{R}^2$

(iii) $0 = (0,0)$

(iv) If $v = (a,b)$ then $-v = (-a,-b)$ because

$v + (-v) = (a,b) + (-a,-b) = (0,0) = 0$

xample $\mathbb{R}^n(\mathbb{R})$ is linear space which is called Eucleaden space, Every $v \in \mathbb{R}^n$ is n-tuple

(i) $v = (a_1, a_2, —, a_n)$

(ii) $v_1 + v_2 = (a_1, —, a_n) + (b_1, —, b_n) = (a_1+b_1, —, a_n+b_n)$

(iii) $\lambda v = \lambda(a_1, —, a_n) = (\lambda a_1, —, \lambda a_n)$

(v) $0 = (0, —, 0)$

(iv) $-v = (-a_1, —, -a_n)$ where $v = (a_1, —, a_n)$

Example Let $F_5(x)$ be the set of all Polynomials of degree five. $F_5(x)(\mathbb{R})$ is not Linear space because

$$f(x) = x^5 + 2x - 3 \in F_5(x)$$
$$g(x) = -x^5 + 3x^2 \in F_5(x)$$

But $f(x) + g(x) = 3x^2 + 2x - 3 \notin F_5(x)$

Example $P_n(x)$ is the set of all Polynomials of degree $\leq n$. Then $P_n(x)(\mathbb{R})$ is Linear space such that

(1) $(a_n x^n + a_{n-1} x^{n-1} + \dots + a_0) + (b_n x^n + b_{n-1} x^{n-1} + \dots + b_0)$
$= (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \dots + (a_0 + b_0)$

(2) $\lambda (a_n x^n + a_{n-1} x^{n-1} + \dots + a_0) = \lambda a_n x^n + \lambda a_{n-1} x^{n-1} + \dots + \lambda a_0$

(3) $0 = 0$ (as polynomial)

(4) $-(a_n x^n + \dots + a_0) = -a_n x^n - a_{n-1} x^{n-1} - \dots - a_0$

Remark

we write Linear spaces such as $\mathbb{R}^n$, $P_n(x)$ rather than $\mathbb{R}^n(\mathbb{R})$, $P_n(x)(\mathbb{R})$.

Definition

Let $V(F)$ be Linear space where v_1, v_2 and $v_n \in V$. we say that w is Linear Combination of $v_1, v_2, \ldots, v_n$ if there are $\lambda_1, \ldots, \lambda_n \in F$ such that

$$w = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$$

Example: Let
$$\left.\begin{array}{l} v_1 = (1,3,2,1) \\ v_2 = (2,-2,-5,4) \\ v_3 = (2,-1,3,6) \end{array}\right\} \text{ be vectors of } \mathbb{R}^4.$$

Is $u = (2,5,-4,0)$ a Linear combination of v_1, v_2 and v_3 ?

solution suppose that

$$(2,5,-4,0) = \lambda_1 (1,3,2,1) + \lambda_2 (2,-2,-5,4) + \lambda_3 (2,-1,3,6)$$

we can conclude the following system

$$\begin{aligned} \lambda_1 + 2\lambda_2 + 2\lambda_3 &= 2 \\ 3\lambda_1 - 2\lambda_2 - \lambda_3 &= 5 \\ 2\lambda_1 - 5\lambda_2 + 3\lambda_3 &= -4 \\ \lambda_1 + 4\lambda_2 + 6\lambda_3 &= 0 \end{aligned}$$

If the system is consistent then there are values of λ_1, λ_2 and λ_3, and hence u is Linear combination of v_1, v_2 and v_3.

Now, the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{after}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice here (number of Equation = number of veriables)
So, we have unique solution $\lambda_3 = -1$, $\lambda_2 = 1$, $\lambda_1 = 2$

Therefore $\boxed{u = 2v_1 + v_2 - v_3}$ ∎

** Linear subspaces

Definition Let $V(F)$ be a Linear space and $\phi \neq W \subseteq V$. we say that W is Linear subspace of V if $W(F)$ is Linear space.

Example Every Linear space V has two trivial subspace which are : 0 and V

Criterion Let V be a Linear space. Then

W is Linear subspace of V $\iff$
(i) $\phi \neq W \subseteq V$
(ii) $\forall\, v_1, v_2 \in W$ then $v_1 + v_2 \in W$
(iii) $\forall\, v \in W$ and $\lambda \in F$, then $\lambda v \in W$.

