Linear Differential Equations of Higher Order

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General Solution of Homogeneous Linear Differential Equations

Definition

The general linear differential equations of order n is an equation that can be written

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \tag{1}$$

where $a_n(x),\ a_{n-1}(x),\ a_1(x)$ and $a_0(x)$ are functions of $x \in \mathrm{I} = (a, b)$, and they are called **coefficients**.
Equation (1) is called homogeneous linear differential equation if the function $g(x)$ is zero for all $x \in (a, b)$.
If $g(x)$ is not equal to zero on I, the equation (1) is called non-homogeneous linear differential equation.

Initial-Value Problem (IVP)

An n-th order initial-value problem associate with (1) takes the form:
Solve:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

subject to:

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_2,\ \ldots,\ y^{n-1}(x_0) = y_{n-1}. \qquad (2)$$

Here (2) is a set of **initial conditions**.

Boundary-Value Problem (BVP)

Remark (Initial vs. Boundary Conditions)

Initial Conditions: all conditions are at the **same** $x = x_0$.
Boundary Conditions: conditions can be at **different** x.

Remark (Number of Initial/Boundary Conditions)

Usually a n-th order ODE requires n initial/boundary conditions to specify an unique solution.

Remark (Order of the derivatives in the conditions)

Initial/boundary conditions can be the value or the function of 0-th to $(n-1)$-th order derivatives, where n is the order of the ODE.

Example (Second-Order ODE)

Consider the following second-order ODE

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{d^{'}y}{dx^{'}} + a_0(x)y = g(x), \tag{3}$$

- IVP: Solve (3) s.t. $y(x_0) = y_0; y^{'}(x_0) = y_1.$
- BVP: Solve (3) s.t. $y(a) = y_0; y(b) = y_1.$
- BVP: Solve (3) s.t. $y(b) = y_0; y(a) = y_1.$

Existence and Uniqueness of the Solution to an IVP

Theorem

For the given linear differential equations of order n

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \quad (4)$$

which is normal on an interval I*. Subject to*

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_2,\ \ldots,\ y^{n-1}(x_0) = y_{n-1}. \quad (5)$$

If $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ *and* $R(x)$ *are all continuous on an interval* I*,* $a_n(x)$ *is not a zero function on* I*, and the initial point* $x_0 \in I$*, then the above IVP has a unique solution in* I*.*

Example (1)

Discuss the Existence of unique solution of IVP

$$\begin{cases} (x^2+1)y'' + x^2y' + 5y = \cos(x) \\ y(3) = 2 \quad , \quad y'(3) = 1. \end{cases}$$

Solution The functions

$$a_2(x) = x^2 + 1, a_1(x) = x^2, a_0(x) = 5.$$

and

$$R(x) = \cos(x).$$

are continuous on $I = \mathbb{R} = (-\infty, \infty)$ and $a_2(x) \neq 0$ for all $x \in \mathbb{R}$, the point $x_0 = 3 \in I$. Then the previous Theorem assures that the IVP has a unique solution on $\mathbb{R}$.

Example (2)

Find an interval I for which the initial values problem (IVP)

$$\begin{cases} x^2y'' + \frac{x}{\sqrt{2-x}}y' + \frac{2}{\sqrt{x}}y = 0 \\ y(1) = 0 \quad , \quad y'(1) = 1. \end{cases}.$$

has a unique solution around $x_0 = 1$.

Solution The function

$$a_2(x) = x^2,$$

is continuous on $\mathbb{R}$ and $a_2(x) \neq 0$ if $x > 0$ or $x < 0$. But $x_0 = 1 \in I_1 = (0, \infty)$. The function

$$a_1(x) = \frac{x}{\sqrt{2-x}} ,$$

is continuous on $I_2 = (-\infty \ , 2)$ and the function

$$a_0(x) = \frac{2}{\sqrt{x}},$$

is continuous on $I_1 = (0, \infty)$.

Then the (IVP)has a unique solution on $I_1 \cap I_2 = (0, 2) = I$. We can take any interval $I_3 \subset (0, 2)$ such that $x_0 = 1 \in I_3$. So I is that the largest interval for which the (IVP) has a unique solution.

Example (3)

Find an interval I for which the IVP

$$\begin{cases} (x-1)(x-3)y'' + xy' + y = x^2 \\ y(2) = 1 \quad , \quad y'(2) = 0 \end{cases} .$$

has a unique solution about $x_0 = 2$.

Solution The functions

$$a_2(x) = (x-1)(x-3)\ ,\ a_1(x) = x\ , a_0(x) = 1\ , R(x) = x^2,$$

are continuous on $\mathbb{R}$. But $a_2(x) \neq 0$ if $x \in (-\infty, 1)$ or $x \in (1,3)$ or $x \in (3, \infty)$. As $x_0 = 2$ so we take $I = (1,3)$. Then the IVP has a unique solution on $I = (1,3)$

Exercises

1. Discuss the Existence of unique solution of IVP

$$\begin{cases} (|x|+3)y'' + x^3 y' + 5y = \sin(x) \\ y(2) = 1 \quad , \quad y'(2) = 0. \end{cases}$$

2. Find an interval I for which the IVP

$$\begin{cases} (x-2)y'' + 3y = x \\ y(0) = 0 \quad , \quad y'(0) = 1 \end{cases}.$$

   has a unique solution about $x_0 = 0$.

3. Find an interval I for which the IVP

$$\begin{cases} y'' + (\tan x)y = e^x \\ y(0) = 1 \quad , \quad y'(0) = 0 \end{cases}.$$

   has a unique solution about $x_0 = 0$.

Linear Dependence and Independence of Functions

Definition

A set of functions $\{f_1(x), f_2(x) \ldots, f_n(x)\}$ are **linearly dependent** on an interval I if $\exists\ c_1, c_2, \ldots, c_n$ not all zero i.e. $(c_1, c_2, \ldots, c_n) \neq (0, 0, \ldots, 0)$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0, \quad \forall\ x \in I$$

that is, the linear combination is a zero function.

If the set of functions is not linearly dependent, it is **linearly independent**, i.e. when $c_1, c_2, \ldots, c_n$ all zero i.e. $(c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0)$.

Example (1)

Show that $f_1(x) = \cos(2x)$, $f_2(x) = 1$, $f_3(x) = \cos^2(x)$ are linearly dependent on $\mathbb{R}$.

