King Saud University
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254 Math Exercises
Approximation Function
Ch. (4)

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Lagrange Interpolating Polynomials

$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1); \quad x_0, x_1 \; given$$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad , L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$; x_0, x_1, x_2 \; given$$

$$L_0 = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \quad , L_1 = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2 = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

In general

$$P_n(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + \cdots + L_n(x)f(x_n)$$

Error Formula:

$$|f(x) - P_n(x)| \leq \frac{f^{(n+1)}(\eta)}{n!}(x - x_0)(x - x_1) \ldots (x - x_n)$$

$$whrer \; f^{(n+1)}(\eta) = \max_{x_0 \leq x \leq x_n} |f(x)|$$

Example: Given $f(0) = 1$, $f(1) = 1.5$, $f(2) = 2$, $f(3) = 2.5$ *and* $f(4) = 3$. Use the quadratic Lagrange interpolating formula to approximate $f(3.5)$.

Solution:

$P_2(x) = L_0(x)\, f(x_0) + L_1(x)\, f(x_1) + L_2(x)\, f(x_2)$

since $x = 3.5$

$\Rightarrow x_0 = 2,\ x_1 = 3,\ x_2 = 4$

$\Rightarrow P_2(3.5) = L_0(3.5)\, f(2) + L_1(3.5)\, f(3) + L_2(3.5)\, f(4)$

$= \dfrac{(3.5-3)(3.5-4)}{(2-3)(2-4)}(2) + \dfrac{(3.5-2)(3.5-4)}{(3-2)(3-4)}(2.5) + \dfrac{(3.5-2)(3.5-3)}{(4-2)(4-3)}(3)$

$= (-0.125)(2) + (0.75)(2.5) + (0.375)(3)$

$\Rightarrow P_2(3.5) = 2.75$

Example: Let $p_2(x)$ be the quadratic Lagrange interpolating polynomial for data: $(0\,,0), (1\,,\alpha), (2\,,3)$. Find the value if α if the coefficient of x in $p_2(x)$ is 3.

Solution:

$P_2(x) = L_0(x) f(0) + L_1(x) f(1) + L_2(x) f(2)$

$x_0 = 0 \quad ; \quad x_1 = 1 \quad , \quad x_2 = 2$

$f(x_0) = f(0) = 0 \quad , \quad f(x_1) = f(1) = \alpha \quad , \quad f(x_2) = f(2) = 3$

0

$\Rightarrow P_2(x) = \cancel{L_0(x)(0)} + L_1(x)(\alpha) + L_2(x)(3)$

$$= \frac{(x-0)(x-2)}{(1-0)(1-2)}(\alpha) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(3)$$

$$= -\alpha(x^2 - 2x) + \frac{3}{2}(x^2 - x)$$

$$= -\alpha x^2 + 2\alpha x + \tfrac{3}{2}x^2 - \tfrac{3}{2}x$$

$$P_2(x) = \left(-\alpha + \tfrac{3}{2}\right)x^2 + \left(2\alpha - \tfrac{3}{2}\right)x$$

Now since the cofficient of x in P_2 is 3

$\Rightarrow$

$$2\alpha - \tfrac{3}{2} = 3 \Rightarrow 2\alpha = \tfrac{9}{2}$$

$$\Rightarrow \alpha = \frac{9}{4}$$

Example: Let $p_2(x)$ be the quadratic Lagrange interpolating polynomial for data: $(1, 2), (2, 3), (3, \alpha)$. Find the value of α if the coefficient of x^2 in $p_2(x)$ is 1. Find the approximation of $f(2.5)$ using quadratic polynomial $p_2(2.5)$

Solution:

$x_0 = 1 \quad ; \quad x_1 = 2 \quad , x_2 = 3$

$f(1) = 2 \quad f(2) = 3 \quad f(3) = \alpha$

$P_2(x) = L_0(x)(2) + L_1(x)(3) + L_2(x)(\alpha)$

Now

$$L_0(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}(x^2 - 5x + 6)$$

$$L_1(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = -(x^2 - 4x + 3)$$

$$L_2(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{1}{2}(x^2 - 3x + 2)$$

$$\Rightarrow P_2(x) = \tfrac{1}{2}(x^2-5x+6)(2) - (x^2-4x+3)(3) + \tfrac{1}{2}(x^2-3x+2)(\alpha)$$

$$= x^2 - 5x + 6 - 3x^2 + 12x - 9 + \alpha/2\, x^2 - 3\alpha/2\, x + \alpha.$$

$$= (1 - 3 + \alpha/2)x^2 + (-5 + 12 - 3\alpha/2)x + (-3 + \alpha)$$

$$P_2(x) = (-2 + \alpha/2)x^2 + (7 - \tfrac{3\alpha}{2})x + (-3 + \alpha)$$

Now;

since the coffiicient of x^2 is 1

$\Rightarrow -2 + \alpha/2 = 1$

$\Rightarrow \alpha/2 = 3 \Rightarrow \alpha = 6$

Now using the value of α to approximate $f(x)$ at $x = 2.5$

$P_2(2.5) = (-2 + 6/2)(2.5)^2 + (7 - 9)(2.5) + (-3 + 6)$

$= (2.5)^2 - 2(2.5) + 3 = 4.25$

Example: Use the quadratic Lagrange Interpolating Polynomial by selection the best points from

$$x = 1.7\,, 1.8\,, 1.9\,, 2.1\,, 2.3\,, 2.4\,, 3.1\,, 4.2\,, 5.9$$

On the function $f(x) = \sqrt{2x+1}$ to estimate $\sqrt{5}$. Also compute the error bound and absolute error.

