King Saud University

College of Science

Department of Mathematics

254 Math Exercises
Differentiation and Integration
Ch. (5)

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Math 254 Khaled A Tanash

First Derivative Formula

Two-Points Formula

1- Forward –difference

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h}$$

2- Backward –difference

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h}$$

$$Error\ bound\ \leq \frac{h}{2}\ max|f''(\eta)|$$

$$\eta \in (x_0\ , x_0 + h), \qquad forward$$

$$\eta \in (x_0 - h\ , x_0), \qquad backward$$

Three-Points Formula

1- Three points central

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

$$Error\ bound\ \leq \frac{h^2}{6} max\left|f^{(3)}(\eta)\right|, \eta \in (x_0 - h, x_0 + h)$$

2- Three points forward

$$f'(x_0) = \frac{-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)}{2h}$$

$$Error\ bound\ \leq \frac{h^2}{3} max\left|f^{(3)}(\eta)\right|\ ,\ \ \eta \in (\ x_0\ , x_0 + 2h)$$

3- Three points backward

$$f'(x_0) = \frac{f(x_0 - 2h) - 4f(x_0 - h) + 3f(x_0)}{2h}$$

$$Error\ bound\ \leq \frac{h^2}{3} max\left|f^{(3)}(\eta)\right|\ ,\ \ \eta \in (\ x_0 - 2h\ , x_0)$$

The best formula is Central

Second Derivative Formula

Three-point central formula

$$f'(x_0) = \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2}$$

$$Error\ bound\ \leq \frac{h^2}{12} max\left|f^{(4)}(\eta)\right|\ , \eta \in (\ x_0 - h\ , x_0 + h)$$

Example: Let $f(x) = x^2 cosx$, then

a- Compute the approximation value of $f'(x) at\ x = 1$, taking $h = 0.1$ using the 2-point difference formula forward and backward. Also compute the error bound

b- What the best maximum value of the step size h required to obtain the approximate value of $f'(x)$ correct to two decimal place

Solution:

a) $x_0 = 1$ and $h = 0.1$

Now 2-Point forward

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

$$f'(1) = \frac{f(1+0.1) - f(1)}{0.1} = \frac{(1.1)^2 \cos(1.1) - \cos 1}{0.1}$$

$$= \frac{0.5489 - 0.5403}{0.1} = 0.086$$

to find the error bound

$$E_F(f,h) = \frac{h}{2} \max |f''(\eta)| \ ; \eta \in (1, 1.1)$$

$$f'(x) = 2x \cos x - x^2 \sin x$$

$$f''(x) = 2\cos x - 2x \sin x - 2x \sin x - x^2 \cos x$$

$$= 2\cos x - 4x \sin x - x^2 \cos x$$

$$M \leq \max_{1 \leq x \leq 1.1} |2\cos x - 4x\sin x - x^2 \sin x| = 3.5630$$

$$\Rightarrow E_F(f,h) \leq \frac{0.1}{2}(3.563) = 0.1782$$

b) the accuracy is 10^{-2}

$$\Rightarrow E_F(f,h) = \frac{h}{2} f''(\eta) \leq 10^{-2}$$

$$\Rightarrow \frac{h}{2}(3.563) \leq 10^{-2}$$

$$\Rightarrow h \leq \frac{2}{3.563} \cdot 10^{-2} = 0.0056$$

Example: Let $f(x) = x^2 cosx$, then

a- Compute the approximation value of $f'(x) at\ x = 1$, taking $h = 0.1$ using the 3-point difference formula central. Also compute the error bound

b- What the best maximum value of the step size h required to obtain approximate value of $f'(x)$ correct to two decimal place

Solution:

$x_0 = \quad , h = 0.1$

$f'(x_0) = \dfrac{f(x_0+h) - f(x_0-h)}{2h} =$

$\Rightarrow f'(1) = \dfrac{f(1.1) - f(0.9)}{0.2} = \dfrac{(1.1)^2 \cos(1.1) - (0.9)^2 \cos(0.9)}{0.2}$