Solution We know that

$$f_3(x) = \cos^2(x) = \frac{1+\cos(2x)}{2} = \frac{1}{2}\ f_2(x) + \frac{1}{2}\ f_1(x)$$

for all $x \in \mathbb{R}$. Then there exist $c_1 = c_2 = \frac{1}{2}$ and $c_3 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \qquad \text{for all } x \in \mathbb{R}.$$

So f_1 , f_2 and f_3 are linearly dependent on $\mathbb{R}$.

Example (2)

Show that

$$f_1(x) = 1,\ f_2(x) = \sec^2(x) \text{ and } f_3(x) = \tan^2(x)$$

are linearly dependent on $\left(0, \frac{\pi}{2}\right)$.

Solution We know that

$$f_2(x) = \sec^2(x) = 1 + \tan^2(x) = f_1(x) + f_3(x)$$

hence

$$f_1(x) - f_2(x) + f_3(x) = 0 \text{ for all } x \in \left(0, \frac{\pi}{2}\right).$$

So there exist $c_1 = c_3 = 1$ and $c_2 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \qquad \text{for all } x \in \left(0, \frac{\pi}{2}\right).$$

So f_1 , f_2 and f_3 are linearly dependent on $\left(0, \frac{\pi}{2}\right)$.

Example (3)

Show that $f_1(x) = x$ and $f_2(x) = x^2$ are linearly independent on $I = [-1, 1]$.

Solution Let c_1 , $c_2 \in \mathbb{R}$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0, \quad \text{for all } x \in I \ .$$

We have to prove that $c_1 = c_2 = 0$. As

$$c_1 x + c_2 x^2 = 0 \quad \text{for all } -1 \leq x \leq 1,$$

then for $x = 1$ and $x = -\frac{1}{2}$ we have

$$c_1 + c_2 = 0,$$

and

$$-\frac{1}{2}c_1 + \frac{1}{4}c_2 = 0,$$

which implies that $c_1 = c_2 = 0$. Then f_1 and f_2 are linearly independent on I.

Example (4)

Show that

$$f_1(x) = \sin(x)\ , f_2(x) = \sin(2x).$$

are linearly independent on $I = [0, \pi)$.

Solution Let c_1 , $c_2 \in I$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \text{ for all} x \in I.$$

We have to show that $c_1 = c_2 = 0$. In fact for $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$ we have

$$\begin{cases} c_1 \sin\left(\frac{\pi}{4}\right) + c_2 \sin\left(\frac{\pi}{2}\right) = 0 \\ c_1 \sin\left(\frac{\pi}{3}\right) + c_2 \sin\left(2\frac{\pi}{3}\right) = 0 \end{cases}$$

hence

$$\frac{1}{\sqrt{2}} c_1 + c_2 = 0 \ , \quad \frac{\sqrt{3}}{2} c_1 + \frac{\sqrt{3}}{2} c_2 = 0,$$

which implies that $c_1 = c_2 = 0$. Then f_1 and f_2 are linearly independent on I.

Example (5)

Show that

$$f_1(x) = x^2 \text{ and } f_2(x) = x\,|x|$$

(i) linearly dependent on $[0, 1]$
(ii) linearly independent on $[-1, 1]$

Solution

(i) on $[0, 1]$ we have

$$f_1(x) = f_2(x) = x^2,$$

hence

$$f_1(x) - f_2(x) = 0 \quad \text{for all} \quad 0 \leq x \leq 1.$$

So there exist $c_1 = 1$, $c_2 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \qquad \text{for all} \quad 0 \leq x \leq 1.$$

Then f_1 and f_2 are linearly dependent on $[0, 1]$.

(ii) Let c_1 , $c_2 \in \mathbb{R}$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \text{ for all } -1 \leq x \leq 1,$$

hence

$$c_1 x^2 + c_2 x\,|x| = 0 \text{ for all} -1 \leq x \leq 1.$$

Now for $x = 1$ and $x = -1$ we have $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$ which implies that $c_1 = c_2 = 0$. Then f_1 and f_2 are linearly independent on $[-1, 1]$.

Exercises

1. Determine whether the functions

$$f_1(x) = x \ , f_2(x) = x^2, \ \ f_3(x) = 4x - x^2.$$

are linearly dependent or independent on $(-\infty, \infty)$.

2. Determine whether the functions

$$f_1(x) = e^x \ , f_2(x) = e^{-x}, \ \ f_3(x) = \cosh x.$$

are linearly dependent or independent on $(-\infty, \infty)$.

3. Determine whether the functions

$$f_1(x) = x \ , f_2(x) = x^2 - 1, \ \ f_3(x) = x^2 + 2x + 1.$$

are linearly dependent or independent on $[0, 1]$.

Criterion of Linearly Independent Solutions

Consider the homogeneous linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0,$$

Given n solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$, we would like to test if they are independent or not.

Note: In Linear Algebra, to test if n vectors $\{v_1, v_2, \ldots, v_n\}$ are linearly independent, we can compute the determinant of the matrix.

$$\mathrm{V} := [v_1 \ v_2 \ \ldots \ v_n].$$

If the determinant of $\mathrm{V} = 0$, they are linearly dependent; if the determinant of $\mathrm{V} \neq 0$, they are linearly independent.

Definition

For n functions $W(f_1, f_2, \ldots, f_n)$ which are $n-1$ times differentiable on an interval I, the **Wronskian** $W(x, f_1, f_2, \ldots, f_n)$ as a function on I is defined by

$$W(x, f_1, f_2\ , ..., f_n) = \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ f_1'' & f_2'' & ... & f_n'' \\ ... & ... & ... & ... \\ f_1^{n-1} & f_2^{n-1} & .... & f_n^{n-1} \end{vmatrix}$$

To test the linear independence of n solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ to

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (6)$$

we can use the following theorem.

Theorem

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be n solutions to the homogeneous linear DE (6) on an interval I. They are **linearly independent** *on I*

$$\Longleftrightarrow W(x, f_1, f_2\ , ..., f_n) := \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ f_1'' & f_2'' & ... & f_n'' \\ ... & ... & ... & ... \\ f_1^{n-1} & f_2^{n-1} & .... & f_n^{n-1} \end{vmatrix} \neq 0.$$

Example (1)

Prove that $f_1(x) = x^2,\ f_2(x) = x^2\ln(x)$ are linearly independent on $(0, \infty)$.

Solution We have that

$$W(f_1, f_2) = \begin{vmatrix} x^2 & x^2\ln(x) \\ 2x & 2x\ln(x) + x \end{vmatrix} = 2x^3\ln(x) + x^3 - 2x^3\ln(x) = x^3 \neq 0$$

for all $x \in (0, \infty)$,

then f_1 and f_2 are linearly independent on $(0, \infty)$.

