Fourier Series

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Fourier Series

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Orthogonal Functions

Firstly, we will introduce a tool called inner product to define orthogonal functions and sets of orthogonal functions.

Definition

The inner product of two functions f and g on the interval $[\alpha, \beta]$ is the scalar (real number)

$$(f, g) = \int_{\alpha}^{\beta} f(x)g(x) \ dx.$$

Definition

We say that The two functions f and g are orthogonal functions on the interval $[\alpha, \beta]$ if

$$(f, g) = \int_{\alpha}^{\beta} f(x)g(x) \ dx = 0.$$

Example (1)

The two functions $f(x) = \cos x$ and $g(x) = \sin x$ are orthogonal on the interval $[-\pi, \pi]$ since

$$(f, g) = \int_{-\pi}^{\pi} \cos x. \sin x \, dx = 0.$$

Example (2)

The two functions $f(x) = x$ and $g(x) = e^{|x|}$ are orthogonal on any symmetric interval $[-A, A]$, where A is a positive real constant. By using integration by parts, It can be easily checked that

$$(f, g) = \int_{-A}^{A} x e^{|x|} dx = 0.$$

Definition

We say that The set of functions $\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots\}$ is orthogonal on the interval $[\alpha, \beta]$ if

$$(\varphi_n(x), \varphi_m(x)) = \int_\alpha^\beta \varphi_n(x)\varphi_m(x) \, dx = 0, \quad n \neq m.$$

Definition

We define the norm (length) of function f in terms of the inner product as the quantity

$$||f|| = \sqrt{(\varphi_n, \varphi_n)} = \left(\int_\alpha^\beta \varphi_n^2(x) \, dx \right)^{1/2}.$$

Definition

If $\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots\}$ is an orthogonal set of function on the interval $[\alpha, \beta]$ with the property $||\varphi_n|| = 1$ for $n = 1, 2, \ldots$, then the set $\{\varphi_n(x)\}_{n \geq 1}$ is said to be an orthonormal set on the interval.

$$(\varphi_n(x), \varphi_m(x)) = \int_\alpha^\beta \varphi_n(x)\varphi_m(x) \ dx = 0, \quad n \neq m.$$

Definition

A set of real-valued functions $\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots\}$ is said to be orthogonal with respct to weight function $w(x) > 0$ on the interval $[\alpha, \beta]$ if We define the norm (length) of function f in terms of the inner product as the quantity

$$(\varphi_n, \varphi_m)_{w(x)} = \int_\alpha^\beta w(x)\varphi_n(x)\varphi_m(x) \ dx = 0, \quad n \neq m.$$

Example (3)

Show that the set of functions
$\{1, \sin x, \cos x, \sin 2x, \cos 2x, ..., \sin mx, \cos mx, ..\}$ is orthogonal on the interval $[-\pi, \pi]$. Find the corresponding orthonormal set on $[-\pi, \pi]$.

Solution We have to show that

$$
\begin{aligned}
(1, \sin nx) &= 0, \ (1, \cos nx) = 0, \ (\sin nx, \sin mx) = 0, \\
(\cos nx, \cos mx) &= 0, (\sin nx, \cos mx) = 0, \ \forall n \neq m.
\end{aligned}
$$

$$
\begin{aligned}
(1, \sin nx) &= \int_{-\pi}^{\pi} \sin nx \, dx = -\frac{1}{n} \cos nx|_{-\pi}^{\pi} = 0, \\
(1, \cos nx) &= \int_{-\pi}^{\pi} \cos nx \, dx = \frac{1}{n} \sin nx|_{-\pi}^{\pi} = 0, \\
(\sin nx, \sin mx) &= \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \\
&= \int_{-\pi}^{\pi} \frac{\cos(n-m)x - \cos(n+m)x}{2} dx = 0, \ n \neq m,
\end{aligned}
$$

$$
\begin{aligned}
(\cos nx, \cos mx) &= \int_{-\pi}^{\pi} \cos nx \cos mx\, dx \\
&= \int_{-\pi}^{\pi} \frac{\cos(n-m)x + \cos(n+m)x}{2} dx = 0, \ n \neq m, \\
(\sin nx, \cos mx) &= \int_{-\pi}^{\pi} \sin nx \ \cos mx \ dx \\
&= \int_{-\pi}^{\pi} \frac{\sin(n-m)x + \sin(n+m)x}{2} dx = 0.
\end{aligned}
$$

