Power Series and Analytic Function

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Power Series and Analytic Function

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- Differentiation and Integration of a Power Series
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- Ordinary Point and Singular Point
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A power series in $x - x_0$ is an infinite series of form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots, \qquad (1)$$

where the coefficients a_n are constants.

- The series (1) converges at the point $x = \alpha$ if

$$\lim_{n\to\infty} S_n(x) = \sum_{n=0}^{\infty} a_n(\alpha - x_0)^n$$

  exists.
- The series (1) diverges at the point $x = \alpha$ if

$$\lim_{n\to\infty} S_n(x) = \sum_{n=0}^{\infty} a_n(\alpha - x_0)^n$$

  does not exist.

Differentiation and Integration of a Power Series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

- $f'(x) = \sum_{n=0}^{\infty} a_n n (x - x_0)^{n-1}$, and

  $f''(x) = \sum_{n=0}^{\infty} a_n n(n-1)(x - x_0)^{n-2}$
- $\int f(x)\ dx = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+1}/(n+1)$

Power Series Solutions for Homogeneous Second-order Linear ODE with Nonconstant Coefficients

A general homogeneous second-order ODE has the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{2}$$

which we will write in standard form

$$y'' + p(x)y' + q(x)y = 0, \tag{3}$$

where $p(x) = \frac{a_1(x)}{a_2(x)}$ and $q(x) = \frac{a_0(x)}{a_2(x)}$.

- A point $x = x_0$ is an **ordinary point** of the differential equation if $p(x)$ and $q(x)$ are analytic as $x = x_0$

- If $p(x)$ or $q(x)$ is not analytic at $x = x_0$ then we say that $x = x_0$ is a **singular point**.

Considering the defnitions of $p(x)$ and $q(x)$ above, we see that typically the points where $a_2(x) = 0$ are the singular points of the ODE.

Example

Locate the ordinary points, regular singular points and irregular singular points of the differential equation

$$(x^4 - x^2)y'' + (2x+1)y' + x^2(x+1)y = 0$$

Solution

We have $a_2(x) = x^4 - x^2$, $a_1(x) = 2x + 1$, $a_0(x) = x^2\,(x+1)$, and so

$$a_1(x)/a_2(x) = \frac{2x+1}{x^4 - x^2} = \frac{2x+1}{x^2(x-1)(x+1)}$$

and

$$a_0(x)/a_2(x) = \frac{x^2\,(x+1)}{x^4 - x^2} = \frac{1}{x-1}.$$

We can see that every real number except $0, 1$ and -1 is an ordinary point the differential equation. To see which of the singular points $0, 1$ and -1 is a regular singular point and which is an irregular singular point for the differential.

we need to examine the two functions: $(x-x_0)a_1(x)/a_2(x)$, and $(x-x_0)^2a_0(x)/a_2(x)$ at the points $0, 1$ and -1.

At $x_0 = 0$, we have

$$(x-x_0)a_1(x)/a_2(x) = \frac{2x+1}{x(x-1)(x+1)},$$

and

$$(x-x_0)^2a_0(x)/a_2(x) = \frac{x^2}{x-1}.$$

The first function is not analytic at $x_0 = 0$, hence we conclude that $x_0 = 0$ is an irregular singular point.

At $x_0 = 1$, we have

$$(x-x_0)a_1(x)/a_2(x) = \frac{2x+1}{x^2(x+1)},$$

and

$$(x-x_0)^2a_0(x)/a_2(x) = x-1.$$

Since both of these expressions are analytic at $x_0 = 1$, we conclude that $x_0 = 1$ is a regular singular point.

Finally, for $x_0 = -1$, we have

$$(x - x_0)a_1(x)/a_2(x) = \frac{2x + 1}{x^2(x - 1)},$$

and

$$(x - x_0)^2 a_0(x)/a_2(x) = \frac{(x + 1)^2}{x - 1}.$$

Since both of these functions are analytic at $x_0 = -1$, we conclude that $x_0 = -1$ is a regular singular point for the differential equation.

Exercises

Locate the ordinary points, regular singular points and irregular singular points of the differential equation

- $xy'' - (2x-1)y' + y = 0$
- $x(x^2+1)^3y'' + y' - 8xy = 0$
- $x^3(1-x^2)y'' + (2x-3)y' + xy = 0$

Power Series Solutions about an Ordinary Point

We now wish to find a series solution by expanding about an ordinary point $x = x_0$ of an ODE using the following method:

- Assume a solution of the form $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$
- Substitute the series into the ODE.
- Obtain an equation relating the coefficients, called a recurrence relationship.
- Apply any initial conditions.

Example (1)

Find the general solution of the differential equation

$$y' - 2xy = 0 \tag{4}$$

about the ordinary point $x_0 = 0$.