Example we know that $\mathbb{R}(\mathbb{R})$ is Linear space
Let $W = \{3a \,;\, a \in \mathbb{R}\}$. Then W is Linear subspace for

(i) $0 = 3(0) \in W \Rightarrow W \neq \phi$

(ii) Let $v_1, v_2 \in W \Rightarrow v_1 = 3a \wedge v_2 = 3b$

Now, $v_1 + v_2 = 3a + 3b = 3(a+b) \in W$

(iii) Let $v \in W$ and $\lambda \in \mathbb{R}$. Then

$\lambda v = \lambda(3a) = 3(\lambda a) \in W$ ∎

Example : Let $W = \{(x, 0) \,;\, x \in \mathbb{R}\}$. Then W is Linear subspace of $\mathbb{R}^2$ For :

(i) $0 = (0,0) \in W \Rightarrow W \neq \phi$

(ii) Let $v_1 = (a,0)$ and $v_2 = (b,0) \in W$. Then
$v_1 + v_2 = (a+b, 0) \in W$

(iii) Let $v = (a,0)$ and $\lambda \in \mathbb{R}$. Then
$\lambda v = \lambda(a,0) = (\lambda a, 0) \in W$

Example Let W be the set of all points lie on the line $y = 3x$. Then W is linear subspace of $\mathbb{R}^2$.
For:

we can describe W as follows:

$$W = \{(x, 3x) \; ; \; x \in \mathbb{R}\}$$

(i) $0 = (0,0) \in W \Rightarrow W \neq \phi$

(ii) Let $v_1, v_2 \in W \Rightarrow v_1 = (a, 3a)$, $v_2 = (b, 3b)$

So, $v_1 + v_2 = (a+b, 3a+3b) = (a+b, 3(a+b)) \in W$

(iii) Let $v \in W$ and $\lambda \in \mathbb{R} \Rightarrow \lambda v = \lambda(x, 3x)$

$= (\lambda x, \lambda 3x) = (\lambda x, 3(\lambda x)) \in W$

Example Let W_1 and W_2 be two linear subspace of V. Then

① $W_1 \cap W_2$ is linear subspace. For

(i) as $0 \in W_1$ and $0 \in W_2 \Rightarrow 0 \in W_1 \cap W_2 \Rightarrow$
$W_1 \cap W_2 \neq \phi$

(ii) Let $v_1, v_2 \in W_1 \cap W_2 \rightarrow$ $v_1, v_2 \in W_1$ and $v_1, v_2 \in W_2$ $\rightarrow$ $v_1 + v_2 \in W_1$ and $v_1 + v_2 \in W_2$

So, $v_1 + v_2 \in W_1 \cap W_2$

(iii) Let $\lambda \in F$ and $v \in W_1 \cap W_2 \rightarrow$ $\lambda \in F$ and $v \in W_1$ and $\lambda \in F$ and $v \in W_2$

$\Rightarrow$ $\lambda v \in W_1$ and $\lambda v \in W_2$, so, $\lambda v \in W_1 \cap W_2$

② $W_1 \cup W_2$ is not necessarily linear subspace
for example $W_1 = \{3x \; ; \; x \in \mathbb{R}\}$ and $W_2 = \{2x \; ; \; x \in \mathbb{R}\}$ are two linear subspace of $\mathbb{R}$.
Let $v_1 = 2$ and $v_2 = 3$. clearly $v_1, v_2 \in W_1 \cup W_2$
but $v_1 + v_2 = 2 + 3 = 5 \notin W_1 \cup W_2$

⑦

Remark

$M_n(\mathbb{R})$ is the set of all squar matrices of size $n \times n$. we know the sum of matrices and product scalar of matrix will make $M_n(\mathbb{R})$ is a Linear space.

Example : $W = \{ A \in M_2(\mathbb{R}) : A \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} A \}$ is a Linear subspace. For :

(i) Since $0 \cdot \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \cdot 0 = 0$, $0 \in W$. So, $W \neq \phi$

(ii) Let $A, B \in W$. The goal is to prove $A+B \in W$, i.e. $(A+B)\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}(A+B)$. For that

$$
\begin{aligned}
L.H.S &= (A+B)\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= A\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} + B\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} A + \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} B \quad (\text{since } A, B \in W) \\
&= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}(A+B) = R.H.S
\end{aligned}
$$

(iii) Let $A \in W$ and $\lambda \in \mathbb{R}$. The goal is to prove that $\lambda A \in W$; i.e., $(\lambda A)\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}(\lambda A)$ for that

$$
\begin{aligned}
L.H.S &= \lambda A\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \lambda \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} A \quad (\text{since } A \in W) \\
&= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}(\lambda A) \quad (\text{since } \lambda \text{ is scalar}) \\
&= R.H.S
\end{aligned}
$$

⑧ Let $A = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \sigma \end{bmatrix} ; \alpha+\beta = \gamma+\sigma \right\} \subseteq M_2(\mathbb{R})$. Then A is Linear subspace of $M_2(\mathbb{R})$.