Example (2)

It is easy to see that the functions

$$y_1 = x \ , y_2 = x^2,$$

and

$$y_3 = x^3.$$

are solutions of the differential equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0.$$

Show that y_1 , y_2 and y_3 are linearly independent on $(0, \infty)$.

Solution Here we have $a_3(x) = x^3 \neq 0$ for all $x > 0$ or $x < 0$. By using the Wronskian we have

$$W(y_1, y_2 \ , \ y_3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3 \neq 0.$$

for all $x \in (0, \infty)$ or for all $x \in (-\infty, 0)$. So y_1 , y_2 and y_3 are linearly independent on $(0, \infty)$.

Example

Show that the functions: $f_1(x) = x$, $f_2(x) = x - 1$ and $f_3(x) = x + 3$ are linearly dependent or linearly independent on $\mathbb{R}$.

$$W(f_1, f_2\ ,\ f_3) = \begin{vmatrix} x & x-1 & x+3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0, \text{ for all} x\ \in \mathbb{R}.$$

so these functions might be linearly dependent or linearly independent on $\mathbb{R}$.

Now we need to do the following steps to determine if they are linearly dependent or linearly independent on $\mathbb{R}$.
Let c_1, c_2 and$c_3 \in \mathbb{R}$ thus $c_1 x + c_2(x+1) + c_3(x+3) = 0$ for all$x \ \in \mathbb{R}$.
For $x = 0, x = 1$ and$x = -1$ we have
$-c_2 + 3c_3 = 0$
$c_1 + 4c_3 = 0$
$-c_1 - 2c_2 + 2c_3 = 0$

$$\begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & 4 \\ -1 & -2 & 2 \end{vmatrix} = +1(2+4) + 3(-2) = 6 - 6 = 0$$

so these equations have infinity solutions.
$c_1 = 1, c_2 = -3/4$ and$c_3 = -1/4$, thus we have
$(1)x - 3/4(x-1) - 1/4(x+3) = 0$ for all$x \ \in \mathbb{R}$ So, f_1, f_2 , f_3 are linearly dependent on $\mathbb{R}$.

Exercises

1. Show by computing the Wronskian that the functions

$$f_1(x) = x\ , f_2(x) = xe^x,\ \ f_3(x) = x^2e^x.$$

are linearly dependent or independent on $(0, \infty)$.

2. Show that the functions

$$y_1 = \cosh(2x)$$

and

$$y_2 = \sinh(2x)$$

are solutions of the differential equation

$$y'' - 4y' = 0.$$

Show that y_1 and y_2 are linearly independent on $(-\infty, \infty)$.

Fundamental Set of Solutions

Definition

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (7)$$

Any set $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ of n linearly independent solutions to the homogeneous linear n-th order DE (7) on an interval I is called a **fundamental set of solutions**.

Theorem

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ *be a fundamental set of solutions to the homogeneous linear* n*-th order DE (7) on an interval* I*. Then the* **general solution to** *(7) is*

$$y(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x),$$

where $\{c_i \mid (i = 1, 2, \ldots, n)\}$ *are arbitrary constants.*

Example (1)

Verify that $y_1 = e^{2x}$ and $y_2 = e^{-3x}$ form a fundamental set of solutions of the differential equation

$$y'' + y' - 6y = 0.$$

and find the general solution.

Solution Substituting

$$y_1 = e^{2x}, y_1' = 2e^{2x}, y_1'' = 4e^{2x},$$

in the differential equation we have

$$4e^{2x} + 2e^{2x} - 6e^{2x} = 0.$$

Hence $y_1 = e^{2x}$ is a solution of the differential equation. By the same method we can prove that $y_2 = e^{-3x}$ is also a solution of the differential equation.

Now we have

$$W(e^{2x}, e^{-3x}) = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix} = -5e^{-x} \neq 0 \text{ for all } x \in \mathbb{R}.$$

Then y_1 and y_2 are linearly independent on $\mathbb{R}$. From the previous Theorem we deduce the general solution of the differential equation given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

where c_1 ,$c_2 \in \mathbb{R}$.

Example (2)

It is easy to see that the functions

$$y_1 = e^x, y_2 = e^{2x}, \text{ and } y_3 = e^{3x}$$

are solutions of the differential equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

Find the general solution of the differential equation.

Solution Since

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0.$$

for all $x \in \mathbb{R}$.

We deduce that

$$y(x) = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

is the general solution of the differential equation.

Example (3)

Prove that

$$y_1 = x^3 e^x, \text{ and } y_2 = e^x.$$

are solutions of the differential equation

$$xy'' - 2(x+1)y' + (x+2)y = 0$$

where $x > 0$. Find also the general solution of the differential equation.

Solution Substituting

$$y_1 = x^3 e^x, \ y_1' = 3x^2 e^x + x^3 e^x, \ y_1'' = 6xe^x + 6x^2 e^x + x^3 e^x,$$

in the differential equation we have

$$6x^2e^x + 6x^3e^x + x^4e^x - 6x^3e^x - 2x^4e^{xe^x} - 6x^2e^x + -2x^3e^x + x^4e^x + 2x^3e^x = 0.$$

Substituting

$$y_2 = y_2' = y_2'' = e^x,$$

in the differential equation

$$xe^x - 2xe^x - 2e^x + xe^x + 2e^x = 0.$$

Now we have to show that

$$y_1 = x^3e^x,$$

and

$$y_2 = e^x.$$

are linearly independent on $(0\ ,\ \infty)$.

In fact

$$W(x^3e^x, e^x) = \begin{vmatrix} x^3e^x & e^x \\ 3x^2e^x + x^3e^x & e^x \end{vmatrix} = -3x^2e^x \neq 0 \text{ for all } x > 0.$$

Then

$$y_1 = x^3e^x,$$

and

$$y_2 = e^x.$$

are linearly independent on $(0\ ,\ \infty)$ and we conclude that

$$y_c = c_1 x^3 e^x + c_2 e^x.$$

is the general solution of the differential equation.

Exercises

Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval, then find the general solution of the differential equation.

- $y'' - y' - 12y = 0; \quad e^{-3x}, \ e^{4x}$ on $(-\infty, \infty)$
- $x^3y''' + 6x^2y'' + 4xy' - 4y = 0; \quad x, \ x^{-2}, \ x^{-2}\ln x$ on $(0, \infty)$
- $y^{(4)} + y'' = 0; \quad 1, \ x, \ \sin x, \ \cos x$ on $(0, \infty)$

Reduction of order Method (when one solution is given)

It is employed when one solution $y_1(x)$ is known and a second linearly independent solution $y_2(x)$ is desired. The method also applies to n-th order equations.

