Eigen values and Eigen vectors

Def : Let $T : V \longrightarrow V$ be a Linear transformation. The vector $0 \neq v \in V$ is the eigen vector of the eigen value $\lambda \in \mathbb{R}$ if $T(v) = \lambda . v$

Example : Let $I : V \longrightarrow V$ the identity Linear transformation then $\lambda = 1$ is an eigen value and V is the set of all eigen vectors of $\lambda = 1$ because $I(v) = v = 1 . v \quad \forall v \in V$.

Remark : ① The eigen vectors are related to the eigen values.

② There are some Linear transformations without eigen values.

How to find Eigen values and Eigen vectors ?

Let $T : V \longrightarrow V$ be a Linear transformation.

STEP 1 : write standered matrix A of T

STEP 2 : write the characterstic Polynomial $\Delta = |A - \lambda I|$ where I is unite matrix

STEP 3 : Put $\Delta = 0$, and find the eigen values (λ)

STEP 4 : For every eigen value λ, we have to find the space of Eigen vectors ~~through~~ which is the solution $(A - \lambda I) X = 0$

X : column of variables

Ex 1 Let $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$. Find the eigen values and the Eigen vectors space of A.

Solution **STEP 1** : To find Eigen values, Put $\Delta = 0$

$$\Leftrightarrow \left| \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Leftrightarrow \begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda^2 + 3\lambda + 2 = 0$$

$$\Leftrightarrow (\lambda+2)(\lambda+1) = 0$$

$$\Leftrightarrow \lambda = -2 \text{ or } \lambda = -1$$

So, the Eigen values $\lambda = -2$ and $\lambda = -1$

STEP 2 to find the Eigen space of $\lambda = -1$:-

Put $(A + I)X = 0$

$$\Leftrightarrow \left(\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} 3 & -12 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

we will use Jauss,

$$\left[\begin{array}{cc|c} 3 & -12 & 0 \\ 1 & 4 & 0 \end{array}\right]$$

Then we have (after Ellimination) :

$$x_1 = t \ , \ x_2 = \frac{t}{4}$$

So, the space of Eigen vector of $\lambda = -1$ is $\left\{ \begin{bmatrix} t \\ \frac{t}{4} \end{bmatrix} ; t \in \mathbb{R} \right\}$.

The basis is $S = \left\{ \begin{bmatrix} 1 \\ 1/4 \end{bmatrix} \right\}$

STEP 3 we will find the eigen vector space of $\lambda = -2$.

Put $(A + 2I)X = 0$

Complete.

(Ex) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transf- where $T(x,y) = (2x-y, 4x)$. Find the Eigen values of T and their Eigen vectors spaces ?

Solution Step 1: we will find the matrix A of T.

$$B_{\mathbb{R}^2} = \{(1,0), (0,1)\}$$

$$T(1,0) = (2,4) \Rightarrow [T(1,0)]_{B_{\mathbb{R}^2}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$T(0,1) = (-1,0) \Rightarrow [T(0,1)]_{B_{\mathbb{R}^2}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

So, $A = \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix}$

step 2 : To find the eigen values of A, Put

$$\Delta = 0 \Leftrightarrow |A - \lambda I| = 0$$

$$\Leftrightarrow \begin{vmatrix} 2-\lambda & -1 \\ 4 & -\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda^2 - 2\lambda + 4 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{-12}}{2} \notin \mathbb{R}$$

Hence, there is no eigen values, and hence, there is no eigen vector space.

Remark : If we put $\Delta = 0$, we will get (for example) like the following form:

$$(\lambda - a)^n (\lambda - b)^m = 0$$

In this case, we have two eigen values a and b.

* n is the multiplicity of (a) or degree (a)
* m is the multiplicity of (b) or degree (b)

Rule Let λ be an eigen value of degree n. Then the Possibilities of dim of eigen space of is m where $1 \leq m \leq n$.

Remark: (1) If $\deg(\lambda) = 1$ then $\dim(V) = 1$ where V is the eigenvector space of λ.

(2) If $\deg(\lambda) = 2$ then $\dim(V)$ is either 1 or 2 where V is the vector space of λ.

Example Let $T: P_1(x) \longrightarrow P_1(x)$ where $T(a+bx) = -b+ax$. find the eigen values of T.

Sol Step 1: we will find the matrix A of T.

$B = \{1, x\}$

$T(1) = T(1+0x) = x \Rightarrow [T(1)]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(x) = T(0+x) = -1 \Rightarrow [T(x)]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

So, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Step 2 To find the eigen values, Put $\Delta = 0$

$\Rightarrow |A - \lambda I| = 0$

$\Leftrightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$

$\Leftrightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm\sqrt{-1} \notin \mathbb{R}$.