Solution:

take $2x+1 = 5 \Rightarrow x = 2$

$\Rightarrow$ The best points for quadratic poly. are

$x_0 = 1.8\,, \quad x_1 = 1.9\,, \quad x_2 = 2.1$

$f(x_0) = \sqrt{4.6}\,, \quad f(x_1) = \sqrt{4.8}\,, \quad f(x_2) = \sqrt{5.2}$

$$P_2(2) = L_0(2) f(x_0) + L_1(2) f(x_1) + L_2(2) f(x_2)$$

$$= L_0(2) f(1.8) + L_1(2) f(1.9) + L_2(2) f(2.1)$$

Now

$$L_0(2) = \frac{(2-1.9)(2-2.1)}{(1.8-1.9)(1.8-2.1)} = \frac{-1}{3}$$

$$L_1(2) = \frac{(2-1.8)(2-2.1)}{(1.9-1.8)(1.9-2.1)} = 1$$

$$L_2(2) = \frac{(2-1.8)(2-1.9)}{(2.1-1.8)(2.1-1.9)} = \frac{1}{3}$$

$$\Rightarrow \sqrt{5} \simeq P_2(2) = \sqrt{4.6} \times \left(-\tfrac{1}{3}\right) + \sqrt{4.8} \times (1) + \sqrt{5.2} \times \left(\tfrac{1}{3}\right)$$

$$= 2.23608$$

Absolute error $= |\sqrt{5} - P_2(2)|$

$= |\sqrt{5} - 2.23608| = 1.8849 \times 10^{-5}$

To find Error bound

$\Rightarrow |E_2(2)| = \dfrac{f^{(3)}(\xi)}{3!} |(2-1.8)(2-1.9)(2-2.1)|$

where $f^{(3)}(\xi) \leq M = \max\limits_{1.8 \leq x < 2.1} |f^{(3)}(x)|$

$f(x) = \sqrt{2x+1} = (2x+1)^{1/2}$

$f'(x) = \frac{1}{2}(2x+1)^{-1/2} \cdot 2 = (2x+1)^{-1/2}$

$f''(x) = -\frac{1}{2}(2x+1)^{-3/2} \cdot 2 = -(2x+1)^{-3/2}$

$f^{(3)}(x) = +\frac{3}{2}(2x+1)^{-5/2} \cdot 2 = 3(2x+1)^{-5/2} = \dfrac{3}{(\sqrt{2x+1})^5}$

$M = \max\limits_{1.8 \leq x \leq 2.1} \dfrac{3}{(\sqrt{2x+1})^5} = \dfrac{3}{(\sqrt{2(1.8)+1})^5} = 0.0661$

$\Rightarrow |E_2(2)| = \left| \dfrac{0.0661}{6} (2-1.8)(2-1.9)(2-2.1) \right|$

$= 2.2033 \times 10^{-5}$

Example: Write the quadratic Lagrange interpolating polynomials for $f(x) = 6x^2 + 3$ at $x = 0, 2, -2$ .Then compute the expression of the rational function

$$\frac{f(x)}{g(x)} = \frac{6x^2 + 3}{x^3 - 4x}$$

As sum of partial fraction.

Solution:

$x: \quad 0 \;, \quad 2 \;, \quad -2$

$f(x): \quad 3 \quad 27 \quad 27$

$$P_2(x) = L_0(x) f(0) + L_1(x) f(2) + L_2(x) f(-2)$$

$$= \frac{(x-2)(x+2)}{(0-2)(0+2)} \cdot 3 + \frac{(x-0)(x+2)}{(2-0)(2+2)} \times 27 + \frac{(x-0)(x-2)}{(-2-0)(-2-2)} \times 27$$

$$= \frac{-3}{4}(x-2)(x+2) + \frac{27}{8}(x-0)(x+2) - \frac{27}{8}(x-0)(x-2)$$

Now $P_2(x) \simeq f(x)$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{-3/4\,(x-2)(x+2)}{x(x-2)(x+2)} + \frac{27/8\; x(x+2)}{x(x-2)(x+2)} + \frac{27/8\; x(x-2)}{x(x-2)(x+2)}$$

$$= \frac{-3/4}{x} + \frac{27/8}{x-2} + \frac{27/8}{x+2}$$

Example: Using the quadratic Lagrange polynomial express the rational function

$$\frac{2x^2 - 9x - 9}{x^3 - 9x}$$

As sum of partial fraction.

Solution:

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Example: Let $p(x)$ be the Lagrange polynomials of

$f(x) = x^3 + x + 1$ *at the points* $x_i = \beta + (i + 1)h,$

$i = 0,1,2\ , \beta$ *constan and* $h > 0$

Find h such that the error at $x = \beta$ is bounded above by 10^{-3}.