Suppose that $y_1(x)$ is a non-zero solution of the equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \qquad (8)$$

where $a_0(x),\ a_1(x)$ and $a_2(x)$ are continuous functions defined on interval I sauch that $a_2(x) \neq 0$ for all $x \in I$.

The method of reduction of order is used to obtain a second linearly independent $y_2(x)$ solution to this differential equation (8) using our one known solution.

We suppose that the solution of (8) is in the form

$$y = u(x)y_1,$$

where u is a fountain of x and which will be determined and satisfies a linear second-order differential equation (8) by using the following method

$$y = u(x)y_1 \Rightarrow y' = u'y_1 + y_1'u \Rightarrow y'' = u''y_1 + 2u'y_1' + y_1''u.$$

It is best to describe the procedure with a concrete example.

Example (1)

If

$$y_1 = \frac{\sin x}{\sqrt{x}}.$$

is a solution of the differential equation

$$4x^2y'' + 4xy' + (4x^2 - 1)y = 0 \quad on \ 0 < x < \pi.$$

then find the general solution of the differential equation..

Solution The solution of the differential equation is of the form $y = u(x)y_1$ or

$$y = \frac{\sin x}{\sqrt{x}} u \ = (\sin x)\,(x)^{\frac{-1}{2}}\, u,$$

hence

$$y' = (\cos x)(x)^{\frac{-1}{2}} u - \frac{1}{2} \sin x (x)^{\frac{-3}{2}} u + \sin x (x)^{\frac{-1}{2}} u',$$

$$y'' = -\sin x(x)^{\frac{-1}{2}}u - \cos x(x)^{\frac{-3}{2}}u + 2\cos x(x)^{\frac{-1}{2}}u' + \frac{3}{4}\sin x(x)^{\frac{-5}{2}}u - \sin x(x)^{\frac{-3}{2}}u' + \sin x(x)^{\frac{-1}{2}}u''$$

we substitute y , y', and y'' in the arbitrary constant we obtain

$$4x^{\frac{3}{2}}\sin x u'' + \left(8x^{\frac{3}{2}}\cos x\right)u' = 0,$$

hence

$$\sin x u'' + 2\cos x u' = 0.$$

To solve this differential equation we put $w = u'$, then we have $w' = u''$.

Then

$$\int \frac{dw}{w}dx + \int \frac{2\cos x}{\sin x}dx = 0,$$

hence

$$u' = w = \frac{c_1}{\sin^2 x},$$

where $c_1 \neq 0$ is an arbitrary constant. So we have $u = -c_1 \cot x + c_2$, hence

$$y = y_1 u = \frac{\sin x}{\sqrt{x}}(-c_1 \cot x + c_2),$$

or

$$y = c_3 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\sin x}{\sqrt{x}},$$

finally we have

$$y = c_2 y_1 + c_3 y_2,$$

where $c_3 = -c_1$ and c_2 are arbitrary constants, is the general solution of the differential equation and we can prove that

$$y_1 = \frac{\sin x}{\sqrt{x}} \text{ and } y_2 = \frac{\cos x}{\sqrt{x}}$$

are linearly independent on solutions $(0, \pi)$.

General case of Equation (8)

Equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

can be written as the form

$$y'' + p(x)y' + q(x)y = 0, \tag{9}$$

where

$$p(x) = \frac{a_1(x)}{a_2(x)},$$

and

$$q(x) = \frac{a_0(x)}{a_2(x)}.$$

Also, let us suppose that y_1 is a known solution of (9) on I and $y_1(x) \neq 0$ for all $x \in I$.

Thus the second solution of (9) y_2 can be given from

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}\, dx. \tag{10}$$

Example (1)

If

$$y_1 = \frac{\sin x}{\sqrt{x}}$$

is a solution of the differential equation

$$4x^2y'' + 4xy' + (4x^2 - 1)y = 0 \quad \text{on} \quad (0, \pi),$$

then find the second solution .

Solution As

$$y'' + \frac{1}{x}y' + \frac{4x^2 - 1}{4x^2}y = 0.$$

then

$$p(x) = \frac{1}{x},$$

and

$$e^{-\int p(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}.$$

We have

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = \frac{\sin x}{\sqrt{x}} \int \frac{\frac{1}{x}}{\frac{\sin^2(x)}{x}} dx,$$
$$= \frac{\sin x}{\sqrt{x}} \int \frac{dx}{\sin^2(x)} = \frac{-\cos x}{\sqrt{x}}.$$

Hence

$$y_2 = \frac{-\cos x}{\sqrt{x}} \quad \text{or} \quad y_2 = \frac{\cos x}{\sqrt{x}}$$

is the second solution of the differential equation on $(0, \pi)$.

Example (2)

If $y_1 = e^{3x}$ is a solution of the differential equation

$$xy'' + (x-1)y' + (3-12x)y = 0 \quad ; \quad x > 0.$$

Find the general solution.

Solution We have

$$y'' + (1 - \frac{1}{x})y' + (\frac{3}{x} - 12)y = 0.$$

From the formula (10) we can find directly y_2, where

$$\int -p(x)dx = \int (-1 + \frac{1}{x})dx = -x + \ln x,$$

hence

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = e^{3x} \int \frac{e^{-x+\ln x}}{e^{6x}} dx = e^{3x} \int x e^{-7x} dx \\ &= e^{3x} \left[-\frac{1}{7} x e^{-7x} - \frac{1}{49} e^{-7x} \right] \\ &= e^{-4x} \left(\frac{-x}{7} - \frac{1}{49} \right) = -\frac{1}{7} e^{-4x} \left(x + \frac{1}{7} \right). \end{aligned}$$

Then the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 e^{-4x} \left(x + \frac{1}{7} \right)$$

on the interval $(0, \infty)$.

Exercises

Apply the reduction of order method to obtain another linearly independent solution for the following differential equations

- $y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}.$
- $x^2y'' + x^2y' - (x+2)y = 0; \quad y_1 = x^{-1}e^{-x}, \; x > 0.$
- $x^2(1 - \ln x)y'' + xy' - y = 0; \quad y_1 = x, \; x > e.$

Homogeneous Linear Differential Equations with Constant Coefficients

The linear differential equations with Constant Coefficients has the general form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0, \tag{11}$$

which is a homogeneous linear DE with **constant real coefficients**, where each coefficient $a_i, 1 \leq i \leq n$ is real constant and $a_n \neq 0$.

Definition

The polynomial

$$f(m) = a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0, \tag{12}$$

is called the characteristic polynomial for equation (11), and $f(m) = 0$ is called the characteristic equation of the linear differential equations with constant coefficients (11).

