6.6 De Moivre's Theorem and *n*th Roots

What you'll learn about

- The Complex Plane
- Trigonometric Form of Complex Numbers
- Multiplication and Division of Complex Numbers
- Powers of Complex Numbers
- Roots of Complex Numbers

… and why

This material extends your equation-solving technique to include equations of the form $z^n = c$, n an integer and c a complex number.

The Complex Plane

You might be curious as to why we reviewed complex numbers in Section P.6, then proceeded to ignore them for the next six chapters. (Indeed, after this section we will pretty much ignore them again.) The reason is simply because the key to understanding calculus is the graphing of functions in the Cartesian plane, which consists of two perpendicular real (not complex) lines.

We are not saying that complex numbers are impossible to graph. Just as every real number is associated with a point of the real number line, every complex number can be associated with a point of the **complex plane**. This idea evolved through the work of Caspar Wessel (1745–1818), Jean-Robert Argand (1768–1822), and Carl Friedrich Gauss (1777–1855). Real numbers are placed along the horizontal axis (the **real axis**) and imaginary numbers along the vertical axis (the **imaginary axis**), thus associating the complex number $a + bi$ with the point (a, b). In Figure 6.57 we show the graph of $2 + 3i$ as an example.

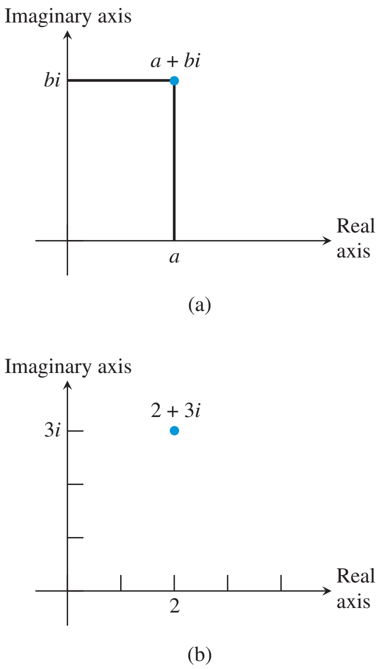


FIGURE 6.57 Plotting points in the complex plane.

EXAMPLE 1 Plotting Complex Numbers

Plot $u = 1 + 3i$, $v = 2 - i$, and $u + v$ in the complex plane. These three points and the origin determine a quadrilateral. Is it a parallelogram?

SOLUTION First notice that $u + v = (1 + 3i) + (2 + i) = 3 + 2i$. The numbers u, v, and $u + v$ are plotted in Figure 6.58a. The quadrilateral is a parallelogram because the arithmetic is exactly the same as in vector addition (Figure 6.58b).

Now try Exercise 1.

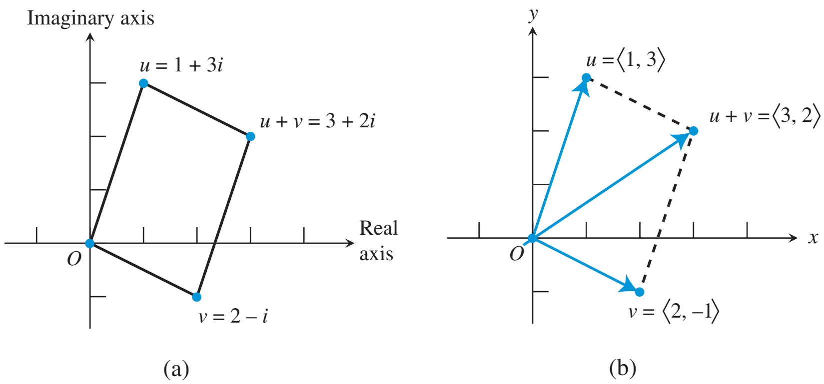


FIGURE 6.58 (a) Two numbers and their sum are plotted in the complex plane. (b) The arithmetic is the same as in vector addition. (Example 1)

Is There a Calculus of Complex Functions?

There is a calculus of complex functions. If you study it someday, it should only be after acquiring a pretty firm algebraic and geometric understanding of the calculus of real functions.

Example 1 shows how the complex plane representation of complex number addition is virtually the same as the Cartesian plane representation of vector addition. Another similarity between complex numbers and two-dimensional vectors is the definition of absolute value.

DEFINITION Absolute Value (Modulus) of a Complex Number

The **absolute value** or **modulus** of a complex number $z = a + bi$ is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

In the complex plane, $|a + bi|$ is the distance of $a + bi$ from the origin.