$= \dfrac{0.5489 - 0.5035}{0.2} = 0.2270$

to find the error bound

$E_C(f,h) = \dfrac{h^2}{6} \max_{0.9 \leq x \leq 1.1} |f^{(3)}(x)|$

$f^{(3)}(x) = -2\sin x - 4\sin x - 4x\cos x - 2x\cos x + x^2 \sin x$

$M \leq \max_{0.9 \leq x \leq 1.1} |f^{(3)}(x)| = 7.4222$

$\Rightarrow$

$$\Rightarrow |E_c(f,h)| \leq \frac{0.01}{6}(7.4222) = 0.0124$$

b) The accuracy is 10^{-2}

$$|E_c(f,h)| = \left|\frac{h^2}{6} f'''(\eta)\right| \leq 10^{-2}$$

$$\Rightarrow \frac{h^2}{6}(7.4222) \leq 10^{-2}$$

$$\Rightarrow h^2 \leq \frac{6}{7.4222} \times 10^{-2}$$

$$\Rightarrow h \leq 0.01$$

Example: When using the 2-point forward formula with $h = 0.2$ for approximating the value, where $f(x) = \ln(x+1)$. Compute the approximation of $f'(1)$. Also compute the error bound.

Solution:

$x_0 = 1 \quad , h = 0.2$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} = f(1.2) - f(1)$$

$$= \frac{\ln(2.2) - \ln 2}{0.2} = 0.4765$$

Now to find the error bound

$$E_F(f,h) \leq \left| \frac{h}{2} \max f''(x) \right| ; \; x \in (1, 1.2)$$

$$f'(x) = \frac{1}{x+1} \quad , \quad f''(x) = \frac{-1}{(x+1)^2}$$

$$\Rightarrow \max_{x \in (1, 1.2)} |f''(x)| = \frac{1}{2^2} = 0.25$$

$$\Rightarrow |E_F(f,h)| \leq \left| \frac{0.2}{2}(0.25) \right| = 0.025$$

Example: Using the table data

x	0.0	0.1	0.2	0.3	0.4	0.45	0.5
$f(x)$	-2.0	0.0	3.0	5.0	8.0	10.0	14.0

To compute the following

1- The best approximate value of $f'(0.3)$ using 3-point difference formula

2- The beast approximate value of $f''(0.4)$

Solution:

1) The beast approximation of $f'(x)$ using 3-point formula is Central

$x_0 = 0.3 \;,\; h = 0.1$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$f'(0.3) = \frac{f(0.4) - f(0.2)}{0.2} = \frac{8-3}{0.2} = 25$$

2) The only formula of 3-point of 2-nd derivative is

$$f''(x_0) = \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2}$$

$$f''(0.4) = \frac{f(0.4-0.1) - 2f(0.4) + f(0.4+0.1)}{(0.1)^2}$$

$$= \frac{f(0.3) - 2f(0.4) + f(0.5)}{0.01} = \frac{5 - 2(8) + 14}{0.01} = 300$$

Example: Using the table data

x	0.9	1.3	1.7	1.9	2.1	2.3
$f(x)$	2.3	2.05	1.8	1.65	1.4	1.15

To compute the following

1- The worst possible for approximation $f'(1.7)$ using 2-point formula

2- The beast possible for approximation $f'(1.9)$ using 3-point formula

3- The worst possible for approximation $f'(1.5)$ using 3-point formula

Solution:

1) choose the max value of h

Forward $h = 0.6$, Backward $h = 0.8$

$\Rightarrow$ the worst Backward , $h = 0.8$

$$\Rightarrow f'(1.7) = \frac{f(1.7) - f(0.9)}{h} = \frac{1.8 - 2.3}{0.8} = -0.625$$

2) choose the min. value of h

central , $h = 0.2$

forward $h = 0.2$ and backword filed

$\Rightarrow$ The beas central $h = 0.2$

$$f'(1.9) = \frac{f(2.1) - f(1.7)}{2(0.2)} = \frac{1.4 - 1.8}{0.4} = \frac{-0.4}{0.4} = -1$$

3) forword and backward are filed

$\Rightarrow$ the worst is the central

with $h = 0.6$

$$f'(1.5) = \frac{f(2.1) - f(0.9)}{2h} = \frac{1.4 - 2.3}{1.2} = -0.75$$

Example: Using data $(0.-2), (0.1,-1), (0.15,1), (0.2,2), (0.3,3)$, find the best approximation of $f'(0.1)$ using 3-point difference formula. So use the same data find the worst approximation of $f''(0.1)$ using 3-point difference formula.