We conclude that if m is a root of equation (54), then

$$y = e^{mx}$$

is a solution of the differential equation (11). Also, Equation (54) has n roots.

Let us summarize the method to solve the differential equation (11)

(1) If all the roots of the characteristic equation are **real roots** then:

(i) If the roots are distinct (i.e. $m_1 \neq m_2 \neq m_3 \neq \cdots \neq m_n$), then the solution of the differential equation (11) is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

(ii) If the roots are equal (i.e. $m_1 = m_2 = m_3 = \cdots = m_n$) (i.e. $m = m_i$ is a root of multiplicity n), then the solution of the differential equation (11) is given by

$$y = c_1 e^{mx} + c_2 x e^{mx} + c_3 x^2 e^{mx} + \cdots + c_n x^{n-1} e^{mx}$$

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) e^{mx}$$

Example (1)

Solve the differential equation

$$y'' - y = 0.$$

Solution For this, the characteristic equation is $m^2 - 1 = 0$ hence $m = \mp 1$.Then $y_1 = e^x$ and $y_2 = e^{-x}$ form the fundamental set of solutions, hence the general solution is

$$y = c_1 e^x + c_2 e^{-x}.$$

Example (2)

Find the general solution of the differential equation

$$y''' - 6y'' + 11y' - 6y = 0\ .$$

Solution For this differential equation the characteristic equation is

$$m^3 - 6m^2 + 11m - 6 = (m-1)(m-2)(m-3) = 0.$$

Then $m = 1, 2, 3$ and $y_1 = e^x$, $y_2 = e^{2x}$ and $y_3 = e^{3x}$ form the fundamental set of solutions, hence the general solution is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

Example (3)

Solve the differential equation

$$y'' - 2y' + y = 0.$$

Solution The characteristic equation for the differential equation is

$$m^2 - 2m + 1 = 0 \ ,$$

so $m = 1$ is a root of multiplicity 2, hence the general solution is

$$y = c_1 e^x + c_2 x e^x.$$

Example (4)

Solve the differential equation

$$y''' - 3y'' + 3y' - y = 0$$

Solution The characteristic equation for the differential equation is $m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0$, so $m = 1$ is a root of multiplicity 3 then the general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x.$$

Now we see the second case

(2) If the characteristic equation has **complex conjugate roots** such as

$$m = \alpha \mp i\beta$$

then he solution of the differential equation of second order is given by

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

Remember:

$$1)\ \sqrt{-1} = i$$

$$2)\ x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

to find the roots of Quadratic equation

$$ax^2 + bx + c = 0$$

Example (5)

Solve the differential equation

$$y'' + 4y' + 5y = 0.$$

Solution The characteristic (auxiliary) equation for the differential equation is $m^2 + 4m + 5 = 0$, now we need to find the roots of this characteristic equation

$$m = \frac{-4 \mp \sqrt{16 - 20}}{2}$$

then $m = -2 \mp i$ hence the general solution is

$$y(x) = c_1 e^{-2x} \cos(x) + c_2 e^{-2x} \sin(x).$$

Example (6)

Solve the differential equation

$$y^{(5)} - 3y^{(4)} + 4y''' - 4y'' + 3y' - y = 0.$$

Solution The characteristic for the differential equation is

$$m^5 - 3m^4 + 4m^3 - 4m^2 + 3m - 1 = 0,$$

then

$$m^5 - 3m^4 + 4m^3 - 4m^2 + 3m - 1 = (m-1)^3(m^2+1) = 0.$$

Thus $m = 1$, 1 ,1 , $\mp i$ where $\sqrt{-1} = i$ and the general solution of the equation has the form

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 \cos x + c_5 \sin x.$$

Example (7)

Solve the initial value problem (IVP)

$$\begin{cases} y'' + y' + y = 0 \\ y(0) = 1 \quad , \quad y'(0) = \sqrt{3}. \end{cases}$$

Solution The characteristic equation for the differential equation is

$$m^2 + m + 1 = 0.$$

Hence

$$m = -\frac{1}{2} \mp \frac{\sqrt{3}}{2} i.$$

So the general solution of a differential equation is

$$y = c_1 e^{\frac{-x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right) + c_2 e^{\frac{-x}{2}} \sin\left(\frac{\sqrt{3}}{2} x\right).$$

from the conditions $y(0) = 1$ and $y'(0) = \sqrt{3}$ we have $c_1 = 1.$ and

$$\frac{-c_1}{2} + c_2\frac{\sqrt{3}}{2} = \sqrt{3}$$

hence $c_1 = 1$ and $c_2 = 2 + \frac{1}{\sqrt{3}}$. So the solution of the IVP is

$$y = e^{\frac{-x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right) + (2 + \frac{1}{\sqrt{3}})e^{\frac{-x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

Exercises

Find the general solution of the following differential equations

- $2y'' + 3y' + y = 0$
- $y'' - y' - 6y = 0$
- $y''' - 4y'' - 5y' = 0$
- $y^{(4)} - 2y'' + y = 0$
- $2y^{(5)} - 7y^{(4)} + 12y''' + 8y'' = 0$

Find the solution of the initial value problems

- $y'' + y' + 2y = 0; \;\; y(0) = y'(0) = 0$
- $y''' + 12y'' + 36y' = 0; \;\; y(0) = 0, \; y'(0) = 1, \; y''(0) = -7$

Cauchy-Euler Differential Equation

A Cauchy-Euler differential equation is in the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0, \qquad (13)$$

where each coefficient $a_i, 1 \leq i \leq n$ are constants and $a_n \neq 0$ i.e. the coefficient $a_n x^n$ should never be zero. Equation (13) is on the interval either $(0, \infty)$ or $(-\infty, 0)$.

Euler differential equation is probably the simplest type of linear differential equation with variable coefficients.

The most common Cauchy-Euler equation is the second-order equation, appearing in a number of physics and engineering applications, such as when solving Laplace's equation in polar coordinates.

It is given by the equation

$$x^2 \frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0 \tag{14}$$

To solve the Cauchy-Euler differential equation, we assume that $y = x^m$, where $x > 0$ and m is a root of a polynomial equation.