To determine the orthonormal set on $[-\pi, \pi]$, we have to divide each element by its norm.

$$
\|1\|^2 = \int_{-\pi}^{\pi} dx = 2\pi,
$$

$$
\begin{aligned}
\|\sin mx\|^2 &= \int_{-\pi}^{\pi} (\sin mx)^2 dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2mx}{2} dx = \pi, \\
\|\cos mx\|^2 &= \int_{-\pi}^{\pi} (\cos mx)^2 dx = \int_{-\pi}^{\pi} \frac{1 + \cos 2mx}{2} dx = \pi.
\end{aligned}
$$

Hence the orthonormal set on $[-\pi, \pi]$:

$$
\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, ..., \frac{\sin mx}{\sqrt{\pi}}, \frac{\cos mx}{\sqrt{\pi}}, .. \right\}.
$$

Example (4)

Show that the functions

$$f(x) = 1, g(x) = 2x, h(x) = 4x^2 - 2$$

are orthogonal with respect to the weight function $w(x) = e^{-x^2}$ on the interval $(-\infty, \infty)$.

$$(1, 2x)_{w(x)} = \int_{-\infty}^{\infty} 2xe^{-x^2}dx = -2\int_{-\infty}^{\infty} e^{-x^2}dx = -2\, e^{-x^2}\Big|_{-\infty}^{\infty} = 0,$$

$$
\begin{aligned}
(1, 4x^2 - 2)_{w(x)} &= \int_{-\infty}^{\infty} (4x^2 - 2)e^{-x^2}dx \\
&= -\int_{-\infty}^{\infty} 2xe^{-x^2}dx - 2\int_{-\infty}^{\infty} e^{-x^2}dx \\
&= -2xe^{-x^2}\Big|_{-\infty}^{\infty} + 2\int_{-\infty}^{\infty} e^{-x^2}dx - 2\int_{-\infty}^{\infty} e^{-x^2}dx \\
&= 0.
\end{aligned}
$$

In the same way and by integration by parts, we find that

$$(2x, 4x^2 - 2)_{w(x)} = 0.$$

Exercises

1. Show that the set of the following functions are orthogonal on the given intervals.
   - $f(x) = \sin^2 x, \ \ g(x) = \cos x, \ \ [0, \pi]$
   - $f(x) = e^x, \ \ g(x) = xe^{-x} - e^{-x}, \ \ [0, 2]$
   - $f(x) = \cos x, \ \ g(x) = e^{-x}, \ \ \left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

2. Show that the set of the following functions are orthogonal with respect to the given weight function on the indicated interval.

$$f(x) = 1 - x, \ \ g(x) = \frac{1}{2}x^2 - 2x + 1; \ \ w(x) = e^{-x}, \ \ [0, \infty)$$

3. Determine the constants λ and δ so that the functions

$$f(x) = \lambda x + 2, \ g(x) = \delta x^2 - 6x + 1 \text{ and } h(x) = x - 1$$

are mutually orthogonal on $[0, 1]$ and then obtain the corresponding orthonormal set.

④ Show that the set of functions

$$\left\{1,\; \cos\frac{n\pi x}{L},\; \sin\frac{n\pi x}{L}\right\};\; m, n = 1, 2, 3, \cdots;\; [0, L]$$

are orthogonal on the given interval and find the norm of each function.