Solution:

$i = 0 \Rightarrow x_0 = \beta + (0+1)h = \beta + h$

$i = 1 \Rightarrow x_1 = \beta + (1+1)h = \beta + 2h$

$i = 2 \Rightarrow x_2 = \beta + (2+1)h = \beta + 3h$

the error bound

$$E_2 \leqslant \frac{\max f^{(3)}(x)}{3!} \left| (x - x_0)(x - x_1)(x - x_2) \right| \leqslant 10^{-3}$$

$\Rightarrow f'(x) = 3x^2 + 1\ ,\ f''(x) = 6x\ ,\ f^{(3)}(x) = 6$

$\Rightarrow \max |f^{(3)}(x)| = 6$

Now error bound at $x = \beta$

$$E_2 \leqslant \frac{6}{3!} \left| \beta - (\beta + h))(\beta - (\beta + 2h)(\beta - (\beta + 3h)) \right| \leqslant 10^{-3}$$

$\Rightarrow |-h(-2h)(-3h)| < 10^{-3}$

$\Rightarrow 6h^3 < 10^{-3} \Rightarrow h^3 < \dfrac{10^{-3}}{6}$

$h \leqslant 0.055032$

Example: Let $p_4(x)$ be the Lagrange polynomials which interpolating

$$f(x) = x^5 + 2x^3 + 10 \textit{ at the points } x_i = \alpha + (i+1)h,$$

$$i = 0,1,2,3,4 \textit{ where } \alpha \textit{ constan and } h > 0$$

Find h such that the error at $x = \alpha$ is bounded above by 10^{-5}.

Solution:

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Example: Let $f(x) = (x+2)\ln(x+2)$ be a function defined on $[3, 4]$ find approximation of $2.5\ln(2.5)$ using quadratic Lagrange polynomials for equally space data points defined over $[3, 4]$. Compute absolute error.

Solution:

equally space data

$\Rightarrow x_0 = 3 \quad , x_1 = 3.5 \quad , x_2 = 4$

$\Rightarrow x$:	3	3.5	4
$f(x)$:	8.0471	9.3761	10.7505

to approximate $2.5\ln(2.5)$ put $x = 0.5$

$$\Rightarrow P_2(0.5) = \frac{(0.5-3.5)(0.5-4)}{(3-3.5)(3-4)}(8.0471) + \frac{(0.5-3)(0.5-4)}{(3.5-3)(3.5-4)}(9.3761)$$

$$+ \frac{(0.5-3)(0.5-3.5)}{(4-3)(4-3.5)}(10.7505)$$

$$= 168.9891 - 328.1635 + 161.2575$$

$$= 2.0831$$

Now the Absolute error

$$= |f(0.5) - P_2(0.5)|$$

$$= |2.5\ln(2.5) - 2.0831| = 0.2076$$

Example: Let $f(x) = 2x + \ln 2x$ be a function defined on $[1\,,3]$. Find the approximation of $(5 + \ln 5)$ using quadratic Lagrange interpolating polynomials for equally space data points defined over $[1\,,3]$. Compute an error bound and the absolute error.

Solution:

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Example: Let $f(x) = \sqrt{x - x^2}$ *and* $p_2(x)$ be the quadratic Lagrange interpolating polynomials on $x_0 = 0,\ x_1 = \alpha,\ x_2 = 1$. Find the largest value of α *in* $(0, 1)$ for which

$$f(0.5) - p_2(0.5) = -0.25$$

Solution:

$$P_2(x) = L_0(x) f(0) + L_1(x) f(\alpha) + L_2(x) f(1)$$

$$= \cancel{L_0(x)}\,(0) + L_1(x)\sqrt{\alpha - \alpha^2} + \cancel{L_2(x)}\, f(0)$$

$$= \frac{(x-0)(x-1)}{(\alpha-0)(\alpha-1)}\sqrt{\alpha-\alpha^2}$$

$$\Rightarrow P_2(0.5) = \frac{(0.5)^2 - (0.5)}{\alpha^2 - \alpha}\sqrt{\alpha - \alpha^2}$$

$$= \frac{-0.25}{\alpha^2 - \alpha}\sqrt{\alpha - \alpha^2}$$

Now

$$f(0.5) - P_2(0.5) = -0.25$$

$$\Rightarrow 0.5 - \frac{-0.25}{\alpha^2 - \alpha}\sqrt{\alpha - \alpha^2} = -0.25$$

$$\Rightarrow 0.5 + \frac{0.25}{-(\alpha-\alpha^2)}\sqrt{\alpha-\alpha^2} = -0.25$$

$$\Rightarrow \frac{-0.25}{\sqrt{\alpha-\alpha^2}} = -0.75$$

$$\Rightarrow \frac{1}{\sqrt{\alpha-\alpha^2}} = \frac{0.75}{0.25} = 3$$

$$\Rightarrow 3\sqrt{\alpha-\alpha^2} = 1 \Rightarrow 9(\alpha-\alpha^2) = 1$$

$$\Rightarrow 9\alpha^2 - 9\alpha + 1 = 0$$

$$\Rightarrow \alpha_1 = 0.1273 \quad , \alpha_2 = 0.872$$

$\Rightarrow$ the largest value of α is 0.872

Example: Construct the table for $(\alpha, Q(\alpha))$ by evaluating the integral

$$Q(\alpha) = \int_1^2 (\alpha - 1)\, dx \,,$$

At $\alpha = 1, 1.5, 2.5, 3.5, 4$. Then use the constructed table to find the best approximation of $Q(3.4)$ by using quadratic Lagrange polynomial. Compute the absolute error.