Trigonometric Form of Complex Numbers

Figure 6.59 shows the graph of $z = a + bi$ in the complex plane. The distance r from the origin is the modulus of z. If we define a direction angle θ for z just as we did with vectors, we see that $a = r \cos \theta$ and $b = r \sin \theta$. Substituting these expressions for a and b gives us the **trigonometric form** (or **polar form**) of the complex number z.

Polar Form

What's in a cis?

Trigonometric (or polar) form appears frequently enough in scientific texts to have an abbreviated form. The expression "$\cos \theta + i \sin \theta$" is often shortened to "cis θ" (pronounced "kiss θ"). Thus $z = r \text{ cis } \theta$.

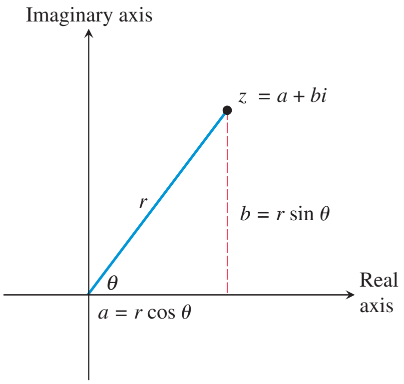


FIGURE 6.59 If r is the distance of $z = a + bi$ from the origin and θ is the directional angle shown, then $z = r(\cos \theta + i \sin \theta)$, which is the trigonometric form of z.

DEFINITION Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is

$$z = r(\cos \theta + i \sin \theta)$$

where $a = r \cos \theta$, $b = r \sin \theta$, $r = \sqrt{a^2 + b^2}$, and $\tan \theta = b/a$. The number r is the *absolute value* or *modulus* of z, and θ is an **argument** of z.

An angle θ for the trigonometric form of z can always be chosen so that $0 \leq \theta \leq 2\pi$, although any angle coterminal with θ could be used. Consequently, the *angle* θ and *argument* of a complex number z are not unique. It follows that the trigonometric form of a complex number z is not unique.

EXAMPLE 2 Finding Trigonometric Forms

Use an algebraic method to find the trigonometric form with $0 \leq \theta < 2\pi$ for the complex number. Approximate exact values with a calculator when appropriate.

(a) $1 - \sqrt{3}i$ **(b)** $-3 - 4i$

SOLUTION

(a) For $1 - \sqrt{3}i$,

$$r = |1 - \sqrt{3}i| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2.$$

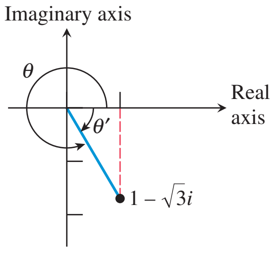


FIGURE 6.60 The complex number for Example 2a.

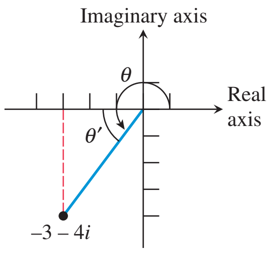


FIGURE 6.61 The complex number for Example 2b.

Because the reference angle θ' for θ is $-\pi/3$ (Figure 6.60),

$$\theta = 2\pi + \left(-\frac{\pi}{3}\right) = \frac{5\pi}{3}.$$

Thus,

$$1 - \sqrt{3}i = 2\cos\frac{5\pi}{3} + 2i\sin\frac{5\pi}{3}.$$

(b) For $-3 - 4i$,

$$|-3 - 4i| = \sqrt{(-3)^2 + (-4)^2} = 5.$$

The reference angle θ' for θ (Figure 6.61) satisfies the equation

$$\tan\theta' = \frac{4}{3}, \quad \text{so}$$

$$\theta' = \tan^{-1}\frac{4}{3} = 0.927....$$

Because the terminal side of θ is in the third quadrant, we conclude that

$$\theta = \pi + \theta' \approx 4.07.$$

Therefore,

$$-3 - 4i \approx 5(\cos 4.07 + i\sin 4.07).$$

Now try Exercise 5.

Multiplication and Division of Complex Numbers

The trigonometric form for complex numbers is particularly convenient for multiplying and dividing complex numbers. The product involves the product of the moduli and the sum of the arguments. (*Moduli* is the plural of *modulus*.) The quotient involves the quotient of the moduli and the difference of the arguments.

Product and Quotient of Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

1. $z_1 \cdot z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.