Solution:

central $h=0.1$; forward $h=0.05$

and Backword filed

$\Rightarrow$ the beast is forward with $h=0.05$

$\Rightarrow$

$$f'(0.1) = \frac{-3f(0.1)+4f(0.15)-f(0.2)}{2h}$$

$$= \frac{-3(-1)+4(1)-2}{2(0.05)} = 50$$

Now the worst approximation of $f''(0.1)$

the only formula 3-point central with largest h

$\Rightarrow$ $h = 0.1$

$$f''(0.1) = \frac{f(0)-2f(0.1)+f(0.2)}{h^2}$$

$$= \frac{-2-2(-1)+2}{(0.1)^2} = 200$$

Example: Consider the following data:

$$(0.2\,,0.39),(0.4\,,1.08),(0.6\,,1.49),(0.8\,,1.78),(1\,,2)$$

Use the numerical formula to compute the best approximation for $f''(0.6)$. The data in this problem were taken from the function $f(x) = \ln x + 2$. Compute the actual errors and also, find error bound of your approximations.

Solution:

The beast is 3-point central with smallest h

$h = 0.2$

$$f''(0.6) = \frac{f(0.4) - 2f(0.6) + f(0.8)}{h^2}$$

$$= \frac{1.08 - 2(1.49) + 1.78}{(0.2)^2} = -3$$

Now to find the actual error

$= |\text{Exact} - \text{approx}| =$

$$f'(x) = \frac{1}{x} \Rightarrow f''(x) = \frac{-1}{x^2} \Rightarrow f''(0.6) = -2.777$$

$\Rightarrow$ actual error $= |-2.777 - (-3)| = 0.222$

error bound $E \leqslant \frac{h^2}{3} \max|f^{(3)}(x)|$; $x \in (0.4, 0.8)$

$\Rightarrow$

max

$$f^{(3)}(x) = \frac{2}{x^3}.$$

$$\Rightarrow \max_{x \in (0.4, 0.8)} \left| \frac{2}{x^3} \right| = \frac{2}{(0.4)^3} = 31.25$$

$$\Rightarrow E \leq \frac{(0.2)^2}{3}(31.25) = 0.4166$$

Example: Let $h = 1$ and f is a function such that $f(x+h) = -f(x-h)$. If $f''(1) = 1.2$, then find the approximation of $f(1)$.

Solution:

$$f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

but $f(x+h) = -f(x-h)$

$$\Rightarrow f''(x) = \frac{f(x-h) - 2f(x) - f(x-h)}{h^2}$$

$$\Rightarrow f''(x) = \frac{-2f(x)}{h^2}$$

$$\Rightarrow f(x) = -\frac{1}{2} h^2 f''(x)$$

$$\Rightarrow f(1) = -\frac{1}{2}(1)(1.2)$$

$$\Rightarrow f(1) = -0.6$$

Example: Let $f(x) = e^x$. Then find the worst approximate value of $f'(0)$ with $h = 0.2$ using 3-point differentiation formula.

Solution:

Now central is the beast

$\Rightarrow$ the worst betwen forword or backword

Forword $f'(x_0) = \dfrac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h}$

$= \dfrac{-3f(0) + 4f(0.2) - f(0.4)}{2(0.2)}$

$= \dfrac{-3 + 4e^{(0.2)} - e^{0.4}}{0.4} = 0.9844$

Backword $f'(x_0) = \dfrac{f(x_0-2h) - 4f(x_0-h) + 3f(x_0)}{2h}$

$= \dfrac{f(-0.4) - 4f(-0.2) + 3f(0)}{2(0.2)}$

$= \dfrac{e^{-0.4} - 4e^{-0.2} + 3e^{0}}{0.4} = 0.9884$

Now

$E_F \leqslant \dfrac{h^2}{3} \max|f^{(3)}(x)|$; $x \in (0, 0.4)$

$\leqslant \dfrac{(0.2)^2}{3}(1.4918) = 0.01989$

$$E_B \leq \frac{h^2}{3} \max |f^{(3)}(x)| \quad ; \quad x \in (-0.4, 0)$$

$$\leq \frac{(0.2)^2}{3}(1) = 0.0026$$

$\Rightarrow$ The worst is Forward.