Fourier Series

Theorem

Suppose that f and f' are piecewise continuous on the interval $[-T, T]$. Further, suppose that f is define outside the interval $[-T, T]$ so that it is periodic with period $2R$. Then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right).$$

Whose coefficients are given by

$$a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos \frac{n\pi x}{T} \, dx, \quad (n = 1, 2, \dots),$$

$$b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin \frac{n\pi x}{T} \, dx, \quad (n = 1, 2, \dots), \quad a_0 = \frac{1}{T} \int_{-T}^{T} f(x) \, dx.$$

Even and Odd Functions

Recall that if $f(x)$ is an even function then $f(-x) = +f(x)$.

Examples

$f(x) = x^4 - x^2$, $f(x) = (2-x)^2$ for $0 < x < 2$ and $f(x) = 2$ for $x \in [-a, a]$.

Recall that if $f(x)$ is an odd function then $f(-x) = -f(x)$

Examples

$f(x) = x^3$, $f(x) = x$.

Two symmetry properties of functions will be useful in the study of Fourier series. A function $f(x)$ that satisfies $f(-x) = f(x)$ for all x in the domain of f has a graph that is symmetric with respect to the $y-axis$. This function is said to be even. For example:

$$f(x) = \sqrt{2 + x^4}, g(x) = e^{-|x|},$$
$$h(x) = \cos x + \ln(1 + x^2),$$
$$k(x) = \left\{ \begin{array}{cc} |\sin x|, & |x| \leq \pi \\ 0, & |x| > \pi \end{array} \right. .$$

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A function f that satisfies $f(-x) = -f(x)$ for all x in the domain of f has a graph that is symmetric with respect to the origin. It is said to be an odd function. For example:

$$f(x) = e^{|x|} \sin x,$$

$$h(x) = \sqrt{1+x^2} \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

$$k(x) = \left\{ \begin{array}{cc} x-1, & 0 < x < 1, \\ x+1, & -1 < x < 0, \\ 0, & |x| > 1 \end{array} \right. ,$$

$$M(x) = x^{1/3} - \sin x.$$

Properties of Symmetric Functions

- If $f(x)$ is an even piecewise continuous function on $[-L, L]$, then

$$\int_{-L}^{L} f(x)\ dx = 2 \int_{0}^{L} f(x)\ dx$$

- If $f(x)$ is an odd piecewise continuous function on $[-L, L]$, then

$$\int_{-L}^{L} f(x)\ dx = 0$$

- For an even function, we have the Fourier coefficients

$$a_n = \frac{2}{T} \int_{0}^{T} f(x) \cos \frac{n\pi x}{T}\ dx, \quad (n = 1, 2, \dots),$$

$$a_0 = \frac{2}{T} \int_{0}^{T} f(x)\ dx,$$

and

$$b_n = 0, \quad (n = 1, 2, \dots)$$

- For an odd function, we have the Fourier coefficients

$$b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin \frac{n\pi x}{T} \ dx, \quad (n = 1, 2, \dots),$$

$$a_n = 0, \quad (n = 0, 1, 2, \dots)$$

- When n is an integer

$$\sin n\pi = 0 \quad \text{and} \quad \cos n\pi = (-1)^n.$$

Example (1)

Assume that there is a Fourier series converging to the function

$$f(x) = \begin{cases} -x, & -T \le x < 0 \\ x, & 0 \le x \le T; \end{cases}$$

$$f(x + 2T) = f(x).$$

Compute the Fourier series for the given function.

Solution The Fourier series has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T}).$$

Since $f(-x) = f(x) \ \forall x \in [-T, T]$, then f is even on $[-T, T]$, hence $b_n = 0, \ (n = 1, 2, ...)$.
We compute to find that

$$a_0 = \frac{2}{T} \int_0^T f(x)dx = T,$$

$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(x)\cos\frac{n\pi x}{T}dx, \quad (n=1,2,...), \\
&= \frac{2}{T}\int_0^T x\cos\frac{n\pi x}{T}dx \\
&= \frac{2T}{(n\pi)^2}\left(\cos n\pi - 1\right), \quad (n=1,2,...),
\end{aligned}$$

Thus the Fourier series for the function f is given by

$$f(x) = \frac{T}{2} - \frac{4T}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{T}.$$

Observe that from the obtained Fourier series, we can deduce that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

This follows from the fact that the Fourier series converges to $f(0) = 0$ at $x = 0$.