2. $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \quad r_2 \neq 0.$

Proof of the Product Formula

$$\begin{aligned} z_1 \cdot z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

You will be asked to prove the quotient formula in Exercise 63.

EXAMPLE 3 Multiplying Complex Numbers

Use an algebraic method to express the product of z_1 and z_2 in standard form. Approximate exact values with a calculator when appropriate.

$$z_1 = 25\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right), \quad z_2 = 14\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

SOLUTION

$$\begin{aligned} z_1 \cdot z_2 &= 25\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right) \cdot 14\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\ &= 25 \cdot 14\sqrt{2}\left[\cos\left(\frac{-\pi}{4} + \frac{\pi}{3}\right) + i\sin\left(\frac{-\pi}{4} + \frac{\pi}{3}\right)\right] \\ &= 350\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \\ &\approx 478.11 + 128.11i \end{aligned}$$

Now try Exercise 19.

EXAMPLE 4 Dividing Complex Numbers

Use an algebraic method to express the product z_1/z_2 in standard form. Approximate exact values with a calculator when appropriate.

$$z_1 = 2\sqrt{2}(\cos 135° + i\sin 135°), \quad z_2 = 6(\cos 300° + i\sin 300°)$$

SOLUTION

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2\sqrt{2}\,(\cos 135° + i\sin 135°)}{6(\cos 300° + i\sin 300°)} \\ &= \frac{\sqrt{2}}{3}[\cos(135° - 300°) + i\sin(135° - 300°)] \\ &= \frac{\sqrt{2}}{3}[\cos(-165°) + i\sin(-165°)] \\ &\approx -0.46 - 0.12i \end{aligned}$$

Now try Exercise 23.

Powers of Complex Numbers

We can use the product formula to raise a complex number to a power. For example, let $z = r(\cos\theta + i\sin\theta)$. Then

$$\begin{aligned} z^2 &= z \cdot z \\ &= r(\cos\theta + i\sin\theta) \cdot r(\cos\theta + i\sin\theta) \\ &= r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)] \\ &= r^2(\cos 2\theta + i\sin 2\theta) \end{aligned}$$

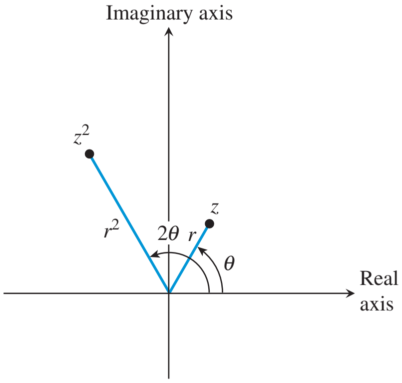


FIGURE 6.62 A geometric interpretation of z^2.

Figure 6.62 gives a geometric interpretation of squaring a complex number: Its argument is doubled and its distance from the origin is multiplied by a factor of r, increased if $r > 1$ or decreased if $r < 1$.

We can find z^3 by multiplying z by z^2:

$$\begin{aligned} z^3 &= z \cdot z^2 \\ &= r(\cos\theta + i\sin\theta) \cdot r^2(\cos 2\theta + i\sin 2\theta) \\ &= r^3[\cos(\theta + 2\theta) + i\sin(\theta + 2\theta)] \\ &= r^3(\cos 3\theta + i\sin 3\theta) \end{aligned}$$

Similarly,

$$z^4 = r^4(\cos 4\theta + i \sin 4\theta)$$
$$z^5 = r^5(\cos 5\theta + i \sin 5\theta)$$
$$\vdots$$

This pattern can be generalized to the following theorem, named after the mathematician Abraham De Moivre (1667–1754), who also made major contributions to the field of probability.

De Moivre's Theorem

Let $z = r(\cos \theta + i \sin \theta)$ and let n be a positive integer. Then

$$z^n = [r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta).$$

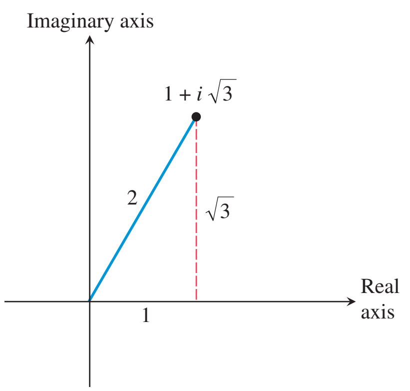


FIGURE 6.63 The complex number in Example 5.

