Fourier Integral

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Fourier Integral

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Formula of Fourier Integral

The Fourier Integral of $f(x)$ defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty A(\lambda)\cos(\lambda x)\ d\lambda + \frac{1}{\pi}\int_0^\infty B(\lambda)\sin(\lambda x)\ d\lambda, \qquad (1)$$

where

$$A(\lambda) = \int_{-\infty}^\infty f(t)\cos(\lambda t)\ dt,$$

and

$$B(\lambda) = \int_{-\infty}^\infty f(t)\sin(\lambda t)\ dt.$$

Formula (1) can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)\ dt d\lambda. \qquad (2)$$

Theorem

If f is absolutely integrable

$$\left(\int_{-\infty}^{\infty} |f(x)| \, dx < \infty \right),$$

and f, f' are piecewise continuous on every finite intreval, then Fourier integral of f converges to $f(x)$ at a point of continuity and converges to

$$\frac{f(x+0) + f(x-0)}{2}$$

at a point of discontinuity.

Example (1)

Express the function

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1, \end{cases}$$

as a Fourier integral. Hence evaluate $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda$ and deduce the value of $\int_0^\infty \frac{\sin\lambda}{\lambda} d\lambda$.

Solution Since

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)dtd\lambda \\ &= \frac{1}{\pi}\int_0^\infty \int_{-1}^1 \cos\lambda(t-x)dtd\lambda \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^\infty \left.\frac{\sin\lambda(t-x)}{\lambda}\right|_{-1}^{1} d\lambda \\
&= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(1-x)-\sin\lambda(-1-x)}{\lambda} d\lambda \\
&= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(1+x)+\sin\lambda(1-x)}{\lambda} d\lambda \\
&= \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda.
\end{aligned}
$$

Hence

$$
\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1, \end{cases}
$$

At $x = \pm 1$, $f(x)$ is discontinuous and the integral has the value $\frac{1}{2}(\frac{\pi}{2}+0) = \frac{\pi}{4}$.

Now by setting $x = 0$, we have

$$
\int_0^\infty \frac{\sin\lambda}{\lambda} d\lambda = \frac{\pi}{2}.
$$

Example (2)

Compute the Fourier integral of the function

$$f(x) = \begin{cases} 0, & -\infty < x < -\pi \\ -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & \pi < x < \infty. \end{cases}$$

Solution We have

$$\begin{aligned}
f(x) &= \frac{1}{\pi}\int_0^\infty \int_{-\pi}^0 -\cos\lambda(t-x)dtd\lambda + \frac{1}{\pi}\int_0^\infty \int_0^\pi \cos\lambda(t-x)dtd\lambda \\
&= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(t-x)}{\lambda}\Big|_{-\pi}^0 d\lambda + \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(t-x)}{\lambda}\Big|_0^\pi d\lambda \\
&= -\frac{1}{\pi}\int_0^\infty \frac{-\sin\lambda x + \sin\lambda(\pi+x)}{\lambda}d\lambda \\
&+ \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(\pi-x) + \sin\lambda x}{\lambda}d\lambda \\
&= 2\int_0^\infty \frac{(1-\cos\lambda\pi)}{\lambda}\sin(\lambda x)d\lambda.
\end{aligned}$$

This Fourier integral converges at the discontinuities points $-\pi, 0, \pi$ respectively to

$$\frac{f((-\pi)^+) + f((-\pi)^-)}{2} = \frac{-1}{2},$$

$$\frac{f(0^+) + f(0^-)}{2} = 0,$$

$$\frac{f(\pi^+) + f(\pi^-)}{2} = \frac{1}{2}.$$

Example (3)

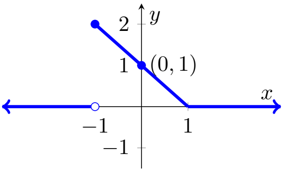
Consider the function

$$f(x) = \begin{cases} 0, & x < -1 \\ 1 - x, & -1 \leq x < 1 \\ 0, & x \geq 1. \end{cases}$$

Sketch the graph of f, find the Fourier integral and deduce the value of

$$\int_0^\infty \frac{\sin \lambda}{\lambda} d\lambda.$$

Solution



(1)

$$
\begin{aligned}
A(\lambda) &= \int_{-\infty}^{\infty} f(t)\cos(\lambda t)dt \\
&= \underbrace{\int_{-1}^{1} (1-t)\cos(\lambda t)dt}_{\text{by parts}} \\
&= \left[\frac{\sin(\lambda t)}{\lambda}(1-t)\right]_{-1}^{1} - \left[\frac{1}{\lambda^2}\cos(\lambda t)\right]_{-1}^{1} \\
&= \frac{-2}{\lambda}\sin(-\lambda) - \frac{\cos(\lambda) - \cos(-\lambda)}{\lambda^2} = \frac{2}{\lambda}\sin(\lambda)
\end{aligned}
$$