Example: The function $f(x)$ satisfies a given equation $f''(x) = x^2 f(x)$ and satisfies the condition

$f(0.5) = 2, f(0.7) = 4$. Use the central- difference formula to find approximation of $f''(0.6)$

Solution:

$$f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

$x = 0.6$, $h = 0.1$ and $f''(x) = x^2 f(x)$

$$\Rightarrow f''(0.6) = \frac{f(0.5) - 2f(0.6) + f(0.7)}{(0.1)^2}$$

$$\Rightarrow (0.6)^2 f(0.6) = \frac{2 - 2f(0.6) + 4}{0.01}$$

$$\Rightarrow 0.0036\, f(0.6) = 6 - 2f(0.6)$$

$$\Rightarrow 2.0036\, f(0.6) = 6$$

$$\Rightarrow f(0.6) = 2.9946$$

Now

$$f''(0.6) = \frac{2 - 2(2.9946) + 4}{0.01} = 1.08$$

Example: Let $f(x) = x^5 + 1$ defined in the interval $[0.1\,, 0.2]$. Use the error formula of three-point formula for the approximation of $f''(0.15)$ to find the unknown point $\eta \in (0.1\,, 0.2)$.

Solution:

the error formula of 3-point central formula for $f''(0.15)$ is

$$E = \text{exact} - \text{Appro:} = -\frac{h^2}{12} f^{(4)}(\eta)$$

the exact value of $f''(0.15)$

$f'(x) = 5x^4$, $f''(x) = 20x^3$, $f^{(3)}(x) = 60x^2$

$f^{(4)}(x) = 120x$

$f''(0.15) = 20(0.15)^3 = 0.0675$ the exact

Now $h = 0.05$

$$f''(0.15) = \frac{f(0.2) - 2f(0.15) + f(0.1)}{(0.05)^2} = 0.071$$

$\Rightarrow E = 0.0675 - 0.071 = -0.0035$

$\Rightarrow -0.0035 = -\frac{(0.05)^2}{12}(120\eta)$

$\eta = 0.1520$

Example: Let $f(x) = x^3 + 1$ defined in the interval $[0.1\,, 0.2]$. Use the error formula of two-point formula for the approximation of $f'(0.1)$ to find the unknown point $\eta \in (0.1\,, 0.2)$.

Solution:

Numerical Integration

Trapezoidal Rule

- Simple Trapezoidal Rule

$$\int_a^b f(x)dx \approx T_1(f) = \frac{h}{2}[f(x_0) + f(x_1)] = \frac{h}{2}[f(a) + f(b)]$$

$$local\ error\ E_{T_1}(f) \leq \left|\frac{h^3}{12}f''(\eta)\right| = \frac{h^3}{12}\max_{x\in(a,b)}|f''(x)|$$

$$where\ h = b - a$$

- Composite Trapezoidal Rule

$$\int_a^b f(x)dx \approx T_n(f) = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right]$$

$$where\ h = \frac{b-a}{n}, \quad x_i = a + ih, \qquad i = 0,1,2,\dots,n$$

$$local\ error\ E_{T_n}(f) \leq \left|\frac{h^3}{12}nf''(\eta)\right| = \frac{h^3}{12}n\max_{x\in(a,b)}|f''(x)|$$

Simpson's Rule

- Simple Simpson's Rule

$$\int_a^b f(x)dx \approx S_2(f) = \frac{h}{3}[f(a) + 4f(a+h) + f(b)]$$

$$local\ error\ E_{S_2}(f) \leq \left|\frac{h^5}{90} f^{(4)}(\eta)\right| = \frac{h^5}{90} \max_{x\in(a,b)} \left|f^{(4)}(x)\right|$$

$$where\ h = \frac{b-a}{2}\ ,\ \ n = 2$$

- Composite Simpson's Rule

$$\int_a^b f(x)dx \approx S_n(f) =$$

$$= \frac{h}{3}\left[f(a) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + f(b)\right]$$

$$where\ h = \frac{b-a}{n} \quad x_i = a + ih\,, \qquad i = 0,1,2,\ldots,n\ and\ \boldsymbol{n\ even}$$

$$local\ error\ E_{S_n}(f) \leq \left|\frac{h^5}{180} n f^{(4)}(\eta)\right| = \frac{h^5}{180} n \max_{x\in(a,b)} \left|f^{(4)}(x)\right|$$