Proof: (i) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in A \Rightarrow A \neq \phi$

(ii) Let $v_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \sigma_1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \sigma_2 \end{bmatrix} \in W$.

Then, $\alpha_1+\beta_1 = \gamma_1+\sigma_1$ and $\alpha_2+\beta_2 = \gamma_2+\sigma_2$.

Now, $v_1+v_2 = \begin{bmatrix} \alpha_1+\alpha_2 & \beta_1+\beta_2 \\ \gamma_1+\gamma_2 & \sigma_1+\sigma_2 \end{bmatrix} \in A$ because

$$(\alpha_1+\alpha_2)+(\beta_1+\beta_2) = (\alpha_1+\beta_1)+(\alpha_2+\beta_2)$$
$$= (\gamma_1+\sigma_1)+(\gamma_2+\sigma_2)$$
$$= (\gamma_1+\gamma_2)+(\sigma_1+\sigma_2)$$

(iii) Let $v = \begin{bmatrix} \alpha & \beta \\ \gamma & \sigma \end{bmatrix} \in W$ and $\lambda \in \mathbb{R}$.

Now, $\lambda v = \begin{bmatrix} \lambda\alpha & \lambda\beta \\ \lambda\gamma & \lambda\sigma \end{bmatrix}$ where

$$\lambda\alpha+\lambda\beta = \lambda(\alpha+\beta)$$
$$= \lambda(\gamma+\sigma) \quad (\text{since } v \in W)$$
$$= \lambda\gamma+\lambda\sigma$$

So, $\lambda v \in W$. ∎

(9) ** Spanning set

Def : Let V be a vector space. The set $S = \{v_1, \dots, v_n\}$ of vectors of V is called a spanning set of V if every vector $v \in V$ is a Linear combination of vectors of S.

Remarks :

(1) To prove that $S = \{v_1, \dots, v_n\}$ is spanning set of V, Let $v \in V$ and suppose that

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n \quad \text{for some } \lambda_1, \dots, \lambda_n \in \mathbb{R}$$

If there is a solution for $\lambda_1, \dots, \lambda_n$ then v is Linear combination of $S \Rightarrow S$ is spanning set

(2) Let $S = \{v_1, \dots, v_n\}$ be a set of vectors of V we define the set

$$\text{Span}(S) = \{\lambda_1 v_1 + \dots + \lambda_n v_n : \lambda_1, \dots, \lambda_n \in \mathbb{R}\}$$

Notice that

(1) Span(S) is Linear subspace of V

(2) If span$(S) = V$ then S is spanning set of V

Theorem

Let $S = \{v_1, \dots, v_n\}$ be a set of vectors of a Linear space V. Then span(S) is the smallest Linear subspace of V contains S, i.e., if W is Linear subspace contains S, then span$(S) \subseteq W$.

Example Let $\{v_1 = (1,1), v_2 = (1,-2)\} = S$. Does S span $V = \mathbb{R}^2$?

Solution Let $v = (a,b)$ be any vector of $\mathbb{R}^2$, and suppose that there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $(a,b) = \lambda_1 (1,1) + \lambda_2 (1,-2)$

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Then, we have

$$\left.\begin{array}{l}\lambda_1 + \lambda_2 = a \\ \lambda_1 - 2\lambda_2 = b\end{array}\right\}$$ it is non-Homogenous system

Notice that

$$|A| = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3 \neq 0$$

so, A^{-1} existed

so, the system is consistent

Hence, $S = \{v_1, v_2\}$ spans $\mathbb{R}^2$ ☒

Example

Does $\{1, 1-x, 1-x^2\}$ spans $P_2(x)$?

Solution Let ax^2+bx+c be any vector of $P_2(x)$ and suppose that there exists $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$ax^2+bx+c = \lambda_1(1) + \lambda_2(1-x) + \lambda_3(1-x^2)$$

Then we have :

$$\left.\begin{array}{l}\lambda_1 + \lambda_2 + \lambda_3 = c \\ -\lambda_2 = b \\ -\lambda_3 = a\end{array}\right\} \Rightarrow$$ it is clear that the system has unique solution

$\Rightarrow \{1, 1-x, 1-x^2\}$ spans $P_2(x)$.