EXAMPLE 5 Using De Moivre's Theorem

Find $(1 + i\sqrt{3})^3$ using De Moivre's Theorem.

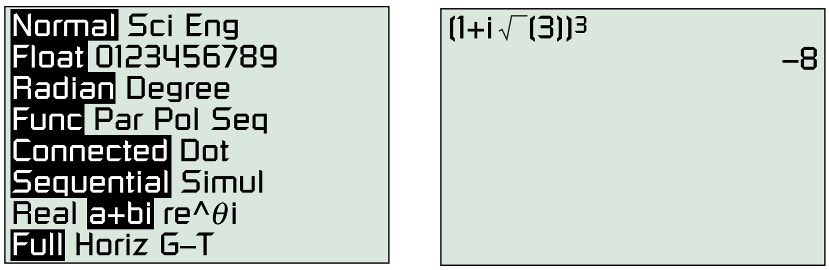
SOLUTION

Solve Algebraically See Figure 6.63. The argument of $z = 1 + i\sqrt{3}$ is $\theta = \pi/3$, and its modulus is $|1 + i\sqrt{3}| = \sqrt{1 + 3} = 2$. Therefore,

$$z = 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$$
$$z^3 = 2^3\left[\cos\left(3 \cdot \frac{\pi}{3}\right) + i \sin\left(3 \cdot \frac{\pi}{3}\right)\right]$$
$$= 8(\cos \pi + i \sin \pi)$$
$$= 8(-1 + 0i) = -8$$

Support Numerically Figure 6.64a sets the graphing calculator we use in complex number mode. Figure 6.64b supports the result obtained algebraically.

Now try Exercise 31.



(a) (b)

FIGURE 6.64 (a) Setting a graphing calculator in complex number mode. (b) Computing $(1 + i\sqrt{3})^3$ with a graphing calculator.

EXAMPLE 6 Using De Moivre's Theorem

Find $[(-\sqrt{2}/2) + i(\sqrt{2}/2)]^8$ using De Moivre's Theorem.

SOLUTION The argument of $z = (-\sqrt{2}/2) + i(\sqrt{2}/2)$ is $\theta = 3\pi/4$, and its modulus is

$$\left|\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1.$$

(continued)

Therefore,

$$z = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$
$$z^8 = \cos\left(8\cdot\frac{3\pi}{4}\right) + i\sin\left(8\cdot\frac{3\pi}{4}\right)$$
$$= \cos 6\pi + i\sin 6\pi$$
$$= 1 + i\cdot 0 = 1$$

Now try Exercise 35.

Roots of Complex Numbers

The complex number $1 + i\sqrt{3}$ in Example 5 is a solution of $z^3 = -8$, and the complex number $(-\sqrt{2}/2) + i(\sqrt{2}/2)$ in Example 6 is a solution of $z^8 = 1$. The complex number $1 + i\sqrt{3}$ is a third root of -8, and $(-\sqrt{2}/2) + i(\sqrt{2}/2)$ is an eighth root of 1.

*n*th Root of a Complex Number

A complex number $v = a + bi$ is an ***n*th root of z** if

$$v^n = z.$$

If $z = 1$, then v is an ***n*th root of unity**.

We use De Moivre's Theorem to develop a general formula for finding the nth roots of a nonzero complex number. Suppose that $v = s(\cos\alpha + i\sin\alpha)$ is an nth root of $z = r(\cos\theta + i\sin\theta)$. Then

$$v^n = z$$
$$[s(\cos\alpha + i\sin\alpha)]^n = r(\cos\theta + i\sin\theta)$$
$$s^n(\cos n\alpha + i\sin n\alpha) = r(\cos\theta + i\sin\theta) \quad (1)$$

Next, we take the absolute value of both sides:

$$|s^n(\cos n\alpha + i\sin n\alpha)| = |r(\cos\theta + i\sin\theta)|$$
$$\sqrt{s^{2n}(\cos^2 n\alpha + \sin^2 n\alpha)} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)}$$
$$\sqrt{s^{2n}} = \sqrt{r^2}$$
$$s^n = r \qquad s > 0, r > 0$$
$$s = \sqrt[n]{r}$$

Substituting $s^n = r$ into Equation (1), we obtain

$$\cos n\alpha + i\sin n\alpha = \cos\theta + i\sin\theta.$$

Therefore, $n\alpha$ can be any angle coterminal with θ. Consequently, for any integer k, v is an nth root of z if $s = \sqrt[n]{r}$ and

$$n\alpha = \theta + 2\pi k$$
$$\alpha = \frac{\theta + 2\pi k}{n}.$$

The expression for v takes on n different values for $k = 0, 1, \ldots, n - 1$, and the values start to repeat for $k = n, n + 1, \ldots$.