Solution:

$$Q(\alpha) = \int_1^2 (\alpha - 1)\, dx = (\alpha - 1)x \Big|_1^2 = 2(\alpha - 1) - (\alpha - 1)$$

$\Rightarrow Q(\alpha) = \alpha - 1$

Now

α :	1	1.5	2.5	3.5	4
$Q(\alpha)$:	0	0.5	1.5	2.5	3

Now the beast points to approximate $Q(3.4)$

$\alpha_0 = 2.5$, $\alpha_1 = 3.5$, $\alpha_2 = 4$

$\Rightarrow Q(3.4) = P_2(3.4)$

$= L_0(3.4) f(2.5) + L_1(3.4) f(3.5) + L_2(3.4) f(4)$

$=$

$$L_0(3.4) = \frac{(3.4-3.5)(3.4-4)}{(2.5-3.5)(2.5-4)} = \frac{1}{25}$$

$$L_1(3.4) = \frac{(3.4-2.5)(3.4-4)}{(3.5-2.5)(3.5-4)} = \frac{27}{25}$$

$$L_2(3.4) = \frac{(3.4-2.5)(3.4-3.5)}{(4-2.5)(4-3.5)} = \frac{-3}{25}$$

$$\Rightarrow P_2(3.4) = \frac{1}{25}(1.5) + \frac{27}{25}(2.5) - \frac{3}{25}(3) = 2.4$$

Now to find the Absolute error

$$|Q(3.4) - P_2(3.4)|$$

$$Q(3.4) = 3.4 - 1 = 2.4$$

$$\Rightarrow |Q(3.4) - P_2(3.4)| = |2.4 - 2.4| = 0$$

Example: the equation $x - 9^{-x} = 0$ has a solution in $[0,1]$. Compute the Lagrange polynomial on $x_0 = 0, x_1 = 0.5, x_2 = 1$. By setting the interpolating polynomial equal to zero and solving the equation, find an approximate solution to the equation.

Solution:

$$f(x) = x - 9^{-x}$$

x_i :	0	0.5	1
$f(x_i)$:	-1	$1/6$	$8/9$

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$L_0(x) = \frac{(x-0.5)(x-1)}{(0-0.5)(0-1)} = 2\left(x^2 - \tfrac{3}{2}x + \tfrac{1}{2}\right) = 2x^2 - 3x + 1$$

$$L_1(x) = \frac{(x-0)(x-1)}{(0.5-0)(0.5-1)} = -4(x^2 - x) = -4x^2 + 4x$$

$$L_2(x) = \frac{(x-0)(x-0.5)}{(1-0)(1-0.5)} = 2\left(x^2 - \tfrac{1}{2}x\right) = 2x^2 - x$$

$$\Rightarrow P_2(x) = (2x^2 - 3x + 1)(-1) + (-4x^2 + 4x)\left(\tfrac{1}{6}\right) + (2x^2 - x)\left(\tfrac{8}{9}\right)$$

$$= -2x^2 + 3x - 1 - \tfrac{2}{3}x^2 + \tfrac{2}{3}x + \tfrac{16}{9}x^2 - \tfrac{8}{9}x$$

$\Rightarrow$

$P_2(x) = -\frac{8}{9}x^2 + \frac{25}{9}x - 1$

Now

$f(x) \simeq P_2(x) = 0$

$\Rightarrow -\frac{8}{9}x^2 + \frac{25}{9}x - 1 = 0$

$\Rightarrow x_1 = 2.7098$ reject since $x_1 \notin [0,1]$

$x_2 = 0.4151$ accepet ; $x_2 \in [0,1]$

$\Rightarrow$ the approximation solution of equation $x - 9^{-x}$ is

0.4151

Example: Let $f(x) = \frac{1}{x}$ be the function defined in the interval $[2, 4]$ and $x_0 = 2, x_1 = 2.5, x_2 = 4$. Compute the value of the unknown point η in the error formula of quadratic Lagrange interpolating polynomial for approximation of $f(3)$ using the given points x_0, x_1, x_2. Also compute an error bound for corresponding error

Solution:

Now;

$P_2(3) = L_0(3) f(2) + L_1(3) f(2.5) + L_2(3) f(4)$

$f(2) = \frac{1}{2}$, $f(2.5) = \frac{2}{5}$, $f(4) = \frac{1}{4}$

$$L_0(3) = \frac{(3-2.5)(3-4)}{(2-2.5)(2-4)} = -\frac{1}{2}$$

$$L_1(3) = \frac{(3-2)(3-4)}{(2.5-2)(2.5-4)} = \frac{4}{3}$$

$$L_2(3) = \frac{(3-2)(3-2.5)}{(4-2)(4-2.5)} = \frac{1}{6}$$

So $f(3) \simeq P_2(3) = (-\frac{1}{2})(\frac{1}{2}) + (\frac{4}{3})(\frac{2}{5}) + (\frac{1}{6})(\frac{1}{4})$

$= 0.325$

Now the error is

$E = |f(3) - P_2(3)| = |0.333 - 0.325| = 0.008$

Now the error formula is

$E = |f(3) - P_2(3)| = \frac{f^{(3)}(\eta)}{3!}(x - x_0)(x - x_1)(x - x_2) \; ; \eta \in (2,4)$