Example (2)

Find a Fourier series to represent the function

$$f(x) = x - x^2$$

from $x = -\pi$ to $x = \pi$. Deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

We write

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

We have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) dx = \frac{-2}{3} \pi^2,$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx = -\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx \\
&= \frac{4}{n^2}(-1)^{n+1},
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}x\sin nx\,dx \\
&= \frac{2}{n}(-1)^{n+1}.
\end{aligned}
$$

Hence

$$
x-x^2 = \frac{-1}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.
$$

By setting $x=0$, we obtain

$$
\frac{-1}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = 0.
$$

From which it follows that

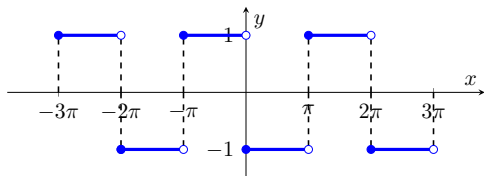$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Example (3)

Let f be 2π-periodic function defined by

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 \le x < \pi. \end{cases}$$

Sketch the graph of f on $[-3\pi, 3\pi]$, find the Fourier series of f, and

deduce the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

Solution We can see that f is an odd function on $(-\pi, \pi)$; $x \ne 0$

Thus,

$$
\begin{aligned}
a_n &= 0, \ n = 0, 1, \cdots \\
b_n &= \frac{-2}{\pi} \int_0^\pi (1) \sin(nx) dx = \frac{x^2}{\pi} \left[\frac{\cos(nx)}{n} \right]_0^\pi = \frac{2}{n\pi} \left((-1)^n - 1 \right),
\end{aligned}
$$

hence, $n = 1, 2, \cdots$

$$
\frac{f(x^+) + f(x^-)}{2} = \sum_{n=1}^\infty \frac{2}{n\pi} \left((-1)^n - 1 \right) \sin(nx), \ -\pi < x < \pi
$$

At $x = \frac{\pi}{2}$, we have

$$
f\left(\frac{\pi}{2} \right) = -1 = \sum_{n=1}^\infty \frac{2}{n\pi} \left((-1)^n - 1 \right) \sin \left(\frac{n\pi}{2} \right)
$$

Note that $\sin \left(\frac{n\pi}{2} \right) = 0$, n is even.

Thus,

$$
\begin{aligned}
-1 &= \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \left((-1)^{2n-1} - 1 \right) \sin\left(\frac{(2n-1)\pi}{2} \right) \\
-1 &= \sum_{n=1}^{\infty} \frac{-4}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi}{2} \right) \\
-1 &= \frac{-4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi}{2} \right) \\
\frac{\pi}{4} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}
\end{aligned}
$$

Exercises

1. Compute the Fourier series for the function

$$f(x) = \begin{cases} \frac{1}{2}\pi + x, & -\pi \le x < 0 \\ \frac{1}{2}\pi - x, & 0 \le x \le \pi; \end{cases}$$

$$f(x + 2T) = f(x) \text{ for all } x \in \mathbb{R}.$$

Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

2. Obtain the Fourier series for the function

$$f(x) = \frac{1}{4}(\pi - x)^2, \ 0 < x < 2\pi,$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Fourier Cosine and Sine Series

Sometimes it is possible to represent a function as a Fourier Cosine or Sine Series.

To do this we use the properties of even and odd functions as defined previously.

To determine a series we usually extend the interval of definition to create a new function that is either even or odd depending on the type of series required.

If we require a Fourier cosine series then the new function created is chosen to be an even function. Similarly, If we require a Fourier sine series then the new function created is chosen to be an odd function.

For example, let $f(x)$ be defined on the interval $[0, L]$.