We summarize this result.

Finding *n*th Roots of a Complex Number

If $z = r(\cos\theta + i\sin\theta)$, then the n distinct complex numbers

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right),$$

where $k = 0, 1, 2, \ldots, n - 1$, are the nth roots of the complex number z.

EXAMPLE 7 Finding Fourth Roots

Find the fourth roots of $z = 5(\cos(\pi/3) + i\sin(\pi/3))$.

SOLUTION The fourth roots of z are the complex numbers

$$\sqrt[4]{5}\left(\cos\frac{\pi/3 + 2\pi k}{4} + i\sin\frac{\pi/3 + 2\pi k}{4}\right)$$

for $k = 0, 1, 2, 3$.

Taking into account that $(\pi/3 + 2\pi k)/4 = \pi/12 + \pi k/2$, the list becomes

$$\begin{aligned}
z_1 &= \sqrt[4]{5}\left[\cos\left(\frac{\pi}{12} + \frac{0}{2}\right) + i\sin\left(\frac{\pi}{12} + \frac{0}{2}\right)\right]\\
&= \sqrt[4]{5}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]\\
z_2 &= \sqrt[4]{5}\left[\cos\left(\frac{\pi}{12} + \frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{12} + \frac{\pi}{2}\right)\right]\\
&= \sqrt[4]{5}\left[\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right]\\
z_3 &= \sqrt[4]{5}\left[\cos\left(\frac{\pi}{12} + \frac{2\pi}{2}\right) + i\sin\left(\frac{\pi}{12} + \frac{2\pi}{2}\right)\right]\\
&= \sqrt[4]{5}\left[\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right]\\
z_4 &= \sqrt[4]{5}\left[\cos\left(\frac{\pi}{12} + \frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{12} + \frac{3\pi}{2}\right)\right]\\
&= \sqrt[4]{5}\left[\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right]
\end{aligned}$$

Now try Exercise 45.

EXAMPLE 8 Finding Cube Roots

Find the cube roots of -1 and plot them.

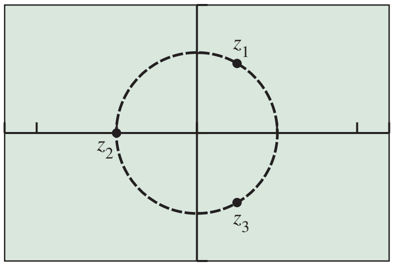
SOLUTION First we write the complex number $z = -1$ in trigonometric form

$$z = -1 + 0i = \cos\pi + i\sin\pi.$$

The third roots of $z = -1 = \cos\pi + i\sin\pi$ are the complex numbers

$$\cos\frac{\pi + 2\pi k}{3} + i\sin\frac{\pi + 2\pi k}{3},$$

(continued)



[–2.4, 2.4] by [–1.6, 1.6]

FIGURE 6.65 The three cube roots z_1, z_2, and z_3 of -1 displayed on the unit circle (dashed). (Example 8)

for $k = 0, 1, 2$. The three complex numbers are

$$z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

$$z_2 = \cos\frac{\pi + 2\pi}{3} + i\sin\frac{\pi + 2\pi}{3} = -1 + 0i,$$

$$z_3 = \cos\frac{\pi + 4\pi}{3} + i\sin\frac{\pi + 4\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Figure 6.65 shows the graph of the three cube roots z_1, z_2, and z_3. They are evenly spaced (with distance of $2\pi/3$ radians) around the unit circle.

Now try Exercise 57.

EXAMPLE 9 Finding Roots of Unity

Find the eight eighth roots of unity.