Example

Does $\left\{ \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix} \right\}$ spans $M_2(\mathbb{R})$?

Solution Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any vector of $M_2(\mathbb{R})$, and suppose that there exists $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \lambda_1 \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}$$

Then, we have

$$\left.\begin{array}{l} -\lambda_2 - 2\lambda_3 = a \\ 2\lambda_1 + 3\lambda_2 + 8\lambda_4 = b \\ \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 = c \\ 2\lambda_2 + 3\lambda_3 + \lambda_4 = d \end{array}\right\}$$ Non-Homogenous square-system

(11) we will examine $|A| = \begin{vmatrix} 0 & -1 & -2 & 0 \\ 2 & 3 & 0 & 8 \\ 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 1 \end{vmatrix}$ $\underline{\underline{-2R_3 + R_2}}$

$$= \begin{array}{c} + \\ - \\ + \\ \\ \end{array} \begin{vmatrix} 0 & -1 & -2 & 0 \\ 0 & 1 & -2 & 4 \\ 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 4 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & -2 & 0 \\ 1 & -2 & 4 \\ 2 & 3 & 4 \end{vmatrix} \quad \frac{R_1 + R_2}{2R_1 + R_3}$$

$$= \begin{vmatrix} -1 & -2 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 4 \end{vmatrix} = (-1) \begin{vmatrix} -4 & 4 \\ -1 & 4 \end{vmatrix}$$

$$= 12 \neq 0$$

So, the system has unique solution

Hence S spans $M_2(\mathbb{R})$ ▨

** Linear Space of Solutions of $AX=0$.

Theorem: Let $AX=0$ be a Homogenous system of Linear equations. Then The solution set of such system is Linear subspace of R^n

Proof Let $S = \{ v \in R^n : v \text{ is a solution} \}$
$= \{ v \in R^n : Av = 0 \}$

(i) $0 \in R^n$ is a solution because $A \cdot 0 = 0 \Rightarrow 0 \in S$
$\Rightarrow S \neq \phi$

(ii) Let $v_1, v_2 \in S$. Our goal is to Prove that $v_1 + v_2 \in S$; i.e., $A(v_1+v_2) = 0$. For that

$$\begin{aligned} L\cdot H\cdot S = A(v_1+v_2) &= Av_1 + Av_2 \\ &= 0 + 0 \quad (\text{as } v_1, v_2 \in S) \\ &= 0 = R\cdot H\cdot S \end{aligned}$$

(iii) Let $v \in S$ and $\lambda \in \mathbb{R}$. Our goal is to Prove that $\lambda v \in S$; i.e. $A(\lambda v) = 0$. For that

$$L\cdot H\cdot S = A(\lambda v) = \lambda(Av) = \lambda(0) = 0 = R\cdot H\cdot S$$

Hence, S is a Linear subspace of $\mathbb{R}^n$.

Example: Find the span set of the space of the solutions of the following system:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{array}\right] \; ?$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice that number of Parameters $= \boxed{3} - \boxed{1} = 2$

Let $x = s$, $y = t$. Then $z = \dfrac{-s+2t}{3}$ (since $x - 2y + 3z = 0$)

So, $S = \left\{ \begin{bmatrix} s \\ t \\ \frac{-s+2t}{3} \end{bmatrix} : s,t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} s \\ 0 \\ \frac{-s}{3} \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ \frac{2t}{3} \end{bmatrix} : s,t \in \mathbb{R} \right\}$

$= \left\{ s \begin{bmatrix} 1 \\ 0 \\ \frac{-1}{3} \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ \frac{2}{3} \end{bmatrix} : s,t \in \mathbb{R} \right\} \Rightarrow \text{span}(V) = \left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{-1}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{2}{3} \end{bmatrix} \right\}$

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$\text{span}(\mathbb{R}^2) = \{(1,0), (0,1)\}$ becaus for every

$(a,b) \in \mathbb{R}^2 \Rightarrow (a,b) = (a,0) + (0,b)$

$= a(1,0) + b(0,1)$.

$\text{span}(\mathbb{R}^3) = \{(1,0,0), (0,0,1), (0,0,1)\}$

$\text{span}(M_2(\mathbb{R})) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\text{span}(P_2(x)) = \{1, x, x^2\}$

$\text{span}(P_1(x)) = \{1, x\}$