Example: If $f(0) = 3, f(1) = \frac{\alpha}{2}, f(2) = \alpha$, and Simpson's rule for $\int_0^2 f(x)dx$ gives 2. Find the value of α

Solution:

$n = 2$; $h = 1$

simple Simpson's rule

$$\int_a^b f(x)\,dx = S_2(f) = \frac{h}{3}\left[f(a) + 4f(a+h) + f(b)\right]$$

$$\Rightarrow S_2(f) = \frac{h}{3}\left[f(0) + 4f(1) + f(2)\right] = 2$$

$$\Rightarrow \frac{1}{3}\left[3 + \frac{\alpha}{2} + \alpha\right] = 2$$

$$\Rightarrow 3 + \alpha/2 + \alpha = 6$$

$$\frac{3\alpha}{2} = 3$$

$$\Rightarrow \alpha = 2$$

Example: When using simple Trapezoidal rule for approximation the integral $\int_1^{1.5} \frac{1}{x} dx$, we have the computed approximation is.

Solution:

$$\int_a^b f(x)\,dx \simeq T_1(f) = \frac{h}{2}\left[f(a) + f(b)\right]$$

$$h = b - a = 0.5 \quad ; \quad f(x) = 1/x$$

$$\Rightarrow T_1(f) = \frac{0.5}{2}\left[f(1) + f(1.5)\right]$$

$$= \frac{1}{4}\left[1 + \frac{2}{3}\right]$$

$$= \frac{1}{4} \cdot \frac{5}{3} = \frac{5}{12}$$

Example: The number of subintervals required to approximate $\int_0^2 \frac{1}{x+4}\,dx$, within the accuracy 10^{-4} by using composite Simpson's rule is.

Solution:

error bound of composite simpson's rule

$$E_{S_n} \leq \frac{h^5}{180}\, n\, f^{(4)}(\eta)$$

$$h = \frac{b-a}{n} = \frac{2}{n}$$

$$f'(x) = -(x+4)^{-2}, \quad f''(x) = 2(x+4)^{-3}$$

$$f'''(x) = -6(x+4)^{-4}, \quad f^{(4)}(x) = \frac{24}{(x+4)^5}$$

$$f^{(4)}(\eta) = \max_{0 \leq x \leq 2} |f^{(4)}(x)| = \frac{24}{4^5} = 0.02343$$

$$|E_{S_n}| \leq \left| \frac{2^5/n^5}{180} \times n \times 0.02343 \right| \leq 10^{-4}$$

$$\Rightarrow \frac{32}{180 n^4} \leq \frac{1}{0.02343} \times 10^{-4}$$

$$\frac{1}{n^4} \leq \frac{180}{32 \times 0.02343} \times 10^{-4}$$

$$n^4 \geq 0.0041653 \times 10^4$$

$$n^4 \geq 41.6533$$

$$n \geq 2.54 \quad \Rightarrow \quad n = 4$$

Example: The number of subintervals required to approximate $\int_0^{0.2} \frac{1}{x+1}\,dx$, within the accuracy 10^{-3} by using composite Trapezoidal rule is.