SOLUTION First we write the complex number $z = 1$ in trigonometric form

$$z = 1 + 0i = \cos 0 + i\sin 0.$$

The eighth roots of $z = 1 + 0i = \cos 0 + i\sin 0$ are the complex numbers

$$\cos\frac{0 + 2\pi k}{8} + i\sin\frac{0 + 2\pi k}{8},$$

for $k = 0, 1, 2, \ldots, 7$.

$$z_1 = \cos 0 + i\sin 0 = 1 + 0i$$

$$z_2 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$z_3 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i$$

$$z_4 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$z_5 = \cos\pi + i\sin\pi = -1 + 0i$$

$$z_6 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

$$z_7 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 - i$$

$$z_8 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

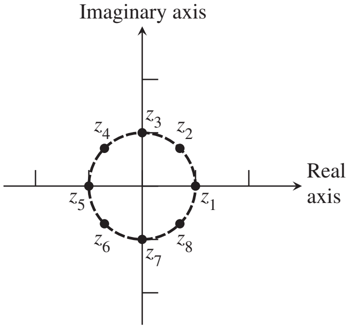


FIGURE 6.66 The eight eighth roots of unity are evenly spaced on a unit circle. (Example 9)

Figure 6.66 shows the eight points. They are spaced $2\pi/8 = \pi/4$ radians apart.

Now try Exercise 59.

QUICK REVIEW 6.6 *(For help, go to Sections P.5, P.6, and 4.3.)*

In Exercises 1 and 2, write the roots of the equation in $a + bi$ form.

1. $x^2 + 13 = 4x$

2. $5(x^2 + 1) = 6x$

In Exercises 3 and 4, write the complex number in standard form $a + bi$.

3. $(1 + i)^5$

4. $(1 - i)^4$

In Exercises 5–8, find an angle θ in $0 \le \theta < 2\pi$ that satisfies both equations.

5. $\sin\theta = \dfrac{1}{2}$ and $\cos\theta = -\dfrac{\sqrt{3}}{2}$

6. $\sin\theta = -\dfrac{\sqrt{2}}{2}$ and $\cos\theta = \dfrac{\sqrt{2}}{2}$

7. $\sin\theta = -\dfrac{\sqrt{3}}{2}$ and $\cos\theta = -\dfrac{1}{2}$

8. $\sin\theta = -\dfrac{\sqrt{2}}{2}$ and $\cos\theta = -\dfrac{\sqrt{2}}{2}$

In Exercises 9 and 10, find all real solutions.

9. $x^3 - 1 = 0$

10. $x^4 - 1 = 0$

SECTION 6.6 EXERCISES

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1 and 2, plot all four points in the same complex plan.

1. $1 + 2i$, $3 - i$, $-2 + 2i$, i

2. $2 - 3i$, $1 + i$, 3, $-2 - i$

In Exercises 3–12, find the trigonometric form of the complex number where the argument satisfies $0 \le \theta < 2\pi$.

3. $3i$

4. $-2i$

5. $2 + 2i$

6. $\sqrt{3} + i$

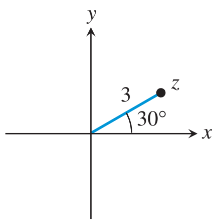
7. $-2 + 2i\sqrt{3}$

8. $3 - 3i$

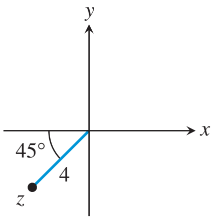
9. $3 + 2i$

10. $4 - 7i$

11.



12.



In Exercises 13–18, write the complex number in standard form $a + bi$.

13. $3(\cos 30° - i \sin 30°)$

14. $8(\cos 210° + i \sin 210°)$

15. $5[\cos(-60°) + i \sin(-60°)]$

16. $5\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

17. $\sqrt{2}\left(\cos\dfrac{7\pi}{6} + i\sin\dfrac{7\pi}{6}\right)$

18. $\sqrt{7}\left(\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right)$

In Exercises 19–22, find the product of z_1 and z_2. Leave the answer in trigonometric form.

19. $z_1 = 7(\cos 25° + i \sin 25°)$
$z_2 = 2(\cos 130° + i \sin 130°)$

20. $z_1 = \sqrt{2}(\cos 118° + i \sin 118°)$
$z_2 = 0.5[\cos(-19°) + i \sin(-19°)]$

21. $z_1 = 5\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$ $z_2 = 3\left(\cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}\right)$

22. $z_1 = \sqrt{3}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$ $z_2 = \dfrac{1}{3}\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$

In Exercises 23–26, find the trigonometric form of the quotient.

23. $\dfrac{2(\cos 30° + i \sin 30°)}{3(\cos 60° + i \sin 60°)}$

24. $\dfrac{5(\cos 220° + i \sin 220°)}{2(\cos 115° + i \sin 115°)}$

25. $\dfrac{6(\cos 5\pi + i \sin 5\pi)}{3(\cos 2\pi + i \sin 2\pi)}$

26. $\dfrac{\cos(\pi/2) + i \sin(\pi/2)}{\cos(\pi/4) + i \sin(\pi/4)}$

In Exercises 27–30, find the product $z_1 \cdot z_2$ and quotient z_1/z_2 in two ways, **(a)** using the trigonometric form for z_1 and z_2 and **(b)** using the standard form for z_1 and z_2.