Now ; $f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f^{(3)}(x) = \frac{-6}{x^4}$

$\Rightarrow f^{(3)}(\eta) = \frac{-6}{\eta^4}$

$\Rightarrow 0.008 = \left| \frac{-6/\eta^4}{3!}(3-2)(3-2.5)(3-4) \right|$

$\Rightarrow 0.008 = \frac{0.5}{\eta^4} \Rightarrow \eta^4 = \frac{0.5}{0.008} = 62.5$

$\Rightarrow \eta = 2.8117 \in (2,4)$

Example: Let $x_0 = 0.5$, $x_1 = 0.7$, $x_2 = 0.9$, $x_3 = 1.1$, $x_4 = 1.3$, $x_5 = 1.5$, $x_6 = 1.7$ *and* $f(x) = \ln(x+3)$. Compute am error bound for the approximation of $\ln(3.6)$ using six degree Lagrange polynomial.

Solution:

to compute error bound for given function $f(x) = \ln(x+3)$ in interval $[0.5, 1.7]$ and put $x = 0.6$

$$E_6 \leq \frac{|f^{(7)}(\xi)|}{7!} |(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)(x-x_6)|$$

where $|f^{(7)}(\xi)| \leq M = \max_{0.5 \leq x \leq 1.7} |f^{(7)}(x)|$

$\Rightarrow f'(x) = \frac{1}{x+3} = (x+3)^{-1}$

$f''(x) = -(x+3)^{-2}$; $f^{(3)}(x) = 2(x+3)^{-3}$

$f^{(4)}(x) = -6(x+3)^{-4}$; $f^{(5)}(x) = 24(x+3)^{-5}$

$f^{(6)}(x) = -120(x+3)^{-6}$; $f^{(7)}(x) = 720(x+3)^{-7}$

$\Rightarrow M = \max_{0.5 \leq x \leq 1.7} \left| \frac{720}{(x+3)^7} \right| = \frac{720}{(0.5+3)^7} = 0.1119$

$\Rightarrow E_6$

$\leq \frac{M}{7!}\left|(0.6-0.5)(0.6-0.7)(0.6-0.9)(0.6-1.1)(0.6-1.3)(0.6-1.5)(0.6-1.7)\right|$

$= \frac{0.1119}{7!} \times (0.001) = 2.2202 \times 10^{-8}$

Example: Let $x_0 < x < x_1$ *and* $h = x_1 - x_0$. By using a suitable Lagrange interpolating polynomial drive the linear polynomial $P_1(x)$ that interpolates the function $f(x)$ at the points $x_0, x_1 \in I$. Prove also that the error in this linear interpolates is bounded above by $\frac{h^2}{8}M$ *where* $M = \max_{x\in I}|f''(x)|$.

Solution:

$$E = |f(x) - P_1(x)| \le \frac{|\max f''(x)|}{2!}|(x - x_0)(x - x_1)|$$

$$M = \max_{x_0 < x < x_1} f''(x)$$

Now let $g(x) = (x - x_0)(x - x_1)$;

to find the maximum value of g

$$g'(x) = x - x_0 + x - x_1 = 0$$

$$2x = x_0 + x_1$$

$$\Rightarrow \quad x = \frac{x_0 + x_1}{2} \quad \text{critical point}$$

$$\left|g\left(\frac{x_0 + x_1}{2}\right)\right| = \left|\left(\frac{x_0 + x_1}{2} - x_0\right)\left(\frac{x_0 + x_1}{2} - x_1\right)\right|$$

$$= \left|\left(\frac{x_0 + x_1 - 2x_0}{2}\right) \cdot \left(\frac{x_0 + x_1 - 2x_1}{2}\right)\right|$$

$$= \left| \frac{(x_1 - x_0)(x_0 - x_1)}{4} \right|$$

but $h = x_1 - x_0$

$$= \left| \frac{h(-h)}{4} \right| = h^2/4$$

$$\Rightarrow E \leq \frac{M}{2} \cdot \frac{h^2}{4} = \frac{Mh^2}{8}$$

Example: Let $f(x) = 0$ be defined on the three numbers $-h\,,0\,,h$ *where* $h \neq 0$. Use Lagrange interpolating polynomial $p(x)$ which interpolate $f(x)$ at the given numbers. Then show that this polynomial can be written in following form:

$$p(x) = \frac{1}{2h^2}[f(-h) - 2f(0) + f(h)]x^2 + \frac{1}{2h}[f(h) - f(-h)]x + f(0)$$

Solution:

$x_0 = -h\,,\quad x_1 = 0\,,\quad x_2 = h\quad ;\; h \neq 0$

$P_2(x) = L_0(x)\,f(x_0) + L_1(x)\,f(x_1) + L_2(x)\,f(x_2)$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-h)}{(-h-0)(-h-h)} = \frac{x^2-xh}{2h^2}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+h)(x-h)}{(0+h)(0-h)} = \frac{x^2-h^2}{-h^2}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+h)(x-0)}{(h+h)(h-0)} = \frac{x^2+xh}{2h^2}$$