(2)

$$\begin{aligned}
B(\lambda) &= \int_{-\infty}^{\infty} f(t)\sin(\lambda t)dt \\
&= \underbrace{\int_{-1}^{1}(1-t)\sin(\lambda t)dt}_{\text{by parts}} \\
&= \left[-\frac{\cos(\lambda t)}{\lambda}(1-t)\right]_{-1}^{1} - \left[\frac{1}{\lambda^2}\sin(\lambda t)\right]_{-1}^{1} \\
&= \frac{2}{\lambda}\cos(-\lambda) - \frac{\sin(\lambda)-\sin(-\lambda)}{\lambda^2} = \frac{2\cos(\lambda)}{\lambda} - \frac{2\sin(\lambda)}{\lambda^2}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{f(x^+)+f(x^-)}{2} \\
= &\ \frac{1}{\pi}\int_0^\infty A(\lambda)\cos(\lambda x)d\lambda + \frac{1}{\pi}\int_0^\infty B(\lambda)\sin(\lambda x)d\lambda \\
= &\ \frac{1}{\pi}\int_0^\infty \left[\frac{2\sin\lambda\cos(\lambda x)}{\lambda} + \left(\frac{2\cos\lambda}{\lambda} - \frac{2\sin\lambda}{\lambda^2}\right)\sin(\lambda x)\right]d\lambda,
\end{aligned}$$

At $x=0$, we have

$$\frac{f(0^+)+f(0^-)}{2} = \frac{1+1}{2} = 1 = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}d\lambda,$$

hence,

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin\lambda}{\lambda}d\lambda$$

Exercises

Find the Fourier integral for the following functions

1.
$$f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0. \end{cases}$$

2.
$$f(x) = \begin{cases} 0, & -\infty < x < -2 \\ -2, & -2 < x < 0 \\ 2, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$$

3.
$$f(x) = \begin{cases} C, & |x| \leq 1 \\ 0, & |x| > 1, \end{cases}$$

where C is a constant such that $C \neq 0$. Deduce the value of the integral $\displaystyle\int_0^\infty \frac{\sin\alpha}{\alpha} d\alpha$.

Fourier Cosine and Sine Series Integrals

The Fourier sine integral is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty C(\lambda) \sin(\lambda x) d\lambda,$$

where

$$C(\lambda) = \int_0^\infty f(t) \sin(\lambda t) dt.$$

The Fourier cosine integral is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty D(\lambda) \cos(\lambda x) d\lambda, \tag{3}$$

where

$$D(\lambda) = \int_0^\infty f(t) \cos(\lambda t) dt.$$

Example

Compute the Fourier integral of the function

$$f(x) = \begin{cases} |\sin x|, & |x| \leq \pi \\ 0, & |x| \geq \pi, \end{cases}$$

and deduce that

$$\int_0^\infty \frac{\cos \lambda\pi + 1}{1 - \lambda^2} \cos\left(\frac{\pi\lambda}{2}\right) d\lambda = \frac{\pi}{2}.$$

Solution We observe that the function f is even on the interval $(-\infty, \infty)$. So It has a Fourier cosine integral given by (3), that is

$$f(x) = \frac{2}{\pi} \int_0^\infty D(\lambda) \cos(\lambda x) d\lambda,$$

where

$$\begin{aligned} D(\lambda) &= \int_0^\infty f(t)\cos(\lambda t)dt = \int_0^\pi \sin t\cos(\lambda t)dt \\ &= \int_0^\pi \frac{\sin t(1-\lambda)+\sin t(1+\lambda)}{2}dt \\ &= \left.\frac{-\cos t(1-\lambda)}{2(1-\lambda)}\right|_0^\pi - \left.\frac{\cos t(1+\lambda)}{2(1+\lambda)}\right|_0^\pi \\ &= \frac{1}{1-\lambda^2}\left[\cos\pi\lambda+1\right]. \end{aligned}$$

Thus

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1-\lambda^2}\left[\cos\pi\lambda+1\right]\cos(\lambda x)d\lambda. \tag{4}$$

Since f is continuous on the whole interval $(-\infty,\infty)$, the above integral converges to the given function $f(x)$. Setting $x=\pi/2$ in (4), we get

$$\int_0^\infty \frac{1}{1-\lambda^2}\left[\cos\pi\lambda+1\right]\cos\left(\frac{\lambda\pi}{2}\right)d\lambda = \frac{\pi}{2}.$$

Exercises

Find the Fourier sine and Fourier cosine integral for the following functions

1. $$f(x) = \begin{cases} x^2, & 0 < x \leq 10 \\ 0, & x > 10, \end{cases}$$

2. $$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ x+1, & 1 < x < 2 \\ 0, & x \geq 2. \end{cases}$$

The Complex Form of Fourier Integral

The complex form of Fourier integral is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \beta(\lambda) e^{-i\lambda x} \, d\lambda,$$

where

$$\beta(\lambda) = \int_{-\infty}^{\infty} f(t) e^{i\lambda t} \, dt$$

Example

Find the complex form of the Fourier integral for the function

$$f(x) = \begin{cases} e^x, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Solution We have

$$\begin{aligned} \beta(\lambda) &= \int_{-\infty}^{\infty} f(t)e^{i\lambda t}dt = \int_{-1}^{1} e^{(i\lambda+1)t}dt \\ &= \frac{1}{(i\lambda+1)}\, e^{(i\lambda+1)t}\Big|_{-1}^{1} = \frac{1}{(i\lambda+1)}\left(e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right) \\ &= \frac{1-i\lambda}{1+\lambda^2}\left[e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right]. \end{aligned}$$

Hence

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-i\lambda}{1+\lambda^2}\left[e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right] e^{-i\lambda x} d\lambda.$$

Exercise

Find the complex form of the Fourier integral for the function

$$f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0. \end{cases}$$

Acknowledgment
This project was supported by King Saud University, Center of Excellence in Learning and Teaching.