Solution:

error bound for Trapezoidal rule

$$E_{T_n} \leq \frac{h^3}{12} \cdot n\, f''(\eta)$$

$$h = \frac{b-a}{n} = \frac{0.2}{n}$$

$$f'(x) = -(x+1)^{-2}, \quad f''(x) = 2(x+1)^{-3} = \frac{2}{(x+1)^3}$$

$$f''(\eta) = \max_{x \in (0, 0.2)} |f''(x)| = \frac{0.2}{1} = 0.2$$

$$\Rightarrow E_{T_n} \leq \left| \frac{(0.2)^3 / n^3}{12} \cdot n (0.2) \right| \leq 10^{-3}$$

$$\Rightarrow \frac{(0.2)^4}{12\, n^2} \leq 10^{-3} \Rightarrow \frac{1}{n^2} \leq \frac{12}{(0.2)^4} \times 10^{-3}$$

$$n^2 \geq 133.33 \times 10^{-6} \times 10^3 \Rightarrow n^2 \geq 133.3 \times 10^{-3}$$

$$n = 0.365$$

$$n = 1$$

Example: Using the data points: $(0,-2),(0.1,-1),(0.15,1),(0.2,2),(0.3,3)$, the best approximate value of the integral $\int_0^{0.3} f(x)dx$, using the composite Trapezoidal rule is.

Solution:

put the points

$0, 0.1, 0.2, 0.3$; $h=0.1$; $n=3$

$$\Rightarrow \int_0^{0.3} f(x)\,dx \simeq T_3(f)$$

$$= \frac{0.1}{2}\left[f(0)+2\big(f(0.1)+f(0.2)\big)+f(0.3)\right]$$

$$= \frac{0.1}{2}\left[-2+2(-1+2)+3\right] = \frac{0.1}{2}\times 3$$

$$= 0.15$$

Example: The upper bound for the error in approximating $\int_0^1 \frac{15}{x+1}\,dx$ using the composite Trapezoidal rule with $n = 5$.

Solution:

$E_{T_n}(f) \leq \left| \frac{h^3}{12} . n . f''(\eta) \right|$; $h = \frac{b-a}{n} \Rightarrow h = 1/5$

$f'(x) = -15(x+1)^{-2}$ and $f''(x) = \frac{30}{(x+1)^3}$

$f''(\eta) \leq \max_{x\in(0,1)} |f''(x)| = \frac{30}{1} = 30$

$\Rightarrow |E_{T_5}| \leq \left| \frac{(1/5)^2}{12} (30) \right| = 0.1$

Example: Estimate the integral $\int_{-1}^{1} \frac{dx}{1+x^2}$ using the Simpson's rule with $n = 8$.

Solution:

$f(x) = \frac{1}{1+x^2}$; $n=8$, $h = \frac{2}{8} = 0.25$

$\int_{-1}^{1} f(x)\,dx = S_8(f)$

$= \frac{h}{3}\left[f(x_0) + 4\left(f(x_1)+f(x_3)+f(x_5)+f(x_7)\right) + 2\left(f(x_2)+f(x_4)+f(x_6)\right) + f(x_8)\right]$

$= \frac{0.25}{3}\left[f(-1) + 4\left(f(-0.75) + f(-0.25) + f(0.25) + f(0.75)\right)\right.$

$\left. + 2\left(f(-0.5) + f(0) + f(0.5)\right) + f(1)\right]$

$= \frac{0.25}{3}\left[0.5 + 4(0.64 + 0.9412 + 09412 + 0.64)\right.$

$\left. + 2(0.8 + 1 + 0.8) + 0.5\right] = 1.5708$

Example: Use the composite Trapezoidal rule for approximation the integral $\int_1^3 \frac{dx}{7-2x}$ with $h = 0.5$. Also compute an error bound.

Solution:

$f(x) = \dfrac{1}{7-2x}$, $h = 0.5$, $n = 4$

$$\int_1^3 f(x)\,dx = T_4(f)$$

$$= \frac{h}{2}\left[f(x_0) + 2\big(f(x_1) + f(x_2) + f(x_3)\big) + f(x_4)\right]$$

$$= \frac{0.5}{2}\left[f(1) + 2\big(f(1.5) + f(2) + f(2.5)\big) + f(3)\right]$$

$$= \frac{0.5}{2}\left[0.2 + 2(0.25 + 0.33 + 0.5) + 1\right] = 0.84$$

Now;

$$|E_{T_4}(f)| \leqslant \frac{h^3}{12}\, n\, f''(\eta)$$

$$f'(x) = -(7-2x)^{-2} \cdot -2 = 2(7-2x)^{-2}$$

$$f''(x) = -4(7-2x)^{-3} \cdot -2 = \frac{8}{(7-2x)^3}$$

$$f''(\eta) = \max_{x\in(1,3)} |f''(x)| = \frac{8}{(7-2)^3} = 0.064$$

$$\Rightarrow E_{T_4}(f) \leqslant \frac{(0.5)^3}{12} \times 4(0.064) = 2.666\times10^{-3}$$

Example: compute the approximation of the integral $I(f)=\int_1^2 \frac{e^{-x}}{x}dx$ when $h=0.2$ using integration rule. Compute the error bound.