- If we require a Fourier cosine series then we create a new function created, $f_e(x)$, which is an even function over the interval $[-L, L]$. That is, we let

$$f_e(x) = \left\{ \begin{array}{rl} f(x), & 0 < x < L, \\ f(-x), & -L \leq x \leq 0; \end{array} \right.$$
$$\text{with} \quad f_e(x + 2L) = f_e(x).$$

- If we require a Fourier sine series then we create a new function created, $f_o(x)$, which is an odd function over the interval $[-L, L]$. That is, we let

$$f_o(x) = \left\{ \begin{array}{rl} f(x), & 0 < x < L, \\ -f(-x), & -L < x < 0, \end{array} \right.$$

and extending $f_o(x)$ to all x using the $2L$ periodicity.

Definition

Let $f(x)$ be piecewise continuous function on the interval $[0, L]$.

- The Fourier cosine series of $f(x)$ on $[0, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx, \quad (n = 0, 1, 2, \dots).$$

- The Fourier sine series of $f(x)$ on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

  where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx, \quad (n = 1, 2, \dots).$$

Example (1)

Compute the Fourier sine series for the function

$$f(x) = \cos \frac{\pi x}{3}, \ 0 < x < 3.$$

Solution We extend $f(x)$ as an odd function on $[-3, 3]$

$$f_o(x) = \left\{ \begin{array}{ll} \cos \frac{\pi x}{3}, & 0 \leq x < 3, \\ -\cos \frac{\pi x}{3} & -3 \leq x < 0. \end{array} \right.$$

The Fourier sine series representation of

$$f(x) = \cos \frac{\pi x}{3}$$

is

$$f(x) = \cos \frac{\pi x}{3} = \sum_{n=1}^{\infty} b_n \sin \frac{n x \pi}{3}, \quad 0 < x < 3,$$

where

$$
\begin{aligned}
b_n &= \frac{2}{3} \int_0^3 \cos \frac{\pi x}{3} \sin \frac{n \pi x}{3} dx \\
&= \frac{1}{3} \int_0^3 \left(\sin \frac{(n+1)\pi x}{3} - \sin \frac{(n-1)\pi x}{3} \right) dx \\
&= \begin{cases} 0, & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)}, & n \text{ even} \end{cases}
\end{aligned}
$$

According to Fourier theorem, equality holds for $0 < x < 3$, but not at $x = 0$ and $x = 3$:

$$
\cos \frac{\pi x}{3} = \frac{8}{\pi} \int_{n=1}^{\infty} \frac{n}{(4n^2 - 1)} \sin \frac{2n x \pi}{3}, \quad 0 < x < 3.
$$

At $x = 0$ and $x = 3$, the Fourier series converges to

$$
\frac{f(0^+) + f(0^-)}{2} = 0
$$

and

$$\frac{f(3^+) + f(3^-)}{2} = 0,$$

respectively.

Example (2)

Compute the Fourier cosine series for the function

$$f(x) = e^{2x}, \ 0 \leq x \leq 1.$$

and deduce that

$$\frac{2}{e^2 - 1} = \sum_{n=1}^{\infty} \frac{4}{4 + n^2\pi^2} \left[e^2(-1)^n - 1 \right]$$

Solution We extend $f(x)$ as an even function on $[-1, 1]$

$$f_e(x) = \left\{ \begin{array}{ll} e^{2x}, & 0 < x < 1, \\ e^{-2x} & -1 < x < 0. \end{array} \right.$$

The Fourier cosine series representation of

$$f(x) = e^{2x},$$

is

$$f(x) = e^{2x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x, \quad 0 \le x \le 1,$$

where

$$a_0 = 2 \int_0^1 e^{2x} dx = e^2 - 1,$$

$$
\begin{aligned}
a_n &= 2 \int_0^1 e^{2x} \cos n\pi x \, dx \\
&= 2 \left[\frac{1}{2} e^{2x} \cos n\pi x \Big|_0^1 + \frac{1}{2} n\pi \int_0^1 e^{2x} \sin n\pi x \, dx \right] \\
&= e^2 (-1)^n - 1 + n\pi \left[\frac{1}{2} n\pi e^{2x} \sin n\pi x \Big|_0^1 - \frac{1}{2} n\pi \int_0^1 e^{2x} \cos n\pi x \, dx \right] \\
&= e^2 (-1)^n - 1 - \frac{1}{2} n^2 \pi^2 \int_0^1 e^{2x} \cos n\pi x \, dx.
\end{aligned}
$$