27. $z_1 = 3 - 2i$ and $z_2 = 1 + i$

28. $z_1 = 1 - i$ and $z_2 = \sqrt{3} + i$

29. $z_1 = 3 + i$ and $z_2 = 5 - 3i$

30. $z_1 = 2 - 3i$ and $z_2 = 1 - \sqrt{3}i$

In Exercises 31–38, use De Moivre's Theorem to find the indicated power of the complex number. Write your answer in standard form $a + bi$ and support with a calculator.

31. $\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)^3$

32. $\left[3\left(\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}\right)\right]^5$

33. $\left[2\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)\right]^3$

34. $\left[6\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)\right]^4$

35. $(1 + i)^5$

36. $(3 + 4i)^{20}$

37. $(1 - \sqrt{3}i)^3$ **38.** $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3$

Use an algebraic method in Exercises 39–44 to find the cube roots of the complex number. Approximate exact solution values when appropriate.

39. $2(\cos 2\pi + i \sin 2\pi)$ **40.** $2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$

41. $3\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$ **42.** $27\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$

43. $3 - 4i$ **44.** $-2 + 2i$

In Exercises 45–50, find the fifth roots of the complex number.

45. $\cos \pi + i \sin \pi$ **46.** $32\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$

47. $2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$ **48.** $2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$

49. $2i$ **50.** $1 + \sqrt{3}\,i$

In Exercises 51–56, find the nth roots of the complex number for the specified value of n.

51. $1 + i, \quad n = 4$ **52.** $1 - i, \quad n = 6$

53. $2 + 2i, \quad n = 3$ **54.** $-2 + 2i, \quad n = 4$

55. $-2i, \quad n = 6$ **56.** $32, \quad n = 5$

In Exercises 57–60, express the roots of unity in standard form $a + bi$. Graph each root in the complex plane.

57. Cube roots of unity **58.** Fourth roots of unity

59. Sixth roots of unity **60.** Square roots of unity

61. Determine z and the three cube roots of z if one cube root of z is $1 + \sqrt{3}i$.

62. Determine z and the four fourth roots of z if one fourth root of z is $-2 - 2i$.

63. Quotient Formula Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, $r_2 \neq 0$. Verify that $z_1/z_2 = r_1/r_2[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$.

64. Group Activity *n*th Roots Show that the nth roots of the complex number $r(\cos \theta + i \sin \theta)$ are spaced $2\pi/n$ radians apart on a circle with radius $\sqrt[n]{r}$.

Standardized Test Questions

65. True or False The trigonometric form of a complex number is unique. Justify your answer.

66. True or False The complex number i is a cube root of $-i$. Justify your answer.

In Exercises 67–70, do not use technology to solve the problem.

67. Multiple Choice Which of the following is a trigonometric form of the complex number $-1 + \sqrt{3}i$?

(A) $2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$ **(B)** $2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$

(C) $2\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$ **(D)** $2\left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right)$

(E) $2\left(\cos \frac{7\pi}{3} + i \sin \frac{7\pi}{3}\right)$

68. Multiple Choice Which of the following is the number of distinct complex number solutions of $z^5 = 1 + i$?

(A) 0 **(B)** 1 **(C)** 3 **(D)** 4 **(E)** 5

69. Multiple Choice Which of the following is the standard form for the product of $\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$ and $\sqrt{2}\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$?

(A) 2 **(B)** -2 **(C)** $-2i$ **(D)** $-1 + i$ **(E)** $1 - i$

70. Multiple Choice Which of the following is not a fourth root of 1?

(A) i^2 **(B)** $-i^2$ **(C)** $\sqrt{-1}$ **(D)** $-\sqrt{-1}$ **(E)** $\sqrt{i}$

Explorations

71. Complex Conjugates The complex conjugate of $z = a + bi$ is $\bar{z} = a - bi$. Let $z = r(\cos \theta + i \sin \theta)$.

(a) Prove that $\bar{z} = r[\cos(-\theta) + i \sin(-\theta)]$.

(b) Use the trigonometric form to find $z \cdot \bar{z}$.

(c) Use the trigonometric form to find $z/\bar{z}$, if $\bar{z} \neq 0$.