Solution:

Now $h=0.2 \Rightarrow n=\frac{b-a}{h}=\frac{1}{0.2}=5$

since n is odd $\Rightarrow$ the best is Trapezoidal rule

$$\int_1^2 f(x)\,dx = T_5(f)$$

$$= \frac{h}{2}\left[f(1)+2\left(f(1.2)+f(1.4)+f(1.6)+f(1.8)\right)+f(2)\right]$$

$$= \frac{0.2}{2}\left[0.3679+2(0.2509+0.1761+0.1262+0.0918)+0.0677\right]$$

$$= 0.1\left[1.7256\right] = 0.1726$$

Now.

$$E\,T_5(f) \le \left|\frac{h^3}{12}\,n\,f''(\eta)\right|$$

$$f'(x) = \frac{e^{-x}}{x}\left[1+\frac{1}{x}\right], \quad f''(x) = \frac{e^{-x}}{x}\left[1+\frac{2}{x}+\frac{2}{x^2}\right]$$

$$f''(\eta) = \max_{x\in(1,2)} |f''(x)| = 5/e$$

$$\Rightarrow E\,T_5(f) \le \frac{(0.2)^3}{12}\times 5\times\frac{5}{e} = 6.1313\times10^{-3}$$

Example: Suppose that $f(0.25) = f(0.75) = \alpha$. Find α if composite Trapezoidal rule with $n = 2$ gives the value of the $\int_0^1 f(x)dx = 2$ and with $n = 4$ gives $\int_0^1 f(x)dx = 1.75$

Solution:

Now for $n=2 \Rightarrow h = 0.5$

$$\int_0^1 f(x)dx = T_2(f) = \frac{h}{2}\left[f(0) + 2f(0.5) + f(1)\right] = 2$$

$$\Rightarrow \frac{0.5}{2}\left(f(0) + 2f(0.5) + f(1)\right) = 2$$

$$\Rightarrow f(0) + 2f(0.5) + f(1) = 8 \quad \cdots (1)$$

for $n = 4 \Rightarrow h = 0.25$

$$\int_0^1 f(x)dx = T_4(f) = \frac{0.25}{2}\left[f(0) + 2\left(f(0.25) + f(0.5) + f(0.75)\right) + f(1)\right] = 1.75$$

$$\Rightarrow f(0) + 2(\alpha + f(0.5) + \alpha) + f(1) = 14$$

$$\Rightarrow f(0) + 2f(0.5) + f(1) + 4\alpha = 14$$

subsitute (1)

$$\Rightarrow 8 + 4\alpha = 14 \Rightarrow 4\alpha = 6$$

$$\Rightarrow \alpha = \frac{3}{2}$$

Example: Evaluate $\int_1^2 \ln x dx$, with $h = 0.25$ using best numerical integration formula. Also compute the error bound.

Solution:

$f(x) = \ln x$, $h = 0.25 \Rightarrow n = 4$

since n is even $\Rightarrow$ the beast method is Simpson's rule

$$\int_1^2 \ln x \, dx = S_4(f)$$

$$= \frac{h}{3}\left[f(1) + 4\left(f(1.25) + f(1.75)\right) + 2\left(f(1.5)\right) + f(2)\right]$$

$$= \frac{0.25}{3}\left[0 + 4(0.2231 + 0.5596) + 2(0.4054) + 0.6931\right]$$

$$= 0.3861$$

Example: The following function tabulated is $f(x) = \ln(x+1)$

x	0.0	.01	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
$f(x)$	0.00	0.10	0.18	0.26	0.34	0.41	0.47	0.53	0.59	0.64	0.69	0.74	0.79