Example (1)

Solve the Cauchy-Euler differential equation

$$x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=0.$$

Solution We substitute

$$y=x^m \implies y'=mx^{m-1} \Longrightarrow y''=m(m-1)x^{m-2}$$

in the differential equation, we obtain

$$x^2[m(m-1)x^{m-2}]+ax[mx^{m-1}]+bx^m=0$$
$$x^m(m^2-m)+amx^m+bx^m=0$$
$$x^m[(m^2-m)+am+b]=0$$
$$x^m[m^2+(1-a)m+b]=0.$$

Since $x^m \neq 0$, then we have

$$m^2 + (1-a)m + b = 0$$

We then can solve for m. There are three particular cases of interest:

Case 1: Two distinct roots, m_1 and m_2. Thus, the solution is given by

$$y = c_1 x^{m_1} + c_2 x^{m_2}.$$

Case 2: One real repeated root, m. Thus, the solution is given by

$$y = c_1 x^m \ln(x) + c_2 x^m.$$

Case 3: Complex roots, $\alpha \pm i\beta$. Thus, the solution is given by

$$y = c_1 x^\alpha \cos\left(\beta \ln(x)\right) + c_2 x^\alpha \sin\left(\beta \ln(x)\right).$$

Example (2)

Solve the Euler differential equation

$$2x^2y'' - 3xy' - 3y = 0. \tag{15}$$

For $x > 0$.

Solution We substitute

$$y = x^m \implies y' = mx^{m-1} \implies y'' = m(m-1)x^{m-2}$$

in the differential equation, we obtain

$$2x^2[m(m-1)x^{m-2}] - 3x[mx^{m-1}] - x^m = 0$$

$$x^m(2m^2 - 2m) - 3mx^m - 3x^m = 0$$

$$x^m[2m^2 - 2m - 3m - 3] = 0$$

$$x^m[2m^2 - 5m - 3] = 0.$$

Since $x^m \neq 0$, then we have

$$2m^2 - 5m - 3 = 0$$

So the roots of this equation are $m_1 = -\frac{1}{2}$, $m_2 = 3$ .Thus, from case 1 we have the solution is given by

$$y(x) = c_1 x^{-\frac{1}{2}} + c_2 x^3.$$

which is the general solution.

Example (3)

Find the general of the differential equation

$$x^2y'' - 3xy' + 13y = 0 \quad ; \quad x > 0.$$

Solution Substituting $y = x^m$ in the equation, we obtain

$$m(m-1) - 3m + 13 = m^2 - 4m + 13 = 0.$$

Then we have two complex roots $m = 3 \mp 3i$ (case 3), hence the the general of the differential equationis

$$y = c_1x^3\cos(3\ln x) + c_2x^3\sin(3\ln x) \quad ; \quad x > 0.$$

If we suppose $x < 0$, then the general of the differential equation is

$$y = c_1(-x)^3\cos(3\ln(-x)) + c_2(-x)^3\sin(3\ln(-x)) \quad ; \quad x < 0.$$

Example (4)

Find the general solution of the differential equation

$$x^4y^{(4)} - 5x^3y''' + 3x^2y'' - 6xy' + 6y = 0 \quad ; \quad x > 0.$$

Solution Substituting $y = x^m$ in the equation, we obtain

$$m(m-1)(m-2)(m-3) - 5m(m-1)(m-2) + 3m(m-1) - 6m + 6 = 0.$$

This implies that

$$(m-1)(m-2)(m^2 - 8m + 3) = 0.$$

The roots of this equation are $m = 1$, $m = 2$, and $m = 4 \mp \sqrt{13}$, then the general solution of the differential equation is

$$y = c_1x + c_2x^2 + c_3x^{4+\sqrt{13}} + c_4x^{4-\sqrt{13}} \quad ; \quad x > 0.$$

Example (5)

Find the general solution of the differential equation

$$x^5y^{(5)} - 2x^3y''' + 4x^2y'' = 0 \quad ; \quad x < 0.$$

Solution Substituting $y = x^m$ in the equation, we obtain

$$m(m-1)(m-2)(m-3)(m-4) - 2m(m-1)(m-2) + 4m(m-1) = 0,$$

$$m(m-1)(m^3 - 9m^2 + 24m - 20) = m(m-1)(m-2)^2(m-5) = 0.$$

So the roots of this equation are $m = 0$, $m = 1$, $m = 2$ repeated two times and $m = 5$, then the general of the differential equation is

$$y = c_1 + c_2(-x) + c_3(-x)^2 + c_4(-x)^2\ln(-x) + c_5(-x)^5.$$

Exercises

Find the general solution of the following differential equations, where we suppose that $x > 0$.

- $x^2y'' - y = 0$
- $x^2y'' + 5xy' + 3y = 0$
- $4x^2y'' + 4xy' - y = 0$
- $x^3y''' + xy' - y = 0$
- $x^3y''' + 4x^2y'' - 8xy' + 8y = 0$
- $(3x+4)^2y'' + 10(3x+4)y' + 9y = 0;\ x > -\dfrac{4}{3}$

General Solutions of Nonhomogeneous Linear Differential Equations

Nonhomogeneous linear n-th order ODE takes the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (16)$$

where $a_n(x),\ a_{n-1}(x),\ a_1(x)$ and $a_0(x)$ are functions of $x \in \mathrm{I} = (a, b)$, such that $a_n(x) \neq 0$ for all $x \in I$, and $g(x) \neq 0$.

Idea

- Find the general solution y_c to the homogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

- Find a solution y_p to the nonhomogeneous equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

- The general solution $y = y_c + y_p$.

Undetermined coefficients

Let us take an example

Example (1)

Find the general solution of the differential equation :

$$y'' - y = -2x^2 + 5 + 2e^x. \quad (*)$$

Solution

1) First we have to find the general solution of the differential equation :

$$y'' - y = 0.$$

For , we have $m^2 - 1 = 0$, hence $m = \mp 1$ then

$$y_c = c_1 e^x + c_2 e^{-x}.$$

2) The form of the particular solution of

$$y'' - y = -2x^2 + 5,$$

is

$$y_{1,p} = Ax^2 + Bx + C,$$

and the form of the particular solution of

$$y'' - y = 2e^x,$$

is

$$y_{2,p} = Dxe^x,$$

because $r = 1$ is a simple root of the characteristic equation. Thus the particular solution of (*) is

$$y_p = y_{1,p} + y_{2,p} = Ax^2 + Bx + C + Dxe^x.$$

Now we have to find the constants A, B, C , and D by substituting y_p and y_p'' in differential equation (*) and we find

$$y_p'' - y_p = -Ax^2 - Bx + 2A - C + 2De^x = -2x^2 + 5 + 2e^x.$$

Equating coefficients of similar terms (because the functions $x^2, 1$ and are e^x linearly independent on $\mathbb{R}$), we obtain the following system of equation $A = 2$, $B = 0$, $2A - C = 5$, and $2D = 2$. Thus we have $A = 2$, $B = 0$, $C = -1$, and $D = 1$.Then the particular solution of (*) is