Hence

$$a_n = \frac{4}{4 + n^2\pi^2} \left[e^2(-1)^n - 1 \right]$$

The Fourier series is then

$$e^{2x} = \frac{e^2 - 1}{2} + \sum_{n=1}^{\infty} \frac{4}{4 + n^2\pi^2} \left[e^2(-1)^n - 1 \right] \cos n\pi x, \quad 0 \le x \le 1.$$

At $x = 0$, we have

$$\frac{1 - e^2}{2} = \sum_{n=1}^{\infty} \frac{4}{4 + n^2\pi^2} \left[e^2(-1)^n - 1 \right].$$

Exercises

Test whether each of the following functions is odd or even, then expand it in a cosine or sine series.

- $f(x) = |\cos x|, \ |x| < \pi$.
- $f(x) = x \cos x, \ |x| < \pi$.
- $f(x) = x^2 |x|, \ |x| < 1$.
- $M(x) = \left\{ \begin{array}{ll} x + 1, & -3 < x < 0 \\ -x + 1, & 0 \leq x < 3. \end{array} \right.$

Complex form of a Fourier Series

We have seen that Fourier Series in the interval $(-T, T)$ of a functon $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right).$$

Thus, from The Euler's formula we have the complex form of Fourier Series of f is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{T}},$$

where

$$c_n = \frac{1}{2T} \int_{-T}^{T} f(x) e^{\frac{in\pi x}{T}} \ dx.$$

Example

Obtain the complex form of the Fourier series for the function $f(x) = e^{\lambda x}$ $-\pi < x < \pi$ in the form

$$e^{\lambda x} = \frac{\sinh \lambda \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{\lambda + in}{\lambda^2 + n^2} e^{inx},$$

and deduce that

$$\frac{\pi}{\lambda \sinh \lambda \pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\lambda^2 + n^2}.$$

Solution We look for the coefficients c_n in the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$,

$$
\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\lambda x} e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(\lambda - in)x} dx \\
&= \frac{1}{2\pi} \left[\frac{e^{(\lambda - in)\pi} - e^{-(\lambda - in)\pi}}{\lambda - in} \right] \\
&= \frac{1}{2\pi} \left[\frac{e^{\lambda\pi}(\cos n\pi - i\sin n\pi) - e^{-\lambda\pi}(\cos n\pi + i\sin n\pi)}{\lambda - in} \right] \\
&= \frac{1}{2\pi(\lambda - in)} \left(e^{\lambda\pi} - e^{-\lambda\pi} \right) \cos n\pi \\
&= \frac{1}{2\pi(\lambda - in)} (e^{\lambda\pi} - e^{-\lambda\pi}) \cos n\pi \\
&= \frac{1}{2\pi(\lambda - in)} (2\sinh \lambda\pi) \cos n\pi \\
&= \frac{(-1)^n \sinh \lambda\pi}{\pi(\lambda - in)} = \frac{(-1)^n (\lambda + in)\sinh \lambda\pi}{\pi(\lambda^2 + n^2)}.
\end{aligned}
$$

Substituting this found c_n in the series to get

$$f(x) = e^{\lambda x} = \frac{\sinh \lambda \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (\lambda + in)}{\lambda^2 + n^2} e^{inx}. \qquad (1)$$

Now by setting $x = 0$ in (1), we obtain

$$\frac{\pi}{\sinh \lambda \pi} = \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{\lambda}{\lambda^2 + n^2} + i \frac{n}{\lambda^2 + n^2} \right).$$

By equating the real part, we have

$$\frac{\pi}{\lambda \sinh \lambda \pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\lambda^2 + n^2}.$$

Exercises

Find the complex form of the Fourier series for:

- $f(x) = e^x, \ |x| < \pi$.
- $f(x) = e^{-x}, \ |x| < 1$.
- $f(x) = \cos(\theta x), \ |x| < \pi$.