Hence, there is no eigen values.

ex Let A be matrix with eigen values $0, 1, -1$ where $deg(0) = 2$, $deg(1) = 1$, $deg(-1) = 3$. write the Characteristic Polynomial of A?

SoL To find eigen values, we have

$$\lambda^2 (\lambda-1)^1 (\lambda+1)^3 = 0$$

So, the charcteristic Polynomial is $\lambda^2(\lambda-1)(\lambda+1)^3$ ∎

ex Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Find the eigen values and their eigen vector spaces?

SoL (step1) Let $\Delta = 0 \iff |A - \lambda I| = 0 \iff \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)^3 = 0 \Rightarrow$ only eigen value of A is $\lambda = 2$ where $deg(2) = 3$

(STEP2) To find eigen vector space of $\lambda = 2$,

Let $(A - 2I)X = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_2 = 0$ and $x_1 = r, x_3 = t$

$\Rightarrow S = \left\{ \begin{bmatrix} r \\ 0 \\ t \end{bmatrix} ; r, t \in \mathbb{R} \right\}$

$\Rightarrow S = \left\{ r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ; r, t \in \mathbb{R} \right\}$

the space of eigen vector with dim = 2.

Rule If v is eigen vector of the eigen value a of a matrix A then $\boxed{\lambda v = A \cdot v}$

for example: In above exersices, we have $\lambda = 2$ eigen value, choose any eigen vector say $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$. Notice that $\lambda v = 2 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$

$Av = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$ ∎

Rule : A is invertible iff 0 is Not eigen value of A.

For example Let the Char- polynomial of A be $(\lambda-1)^2+2$.

If we put $\lambda=0$, $(0-1)^2+2 \neq 0$.

Hence 0 is not eigen value $\Rightarrow$ A is ~~not~~ invertible ■

Rule : If A is invertible and $\boxed{0\neq\lambda}$ is eigen values of A then $\frac{1}{\lambda}$ is eigen values of A^{-1}

Example Let A has the following eigen values : 1, 2, -3.

① show that A is invertible ?

② find eigen values of A^{-1} ?

Sol as 0 is Not eigen value of A $\Rightarrow$ A is invertible

So, the eigen values of A^{-1} are $1, \frac{1}{2}, \frac{-1}{3}$ ■

(Ex) Let λ_1 be eigen value of matrix A and λ_2 be eigen value of matrix B. If ν is eigen vector of λ_1 and λ_2 at the same time. Prove ν is eigen vector of A^2+2B ?

Sol we have $\lambda_1 \nu = A\nu$ .... ①

$\lambda_2 \nu = B\nu$ .... ②

From ①

$A^2\nu = A(A\nu) = A(\lambda_1\nu) = \lambda_1(A\nu) = \lambda_1^2\nu$ .... ③

(i.e. λ_1^2 is eigen ~~vector~~ value of A^2 with eigen vector ν)

From ② $2B\nu = 2\lambda_2\nu$ ...④

($2\lambda_2$ is eigen value of $2B$ with eigen vector ν)

From ③ and ④,

$(A^2+2B)\nu = A^2\nu+2B\nu = \lambda_1^2\nu+2\lambda_2\nu = (\lambda_1^2+2\lambda_2)\nu$ ■

(Ex) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(x, y, z) = (x, y, -2z)$.
find the eigen values of A and their eigen vector spaces ?

Sol (step 1) we will find matrix A of T.

$$B = \{ (1,0,0), (0,1,0), (0,0,1) \}$$

$$(1,0,0) \xrightarrow{T} (1,0,0) \xrightarrow{[T(v_1)]_B} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(0,1,0) \xrightarrow{T} (0,1,0) \xrightarrow{[T(v_2)]_B} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$(0,0,1) \longrightarrow (0,0,-2) \xrightarrow{[T(v_3)]_B} \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

So, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(step 2) we will find eigen values :

$$A = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2(-2-\lambda) = 0$$

$\Rightarrow \lambda = 1$ and $\lambda = -2$ are the eigen values
(Note that $\deg(1) = 2$ and $\deg(-2) = 1$)

(step 3) To find the eigen vector space of $\lambda = 1$,

let $(A - I) = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_3 = 0 \wedge x_1 = t \wedge x_2 = s$

$\Rightarrow S = \left\{ \begin{bmatrix} t \\ s \\ 0 \end{bmatrix} ; t, s \in \mathbb{R} \right\}$

$\Rightarrow S = \left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; t, s \in \mathbb{R} \right\}$

is the eigen space of $\lambda = 1$ with dim $= 2$

Rule : If A is upper or Lower tringular then the eigen values of A is the main diognal

In Last Ex, the eigen values : $\underbrace{1, 1}_{\deg(1) = 2}, \underbrace{-2}_{\deg(-2) = 1}$