$$y_p = 2x^2 - 1 + xe^x,$$

and the general solution of the differential equation of (*) is

$$y = y_c + y_p = c_1 e^x + c_2 e^{-x} + 2x^2 - 1 + xe^x.$$

Some of the Typical forms of the particular integral

Function of x	Form for y_p
ke^{ax}	Ce^{ax}
$kx^n,\ n = 0, 1, 2, \ldots$	$\sum_{i=1}^{n} C_i x^i$
$k\cos(ax)$ or $k\sin(ax)$	$C_1\cos(ax) + C_2\sin(ax)$
$ke^{ax}\cos(bx)$ or $ke^{ax}\sin(bx)$	$e^{ax}\left(C_1\cos(bx) + C_2\sin(bx)\right)$
$\left(\sum_{i=1}^{n} k_i x^i\right)\cos(ax)$ or $\left(\sum_{i=1}^{n} k_i x^i\right)\sin(ax)$	$\left(\sum_{i=1}^{n} C_i x^i\right)\cos(ax) + \left(\sum_{i=1}^{n} R_i x^i\right)\sin(ax)$

Exercises

Find the general solution of the following differential equations.

- $x^2y'' - y = 0$
- $y'' + 4y = sin(2x) + e^x$
- $y'' - 5y' + 4y = e^{2x}(\cos x + \sin x)$

Find only the form of the particular solution of the given differential equation by using the method of undetermined coefficients.

- $y'' - y = e^x + s\sin x$
- $y'' - y = x^2e^x$
- $y^{(6)} - 3y^{(3)} = 3x + 1$
- $y''' - y' = x^5 + \cos x$

Variation of Parameters

This method is used to determine the particular solution y_p of nonhomogeneous differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (17)$$

If we have the nonhomogeneous differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \qquad (18)$$

which has the particular solution

$$y_p = y_1u_1 + y_2u_2,$$

where y_1 and y_2 are the first and the second solution of the homogeneous differential equation, respectively.

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (19)$$

Here we will explain the method to find u_1 and u_2. So, if we have y_1 & y_2, then we will determine as below

$$W(x, y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1',$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix} = -y_2 g(x),$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix} = y_1 g(x).$$

Thus,

$$u_1' = \frac{W_1}{W}$$

and

$$u_2' = \frac{W_2}{W}.$$

Example (1)

Solve the differential equation

$$y'' + y = \csc x \quad ; \quad 0 < x < \pi.$$

Solution

1) The general solution of

$$y'' + y = 0,$$

is

$$y_c = c_1 \sin x + c_2 \cos x.$$

2) The particular solution of

$$y'' + y = \csc x,$$

is the form

$$y_p = y_1 u_1 + y_2 u_2,$$

where

$$y_1 = \sin x \quad and \quad y_2 = \cos x.$$

The functions u_1 and u_2 are determined from the system below

$$W(x, y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1,$$

$$W_1 = \begin{vmatrix} 0 & \cos x \\ \csc x & -\sin x \end{vmatrix} = -\cot x,$$

$$W_2 = \begin{vmatrix} \sin x & 0 \\ \cos x & \csc x \end{vmatrix} = 1,$$

Hence

$$u_1' = \frac{W_1}{W} = \cot x,$$

then

$$u_1 = \ln(\sin x).$$

But

$$u_2' = -1,$$

hence $u_2 = -x$. Therefore we have

$$y_p = y_1 u_1 + y_2 u_2 = \sin x . \ln(\sin x) - x \cos x,$$

and the general solution of the differential equation is

$$y = y_c + y_p = c_1 \sin x + c_2 \cos x + \sin x . \ln(\sin x) - x \cos x.$$

Example (2)

Solve the differential equation

$$y'' - 4y' + 4y = (x+1)e^{2x}.$$

Solution

1) The general solution of

$$y'' - 4y' + 4y = 0,$$

is

$$y_c = c_1 e^{2x} + c_2 x e^{2x}.$$

2) Let

$$y_1 = e^{2x} \ \ and \ \ y_2 = xe^{2x}.$$

So we have

$$W(x, y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x},$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = -x(x+1)e^{4x},$$

and

$$W_2(x, y_1, y_2) = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x},$$

hence

$$u_1' = \frac{W_1}{W} = -x(x+1) = -x^2 - x,$$

so

$$u_1 = -\frac{x^3}{3} - \frac{x^2}{2}$$

But

$$u_2' = \frac{W_2}{W} = x + 1,$$

then

$$u_2 = \frac{x^2}{2} + x.$$

Therefore,

$$y_p = y_1u_1 + y_2u_2 = (-\frac{x^3}{3} - \frac{x^2}{2})e^{2x} + x(\frac{x^2}{2} + x)e^{2x} = (\frac{x^3}{6} + \frac{x^2}{2})e^{2x},$$

and The general solution of the differential equation is

$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + (\frac{x^3}{6} + \frac{x^2}{2})e^{2x}.$$

In this example we can use the undetermined coefficients, where

$$y_p = x^2(A + Bx)e^{2x}.$$

Example (3)

Solve the Differential equation

$$y'' - 3y' + 2y = \frac{1}{1 + e^{-x}}.$$

Solution

1) The general solution of

$$y'' - 3y' + 2y = 0.$$

is

$$y_c = c_1 e^{x} + c_2 e^{2x}.$$

2) Let

$$y_1 = e^{x} \text{ and } y_2 = e^{2x},$$

then

$$W(x, y_1, y_2) = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = e^{3x},$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & e^{2x} \\ \frac{1}{1+e^{-x}} & 2e^{2x} \end{vmatrix} = \frac{-e^{2x}}{1+e^{-x}},$$

$$W_2(x, y_1, y_2) = \begin{vmatrix} e^{x} & 0 \\ e^{x} & \frac{1}{1+e^{-x}} \end{vmatrix} = \frac{e^{x}}{1+e^{-x}},$$

hence

$$u_1' = \frac{W_1}{W} = -\frac{e^{-x}}{1+e^{-x}}$$

and

$$u_1(x) = -\int \frac{e^{-x}}{1+e^{-x}}dx = \ln\big(1+e^{-x}\big).$$

But

$$u_2' = \frac{W_2}{W} = \frac{e^{-2x}}{1+e^{-x}},$$

and

$$u_2 = \int \frac{e^{-2x}}{1+e^{-x}}dx = -(1+e^{-x}) + \ln\big(1+e^{-x}\big),$$

so we have

$$\begin{aligned} y &= y_c + y_p = (c_1 - 1)e^x + (c_2 - 1)e^{2x} + \left(e^x + e^{2x}\right)\ln\left(1 + e^{-x}\right), \\ &= c_3e^x + c_4e^{2x} + \left(e^x + e^{2x}\right)\ln\left(1 + e^{-x}\right). \end{aligned}$$