Find the approximation of $\int_0^{1.2} f(x)dx$ taking $h = 0.4$ by using the best numerical integration rule. Also find that how many subintervals approximate the given integral to within accuracy 10^{-6}

Solution:

$h = 0.4 \Rightarrow n = \frac{1.2}{0.4} = 3$

since n is odd $\Rightarrow$ the beast method is Trapezoidal rule

$$\int_0^{1.2} f(x)\,dx = T_3(f)$$

$$= \frac{h}{2}\left[f(x_0) + 2(f(x_1) + f(x_2)) + f(x_3)\right)$$

$$= \frac{0.4}{2}\left[f(0) + 2(f(0.4) + f(0.8)) + f(1.2)\right]$$

$$= 0.2\left[0 + 2(0.34 + 0.59) + 0.79\right]$$

$$= 0.2\left[2.65\right] = 0.53$$

Now to find number of subintervals to get accurcy 10^{-6}

$$E_{T_n}(f) \leq \left|\frac{h^3}{12} . n\, f''(2)\right| \leq 10^{-6}$$

$f(x) = \ln(x+1) \Rightarrow f'(x) = \dfrac{1}{x+1}$

$\Rightarrow f''(x) = \dfrac{-1}{(x+1)^2}$

$\Rightarrow f''(\xi) = \max\limits_{0 \leq x \leq 1.2} |f''(x)| = \dfrac{1}{1} = 1$

$h = \dfrac{1.2}{n}$

$\Rightarrow E_{T_n}(f) \leq \left| \dfrac{(1.2)^3/n^3}{12} \cdot n(1) \right| \leq 10^{-6}$

$\Rightarrow \dfrac{(1.2)^3}{12n^2} \leq 10^{-6}$

$\Rightarrow \dfrac{1}{n^2} \leq \dfrac{12}{(1.2)^3} \times 10^{-6}$

$\Rightarrow n^2 \geq 0.144 \times 10^{6}$

$n = 379.4$

$n = 380$

Example: How many subintervals approximate the integral $\int_0^2 \frac{1}{x+4}\,dx$ to an accuracy 10^{-5} using the Simpson's rule. Also compute the approximation

Solution:

$$|E_{S_n}(f)| \leq \left|\frac{h^5}{180}\, n\, f^{(4)}(\eta)\right|$$

$h = \frac{2}{n}$, $f'(x) = -(x+4)^{-2}$, $f''(x) = 2(x+4)^{-3}$

$f'''(x) = -6(x+4)^{-4}$ and $f^{(4)}(x) = \frac{24}{(x+4)^5}$

$\Rightarrow f^{(4)}(\eta) = \max_{0\leq x\leq 2} |f^{(4)}(x)| = \frac{24}{4^5} = 0.0234$

$\Rightarrow E_{S_n}(f) \leq \frac{2^5/n^4}{180}(0.0234) \leq 10^{-5}$

$\Rightarrow \frac{1}{n^4} \leq \frac{180}{2^5 \times 0.0234} \times 10^{-5}$

$n^4 \geq 416$

$n \geq 4.52$

$n = 6$

Now to compute the approximation if $n = 6$

$\Rightarrow h = \frac{2}{6} = \frac{1}{3}$

$$\int_0^2 f(x)\,dx = S_6(f)$$

$$= \frac{h}{3}\Big[f(x_0) + 4\big(f(x_1) + f(x_3) + f(x_5)\big) + 2\big(f(x_2) + f(x_4)\big) + f(x_6) \Big]$$

$$= \frac{1}{9}\Big[f(0) + 4\big(f(1/3) + f(1) + f(5/3)\big) + 2\big(f(2/3) + f(4/3)\big) + f(2) \Big]$$

$$= \frac{1}{9}\left[\frac{1}{4} + 4\left(\frac{3}{13} + \frac{3}{15} + \frac{3}{17}\right) + 2\left(\frac{3}{14} + \frac{3}{16}\right) + \frac{1}{6} \right]$$

$$= 0.4055$$

Example: Use the best numerical integration rule to approximate $\int_0^2 \frac{1}{1+2x}\,dx$ when $h = 0.5$. Also compute an error bound for your approximation.

Solution:

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