$\Rightarrow$

$$P_2(x) = \frac{x^2-xh}{2h^2}f(-h) + \frac{x^2-h^2}{-h^2}f(0) + \frac{x^2+xh}{2h^2}f(h)$$

Now

$$P_2(x) = \frac{f(-h)}{2h^2}x^2 - \frac{f(-h)}{2h}x + \frac{f(0)}{-h^2}x^2 + f(0) + \frac{f(h)}{2h^2}x^2 + \frac{f(h)}{2h}x$$

$\Rightarrow$

$$P_2(x) = \left(\frac{f(-h)}{2h^2} - \frac{f(0)}{h^2} + \frac{f(h)}{2h^2} \right) x^2 + \left(\frac{f(h)}{2h} - \frac{f(-h)}{2h} \right) x + f(0)$$

$$= \frac{1}{2h^2} \left[f(-h) - 2f(0) + f(h) \right] x^2 + \frac{1}{2h} \left[f(h) - f(-h) \right] x + f(0)$$

Divided Difference

$$Zeroth\ D.D \qquad f[x_i] = f(x_i)$$

$$1st\ \ D.D \quad f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

$$2nd\ \ D.D \quad f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

$$nth\ \ D.D \quad f[x_i, \dots, x_{i+n}] = \frac{f[x_{i+1}, \dots, x_{i+n}] - f[x_i, \dots, x_{i+n-1}]}{x_{i+n} - x_i}$$

Newton's interpolating divided difference polynomial

$$p_1(x) = f[x_0] + f[x_0, x_1](x - x_0)$$

$$p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

Note:

$$f[x_i, \dots, x_i] = \frac{f^{(n)}(x_i)}{n!} \quad , x_i\ n + 1\ time$$

Example: If the best approximation of $f(1.5)$ using Newton's
quadratic interpolating polynomial is 7 and $f[1,2,3,4] = 8$. Find the Newton's cubic $p_3(1.5)$.

Solution:

$x_0 = 1, \quad x_1 = 2, \quad x_2 = 3, \quad x_3 = 4$

$P_3(x) = P_2(x) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$

Given $P_2(1.5) = 7$ and $f[1,2,3,4] = 8$

$\Rightarrow P_3(1.5) = P_2(1.5) + f[1,2,3,4](1.5-1)(1.5-2)(1.5-3)$

$\Rightarrow P_3(1.5) = 7 + 8(1.5-1)(1.5-2)(1.5-3) = 10$

Example: consider the points $x_0 = 0, x_1 = 0.4, x_2 = 0.7$ and for a function $f(x)$, the divided difference are

$$f[x_2] = 6, f[x_1, x_2] = 10, f[x_0, x_1, x_2] = \frac{50}{7}.$$

Use Linear Newton's polynomial to find the approximation of $f(0.5)$.

Solution:

$P_1(x) = f[x_0] + f[x_0, x_1](x - x_0)$

$f(0.5) \simeq P_1(0.5)$

x_i	0-D.D	1-st D.D	2-nd D.D
0	$f[x_0] = ?$		
0.4	$f[x_1] = ?$	$f[x_0, x_1] = ?$	
0.7	$f[x_2] = 6$	$f[x_1, x_2] = 10$	$f[x_0, x_1, x_2] = 50/7$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = 10$$

$$\Rightarrow \frac{6 - f[x_1]}{0.7 - 0.4} = 10 \Rightarrow f[x_1] = 3$$

the best point is $x_0 = 0.4$, $x_1 = 0.7$

$P_1(0.5) = f[0.4] + f[0.4, 0.7](0.5 - 0.4)$

$= 3 + 10(0.1) = 4$

Example: If a function $f(x)$ satisfies the conditions $f[-1,1] = 1$, $f'(1) = 5$, $f'(-1) = -1$. Find $f[1,-1,1]$

Solution:

Now $f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

$f[1,-1,1] = f[-1,1,1]$

$= \dfrac{f[1,1] - f[-1,1]}{1 - -1}$

$f[1,1] = f'(1) = 5$ and $[-1,1] = 1$

$\Rightarrow f[1,-1,1] = \dfrac{5-1}{2} = 2$

Example: If $f(x) = \frac{1}{x}$ then show that $f[1,1,1,2] = -\frac{1}{2}$

Solution:

$$f[1,1,1,2] = \frac{f[1,1,2] - f[1,1,1]}{2 - 1}$$

$$f[1,1,1] = \frac{f''(1)}{2!}$$

Now $f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$

$$\frac{f''(1)}{2!} = \frac{2}{2} = 1$$

$\Rightarrow f[1,1,1] = 1$

and

$$f[1,1,2] = \frac{f[1,2] - f[1,1]}{2 - 1}$$

$$= \frac{\frac{f(2) - f(1)}{2 - 1} - \frac{f'(1)}{1!}}{}$$

$$= 1/2 - 1 + 1 = 1/2$$

$$\Rightarrow f[1,1,1,2] = \frac{1/2 - 1}{1} = -1/2$$

Example:

Let $f(x) = e^{(x+4)}$ *and* $x_0 = 1, x_1 = 1, x_2 = 2,\ x_3 = 2$ compute $f[1,1,2,2]$ and then find the approximation of $e^{5.5}$ by using cubic Newton's interpolating polynomial and absolute error.