Example (4)

Find the general solution of the differential equation

$$y''' + y' = \tan x \quad ; \quad 0 < x < \frac{\pi}{2}.$$

Solution

1) The the general solution of

$$y''' + y' = 0,$$

is

$$y_c = c_1 + c_2 \cos x + c_3 \sin x.$$

2) Let $y_1 = 1$, $y_2 = \cos x$ and $y_3 = \sin x$. The particular solution of the differential equation has the form

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3.$$

We have

$$W(x, y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1,$$

$$W_1(x, y_1, y_2, y_3) = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x \ ,$$

$$W_2(x, y_1, y_2, y_3) = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = -\sin x \ ,$$

$$W_3(x, y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = \frac{-\sin^2(x)}{\cos x}.$$

Then we have,

$$u_1' = \frac{W_1}{W} = \tan x,$$

and

$$u_1 = \int \tan x dx = -\ln(\cos x)\ .$$

But

$$u_2' = \frac{W_2}{W} = -\sin x\ ,$$

then

$$u_2 = -\int \sin x dx = \cos x.$$

Also

$$u_3' = \frac{W_3}{W} = \frac{-\sin^2(x)}{\cos x},$$

hence,

$$u_3 = -\int \frac{\sin^2(x)}{\cos x}dx = -\int \frac{1-\cos^2(x)}{\cos x}dx = -\ln(\sec x + \tan x) + \sin x.$$

Thus,

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 + u_3y_3, \\ &= -\ln(\cos x) + \cos^2(x) - \sin x \ln(\sec x + \tan x) + \sin^2(x), \\ &= 1 - \ln(\cos x) - \sin x \ln(\sec x + \tan x). \end{aligned}$$

So the general solution of the differential equation is

$$\begin{aligned} y &= y_c + y_p = (c_1 + 1) + c_2 \cos x + c_3 \sin x - \ln(\cos x) \\ &- \sin x \ln(\sec x + \tan x) \end{aligned}$$

$$y = c_4 + c_2 \cos x + c_3 \sin x - \ln(\cos x) - \sin x \ln(\sec x + \tan x).$$

Example (5)

Find the solution of the initial value problem (IVP)

$$\begin{cases} 2x^2y'' + xy' - 3y = x^{-3} \quad ; \quad x > 0 \\ y(1) = 1 \quad , \quad y'(1) = -1. \end{cases}$$

Solution

1) We have to find the general solution of

$$2x^2y'' + xy' - 3y = 0.$$

By substituting $y = x^m$, we have

$$m(m-1) + m - 3 = (2m-3)(m+1) = 0,$$

hence the general solution of the homogeneous differential equation is

$$y_c = c_1\, x^{-1} + c_2\, x^{\frac{3}{2}}.$$

2) Let $y_1 = x^{-1}$, $y_2 = x^{\frac{3}{2}}$, then

$$y_p = u_1 y_1 + u_2 y_2.$$

We have

$$W(x, y_1, y_2) = \begin{vmatrix} x^{-1} & x^{\frac{3}{2}} \\ -x^{-2} & \frac{3}{2}x^{\frac{1}{2}} \end{vmatrix} = \frac{5}{2}x^{-\frac{1}{2}},$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & x^{\frac{3}{2}} \\ \frac{1}{2}x^{-5} & \frac{3}{2}x^{\frac{1}{2}} \end{vmatrix} = -\frac{1}{2}x^{-\frac{7}{2}},$$

$$W_2(x, y_1, y_2) = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{1}{2}x^{-5} \end{vmatrix} = \frac{1}{2}x^{-6}.$$

Then we have

$$u_1' = \frac{W_1}{W} = -\frac{1}{5}x^{-3},$$

and

$$u_1 = \frac{1}{10}x^{-2}.$$

Also we have

$$u_2' = \frac{W_2}{W} = \frac{1}{5}x^{-\frac{11}{2}},$$

hence

$$u_2 = -\frac{2}{45}x^{-\frac{9}{2}}.$$

So

$$y_p = u_1y_1 + u_2y_2 = \frac{1}{10}x^{-3} - \frac{2}{45}x^{-3} = \frac{1}{18}x^{-3}.$$

Then the general solution of the differential equation is

$$y = y_c + y_p = c_1\ x^{-1} \quad + c_2\ x^{\frac{3}{2}} + \frac{1}{18}x^{-3}.$$

We can obtain y_p by substituting $y_p = Ax^{-3}$, which implies $A = \frac{1}{18}$.

3)

$$y'(x) = -c_1 x^{-2} + \frac{3}{2} c_2 x^{\frac{1}{2}} - \frac{1}{6} x^{-4}.$$

From the conditions $y(1) = 1$ and $y'(1) = -1$, we deduce

$$c_1 + c_2 = \frac{17}{18},$$

and

$$-c_1 + \frac{3}{2} c_2 = -\frac{5}{6},$$

which implies $c_1 = \frac{9}{10}$ and $c_2 = \frac{2}{45}$. Thus the solution of the IVP is

$$y = \frac{9}{10} x^{-1} + \frac{2}{45} x^{\frac{3}{2}} + \frac{1}{18} x^{-3}.$$

Exercises

Use the variation of parameters method to find the general solution or initial value problems of the following differential equations.

- $y'' + y = \sec x; \quad 0 < x < \dfrac{\pi}{2}$
- $y'' - 2y' + y = \dfrac{e^x}{x}; \quad x > 0$
- $y'' - 12y' + 36y = e^{6x} \ln x; \quad x > 0$
- $y'' - 2y' + y = \dfrac{e^x}{(e^x + 1)^2}$
- $y'' - y = \dfrac{2}{\sqrt{1 - e^{-2x}}}$
- $y''' + 4y' = \sec 2x; \quad 0 < x < \dfrac{\pi}{4}$
- $2y''' - 6y'' = x^2$

- $y'' + y = \tan x; \;\; y\left(\frac{\pi}{3}\right) = 1, \;\; y'\left(\frac{\pi}{3}\right) = 0$
- $y'' + y = \sec^3(x); \;\; y(0) = 1, \;\; y'(0) = 1$
- $y'' - 2y' + y = \frac{e^x}{x}; \;\; y(1) = e, \;\; y'(1) = 0$