Solution It is clear that $x_0 = 0$ is an ordinary point since there are no finite singular points. The solution of (4) is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{5}$$

We have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

then equation (4) becomes

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0 \tag{6}$$

We first make the same power of x as x^n in both series in (6) by letting $k = n - 1$ in the first series and $k = n + 1$ in the second one, we have

$$\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - \sum_{k=1}^{\infty} 2 a_{k-1} x^k = 0. \tag{7}$$

We now let the index of summation starts by 1 in both series in (7), so that

$$a_1 + \sum_{k=1}^{\infty} \left[(k+1) a_{k+1} - 2 a_{k-1}\right] x^k = 0. \tag{8}$$

For equation (8) to be satisfied, it is necessary that $a_1 = 0$ and

$$(k+1) a_{k+1} - 2 a_{k-1} = 0, \quad \text{for all } k \geq 1. \tag{9}$$

Equation (9) provides a recurrence relation and we write

$$a_{k+1} = \frac{2a_{k-1}}{k+1} \quad \text{for all } k \geq 1 \tag{10}$$

Iteration of (10) then gives for $k = 1$

$$a_2 = a_0.$$

For $k = 2$

$$a_3 = \frac{2}{3}a_1 = 0.$$

For $k = 3$

$$a_4 = \frac{2}{4}a_2 = \frac{1}{2}a_0.$$

For $k = 4$

$$a_5 = \frac{2}{5}a_3 = 0.$$

And for $k = 5$

$$a_6 = \frac{2}{6}a_4 = \frac{1}{3!}a_0,$$

and so on.

Thus from the original assumption, we find

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \\ &= a_0 \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots \right) \\ &= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \quad \text{for all } x \in \mathbb{R}. \\ &= a_0 e^{x^2}. \end{aligned}$$

Example (2)

Solve the initial value problem by the method of power series about the initial point $x_0 = 0$.

$$\begin{cases} (1-x^2)y'' - xy' + 4y = 0 \\ y(0) = 1,\ y'(0) = 0 \end{cases} \tag{11}$$

Solution The two functions

$$a_1(x)/a_2(x) = \frac{-x}{1-x^2} = -\sum_{n=0}^{\infty} x^{2n+1} \quad \text{for } |x| < 1,$$

and

$$a_0(x)/a_2(x) = \frac{4}{1-x^2} = 4\sum_{n=0}^{\infty} x^{2n} \quad \text{for } |x| < 1,$$

are analytic for all $|x| < 1$,then the solution of the differential equation in (11) is given by

$$y = \sum_{n=0}^{\infty} a_n x^n \text{ for } |x| < 1.$$

Hence

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

for all $|x| < 1$. So we have

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0. \quad (12)$$

Let $k = n - 2$ in the first series and $k = n$ in the other series, we get

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=2}^{\infty}k(k-1)a_kx^k - \sum_{k=1}^{\infty}ka_kx^k + 4\sum_{k=0}^{\infty}a_kx^k = 0.$$

All sums in (12) should start by the same index of summation 2, therefore we have

$$\sum_{k=2}^{\infty}\left[(k+2)(k+1)a_{k+2} - (k^2-4)a_k\right]x^k$$
$$+2a_2 + 4a_0 + (6a_3 + 3a_1)x = 0.$$

From this last identity, we conclude that

$$2a_2 + 4a_0 = 0, 6a_3 + 3a_1 = 0$$

and

$$a_{k+2} = \frac{(k^2-4)a_k}{(k+2)(k+1)}, \quad \text{for all } k \geq 2.$$

By using the initial conditions, we would observe that $a_0 = y(0) = 1$ and $a_1 = y'(0) = 0$, then $a_2 = -2$, $a_3 = 0$ and

$$a_{k+2} = \frac{k-2}{k+1} a_k, \quad \text{for all } k \geq 2.$$

So for $k = 2$,

$$a_4 = 0,$$

for $k = 3$,

$$a_5 = 0,$$

for $k = 4$,

$$a_6 = 0$$

for $k = 5$,

$$a_7 = 0,$$

for $k = 6$,

$a_8 = 0$, and so on ,.....

and so on. Then the initial value problem (11) has a unique solution given by

$$\begin{aligned} y = \sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + ... \\ &= 1 - 2x^2 + ... \end{aligned}$$

for all $|x| < 1$.

Exercises

1. Find the general solution of the differential equation

$$y'' + y = 0$$

about the ordinary point $x_0 = 0$.

2. Solve the initial value problem by the method of power series about the initial point $x_0 = 0$.

$$\begin{cases} (1 - x^2)y'' - 2xy' + 6y = 0 \\ y(0) = 1,\ y'(0) = 0 \end{cases}$$

3. Compute the first four coefficients of power serises solution about the given initial point

$$\begin{cases} y'' - 2(x - 1)y' + 2y = 0 \\ y(1) = 1,\ y'(1) = 0 \end{cases}$$