Solution:

$$f[1,1,2,2] = \frac{f[1,2,2] - f[1,1,2]}{2-1}$$

$$\Rightarrow f[1,1,2] = \frac{f[1,2] - f[1,1]}{2-1}$$

$$\Rightarrow f[1,2,2] = \frac{f[2,2] - f[1,2]}{2-1}$$

$$f[1,2] = \frac{f[2] - f[1]}{2-1} = \frac{e^6 - e^5}{1} = 255.01563$$

$$f[1,1] = f'(1) = e^5 = 148.4131$$

$$f[2,2] = f'(2) = e^6 = 403.4288$$

$$\Rightarrow f[1,1,2] = 255.0156 - 148.4131 = 106.6025$$

$$f[1,2,2] = 403.4288 - 255.0156 = 148.4132$$

$$\Rightarrow f[1,1,2,2] = 148.4132 - 106.6025$$

$$= 41.8107$$

Now; The Newton's poly. of degree 3

$P_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$
$\quad + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$

$\Rightarrow$ Put $x = 1.5$

$P_3(1.5) = f[1] + f[1,1](1.5-1) + f[1,1,2](1.5-1)(1.5-1)$
$\quad + f[1,1,2,2](1.5-1)(1.5-1)(1.5-2)$

$\Rightarrow$

$P_3(1.4) = 148.4131 + 148.4131(0.5) + 106.6025(0.5)^2 +$
$\quad + 41.8107(0.5)^2(-0.5) = 244.0439$

The absolute error $= |f(1.5) - P_3(1.5)|$

$= |e^{5.5} - 244.0439| = 0.648$

Example: Let $f(x) = e^{(x+2)}$ and

$x_0 = -0.2, x_1 = -0.1, x_2 = 0, x_3 = 0.1, x_4 = 0.2, x_5 = 0.3$. If $p_4(0.05) = 7.7679$ and $f[-0.2, -0.1, 0, 0.1, 0.2, 0.3] = 0.0649$ then find the approximation of $e^{2.05}$ using fifth degree Newton's interpolating polynomial. Compute error bound and absolute error

Solution:

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Example: Given $f(x) = x\ln(x+1)$ at the points $-1, 0, 1.5, 2, 2.8$ *and* 3. Use quadratic Newton's divided difference formula to find the best approximation of $2.5\ln 3.5$, also compute exact error.

Solution:

put $x = 2.5$

$\Rightarrow$ the beast points

$x_0 = 2, \quad x_1 = 2.8, \quad x_2 = 3$

$\Rightarrow f(x) \simeq P_2(x)$

$\Rightarrow f(2.5) \simeq P_2(2.5)$

$$= f[2] + f[2, 2.8](2.5-2) + f[2, 2.8, 3](2.5-2)(2.5-2.8)$$

$f[2] = f(2) = 2\ln 3 = 2.1972$

$f(2.8) = 2.8\ln 3.8 = 3.738$

$f(3) = 3\ln 4 = 4.1589$

Now

$$f[2, 2.8] = \frac{f(2.8) - f(2)}{2.8 - 2} = 1.926$$

$$f[2.8, 3] = \frac{f(3) - f(2.8)}{3 - 2.8} = 2.1045$$

$$f[2, 2.8, 3] = \frac{f[2.8, 3] - f[2, 2.8]}{3 - 2} = 0.1785$$

$\Rightarrow P_2(2.5)$

$= 2.1972 + 1.926(0.5) + 0.1785(0.5)(-0.3)$

$= 3.1279$

Now the Absolute error

$| f(2.5) - P_2(2.5)|$

$= | 2.5 \ln(3.5) - 3.1279| = 4.007 \times 10^{-3}$

Example: let $f(x) = x + 2\ln(x + 2)$ be given at points $x = 0, 1, 2, 4$ *and* 5. Use the quadratic Newton's interpolating formula to approximate $f(2.5)$,and compute the error bound for your approximation.

Solution:

put $x = 2.5 \Rightarrow$ the beast points is

$x_0 = 1 \;;\; x_1 = 2 \;,\; x_2 = 4$

$f(1) = 3.1972 \;,\; f(2) = 4.7726 \;,\; f(4) = 7.5835$

$P_2(2.5) = f[1] + f[1,2](2.5-1) + f[1,2,4](2.5-1)(2.5-2)$

Now

$f[1] = f(1) = 3.1972 \;,\; f[2] = f(2) = 4.7726 \;;\; f[4] = f(4) = 7.5835$

$f[1,2] = \dfrac{f[2] - f[1]}{2-1} = 1.5754$

$f[2,4] = \dfrac{f[4] - f[2]}{4-2} = 1.4054$

$f[1,2,4] = \dfrac{f[2,4] - f[1,2]}{4-1} = -0.0566$

$\Rightarrow f(2.5) \simeq P_2(2.5)$

$= 3.1972 + 1.5754(2.5-1) + -0.0566(2.5-1)(2.5-2)$

$= 5.5178$

Now;

to compute the errorbound for approximation

$$E = |f(2.5) - P_2(2.5)| \leq \frac{M}{3!} \left| (2.5-1)(2.5-2)(2.5-4) \right|$$

where $M = \max\limits_{1 \leq x \leq 4} f^{(3)}(x)$

$$f(x) = x + 2\ln(x+2)$$

$$f'(x) = 1 + \frac{2}{x+2} = 1 + 2(x+2)^{-1}$$

$$f''(x) = -2(x+2)^{-2}$$

$$f^{(3)}(x) = 4(x+2)^{-3} = \frac{4}{(x+2)^3}$$

Now $\max\limits_{1 \leq x \leq 4} |f^{(3)}(x)| = f(1) = 0.1481$

$\Rightarrow$

$$E \leq \left| \frac{0.1481}{6} (2.5-1)(2.5-2)(2.5-4) \right|$$

$$= 0.0277$$

Example: let $f(x) = 3x^5$ then compute the value of divided difference $f[-1,0,-1]$ and $f[-1,0,2,3,4,5]$