(d) Prove that $-z = r[\cos(\theta + \pi) + i \sin(\theta + \pi)]$.

72. Modulus of Complex Numbers Let $z = r(\cos \theta + i \sin \theta)$.

(a) Prove that $|z| = |r|$.

(b) Use the trigonometric form for the complex numbers z_1 and z_2 to prove that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

Extending the Ideas

73. Using Polar Form on a Graphing Calculator The complex number $r(\cos \theta + i \sin \theta)$ can be entered in polar form on some graphing calculators as $re^{i\theta}$.

(a) Support the result of Example 3 by entering the complex numbers z_1 and z_2 in polar form on your graphing calculator and computing the product with your graphing calculator.

(b) Support the result of Example 4 by entering the complex numbers z_1 and z_2 in polar form on your graphing calculator and computing the quotient with your graphing calculator.

(c) Support the result of Example 5 by entering the complex number in polar form on your graphing calculator and computing the power with your graphing calculator.

74. Visualizing Roots of Unity Set your graphing calculator in parametric mode with $0 \le T \le 8$, Tstep $= 1$, Xmin $= -2.4$, Xmax $= 2.4$, Ymin $= -1.6$, and Ymax $= 1.6$.

(a) Let $x = \cos((2\pi/8)t)$ and $y = \sin((2\pi/8)t)$. Use TRACE to visualize the eight eighth roots of unity. We say that $2\pi/8$ *generates* the eighth roots of unity. (Try both dot mode and connected mode.)

(b) Replace $2\pi/8$ in part (a) by the arguments of other eighth roots of unity. Do any others *generate* the eighth roots of unity?

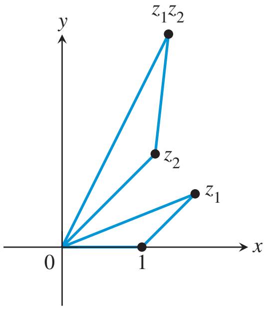
(c) Repeat parts (a) and (b) for the fifth, sixth, and seventh roots of unity, using appropriate functions for x and y.

(d) What would you conjecture about an nth root of unity that generates all the nth roots of unity in the sense of part (a)?

75. Parametric Graphing Write parametric equations that represent $(\sqrt{2} + i)^n$ for $n = t$. Draw and label an *accurate* spiral representing $(\sqrt{2} + i)^n$ for $n = 0, 1, 2, 3, 4$.

76. Parametric Graphing Write parametric equations that represent $(-1 + i)^n$ for $n = t$. Draw and label an *accurate* spiral representing $(-1 + i)^n$ for $n = 0, 1, 2, 3, 4$.

77. Explain why the triangles formed by 0, 1, and z_1, and by 0, z_2, and $z_1 z_2$ shown in the figure are similar triangles.



78. Compass and Straightedge Construction Using only a compass and straightedge, construct the location of $z_1 z_2$ given the location of 0, 1, z_1, and z_2.

In Exercises 79–84, find all solutions of the equation (real and complex).

79. $x^3 - 1 = 0$

80. $x^4 - 1 = 0$

81. $x^3 + 1 = 0$

82. $x^4 + 1 = 0$

83. $x^5 + 1 = 0$

84. $x^5 - 1 = 0$

CHAPTER 6 Key Ideas

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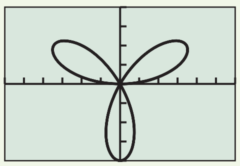
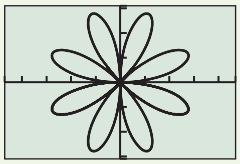
Gallery of Functions

Rose Curves: $r = a \cos n\theta$ and $r = a \sin n\theta$



[–6,6] by [–4, 4]

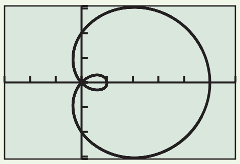
$r = 4 \sin 3\theta$



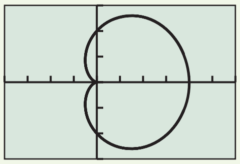
[–4.7, 4.7] by [–3.1, 3.1]

$r = 3 \sin 4\theta$

Limaçon Curves: $r = a \pm b \sin \theta$ and $r = a \pm b \cos \theta$ with $a > 0$ and $b > 0$



Limaçon with an inner loop: $\dfrac{a}{b} < 1$



Cardioid: $\dfrac{a}{b} = 1$