Solution:

$$f[-1,0,-1] = f[-1,-1,0] = \frac{f[-1,0] - f[-1,-1]}{0 - -1}$$

Now $f[-1,0] = \dfrac{f[0] - f[-1]}{1} = \dfrac{0 - 3(-1)^5}{1} = 3$

and

$$f[-1,-1] = f'(-1)$$

$$f'(x) = 15x^4 \Rightarrow f'(-1) = 15$$

$$\Rightarrow f[-1,0,-1] = \frac{3 - 15}{1} = -12$$

Also

$$f[-1,0,2,3,4,5] = \frac{f^{(5)}(x)}{5!} = \frac{3(5 \times 4 \times 3 \times 2 \times 1)}{5!} = 3$$

Example: Consider the points $x_0 = 2, x_1 = 2.5, x_2 = 3, x_3 = 3.5$ and for a function $f(x)$, the divided difference are $f[x_2] = 4, f[x_1, x_2] = 5, f[x_0, x_1, x_2] = 10$ *and* $f[x_0, x_1, x_2, x_3] = \frac{1}{5}$.
Use this information to construct the complete divided difference table for the given points. Also find the approximation of $f(2.9)$ using linear Newton's polynomial.

Solution:

The table of D.D

K	x_K	0-D.D	1-st D.D	2-nd D.D	3-rd D.D
0	2	$f[x_0] = ?$			
1	2.5	$f[x_1] = ?$	$f[x_0, x_1] = ?$		
2	3	$f[x_2] = 4$	$f[x_1, x_2] = 5$	$f[x_0, x_1, x_2] = 10$	
3	3.5	$f[x_3] = ?$	$f[x_2, x_3] = ?$	$f[x_1, x_2, x_3] = ?$	$f[x_0, x_1, x_2, x_3] = 1/5$

Now to complet the table

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

$$\Rightarrow \frac{1}{5} = \frac{f[x_1, x_2, x_3] - 10}{3.5 - 2} \Rightarrow f[x_1, x_2, x_3] = 10.3$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$\Rightarrow 10 = \frac{5 - f[x_0, x_1]}{3 - 2} \Rightarrow f[x_0, x_1] = -5$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

$$\Rightarrow 10.3 = \frac{f[x_2, x_3] - 5}{3.5 \quad -2.5} \Rightarrow f[x_2, x_3] = 15.3$$

$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$$

$$\Rightarrow 15.3 = \frac{f[x_3] - 4}{3.5 - 3} \Rightarrow f[x_3] = 11.65$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

$$\Rightarrow 5 = \frac{4 - f[x_1]}{3 - 2.5} = 5 \Rightarrow f[x_1] = 1.5$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$\Rightarrow -5 = \frac{1.5 - f[x_0]}{2.5 - 2} \Rightarrow f[x_0] = 4$$

Now the complet table

$\Rightarrow$

K	x_K	0-D.D	1-st D.D	2-nd D.D	3-rd D.D
0	2	$f[x_0]=4$			
1	2.5	$f[x_1]=1.5$	$f[x_0,x_1]=-5$		
2	3	$f[x_2]=4$	$f[x_1,x_2]=5$	$f[x_0,x_1,x_2]=10$	
3	3.5	$f[x_3]=11.65$	$f[x_2,x_3]=15.3$	$f[x_1,x_2,x_3]=10.3$	$f[x_0,x_1,x_2,x_3]=1/5$

Now the Newton's D.D. poly of degree one

$P_1(x) = f[x_0] + f[x_0,x_1](x-x_0)$

put $x = 2.9 \Rightarrow x_0 = 2.5,\ x_1 = 3$

$\Rightarrow P_2(2.9) = f[2.5] + f[2.5,3](2.9-2.5)$

$= 1.5 + 5(2.9-2.5)$

$= 3.5$

Example: Let $f(x)$ be defined on x_0, x_1, x_2 such that $h = x_{i+1} - x_i$, $i = 0$, 1 then show that

$$f[x_0, x_1, x_2] = \frac{1}{2h^2}[f(x_2) - 2f(x_1) + f(x_0)]$$

Solution:

$h = x_{i+1} - x_i \Rightarrow x_{i+1} = h + x_i$

$\Rightarrow x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + h + h = x_0 + 2h$

Now

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2) - f(x_1) - f(x_1) - f(x_0)}{h}}{2h}$$

$$= \frac{f(x_2) - 2f(x_1) - f(x_0)}{2h^2}$$

$$= \frac{1}{2h^2}\left[f(x_2) - 2f(x_1) - f(x_0)\right]$$