Linear D.Es of Higher Orders

In general, an n^{th} order L.D.E. is on the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x). \qquad (1)$$

If $g(x)=0$, then Equation (1) is called homogeneous L.D.E, otherwise it is nonhomogeneous.

For example: $x^3\,y''-8x\,y'+5y=0$ is a homogeneous L. D. E., while $x^3\,y''-8x\,y'+5y=e^{2x}-3$ is a non-homogeneous L. D. E.

Solving equation (1) subject to the constraints:

$$y(x_0)=y_0,\, y'(x_0)=y'_0,\ldots,\, y^{(n-1)}=y_0^{(n-1)}, \qquad (2)$$

is an n^{th} order initial value problem.

The specified values given in (2) are called initial conditions.

By solving the I.V.P.

$$\begin{cases} a_n(x)\dfrac{d^n y}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x), & (1) \\ y(x_0) = y_0, y'(x_0) = y'_0, \ldots, y^{(n-1)} = y_0^{(n-1)}, & (2) \end{cases}$$

we mean to determine a function $y(x)$ defined on some interval I containing x_0 and satisfies equation (1) and all the conditions given in (2).

Theorem (Existence and uniqueness)

Let $a_n(x), a_{n-1}(x), ..., a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I and $a_n(x) \neq 0$ for all x in this interval.

If $x = x_0$ is any point in this interval, then there exists a unique solution $y(x)$ defined on the interval I satisfies the initial value problem given in (1)-(2).

Example

It is easy to see that the function $y = 3e^{2x} + e^{-2x} - 3x$ is a solution of the I.V.P.

$$y'' - 4y = 12x, \quad y(0) = 4, \; y'(0) = 1.$$

Since the coefficients $a_2(x),\ a_1(x),\ a_0(x)$ as well as $g(x)$ are continuous and $a_2(x) \neq 0$ on any interval containing $x_0 = 0$. Therefore, in view of the above theorem, this function is the unique solution of this problem on the interval $I = (-\infty, \infty)$.

Example

Find the largest interval on which the I.V.P.

$$\begin{cases} x(x^2-4)y'' - \sqrt{5-x}\,y' + x^3 y = \ln(x+3), \\ y(-1) = 1,\ y'(-1) = 0, \end{cases}$$

has a unique solution.

Solution.

Here we have

$a_2(x) = x(x^2 - 4)$ which is continuous on $(-\infty, \infty)$,

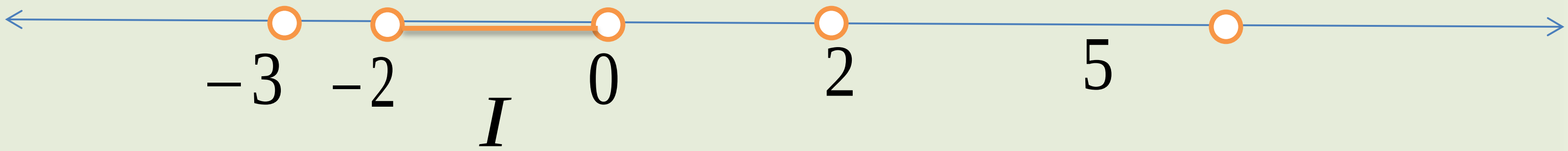
$a_1(x) = -\sqrt{5 - x}$ is continuous on $(-\infty, 5]$,

$a_0(x) = x^3$ is continuous on $(-\infty, \infty)$,

$g(x) = \ln(x + 3)$ is continuous on $(-3, \infty)$,

and $a_2(x) = 0, \ at \ \ x = 0, \pm 2.$

Thus, the functions $a_2(x), a_1(x), a_0(x) \ and \ g(x)$ are all continuous on the interval $I = (-2, 0)$ which contains $x_0 = -1$ and $a_2(x) \neq 0, \ on \ I.$ Hence, the IVP admits a unique solution on the interval I.



Homework

Determine the largest symmetric interval on which the following I.V.P has a unique solution

$$\begin{cases} \ln(x+2)\,y''-\sqrt{9-x^2}\;y'+3y=\tan x, \\ y(0)=1,\; y'(0)=2, \end{cases}$$

Linear dependence

A set of functions $f_1, f_2, \ldots, f_n$ is said to be linearly dependent on an interval ***I***, if there are constants $c_1, c_2, \ldots, c_n$ (at least some of them are not zero) such that $c_1\, f(x)_1 + c_2\, f_2(x) + \ldots + c_n\, f_n(x) = 0,\ \forall\, x \in I\,.$

Example 1. The functions:

$$f_1(x) = x^2,\ \ f_2(x) = e^{-x},\ \ f_3(x) = xe^{-x},\ \ and\ \ f_4(x) = (3-5x)\, e^{-x}$$

Are linearly dependent on $I = (-\infty,\ \infty)$.

Because
$$f_4(x) = (3-5x)e^{-x} = 3e^{-x} - 5xe^{-x}$$
$$= 0\, f_1(x) + 3\, f_2(x) - 5\, f_3(x)$$

$$\Rightarrow \ 0 * f_1(x) + 3 * f_2(x) - 5 * f_3(x) - 1 * f_4(x) = 0$$

for all x in I. Hence there are constants

$$c_1 = 0, c_2 = 3, \ c_3 = -5, \ and \ \ c_4 = -1$$

not all zero such that

$c_1 f(x)_1 + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) = 0$ for all x in I.

Example 2. The functions:

$$f_1(x) = 1, \quad f_2(x) = \cos(2x) \ \ and \ \ \ f_3(x) = \sin^2(x)$$

are linearly dependent on $I = (-\infty, \ \infty)$.

Since, $2\sin^2(x) = [1 - \cos(2x)]$

$$hence, \ 1 - \cos(2x) - 2\sin^2(x) = 0 \ \ for\, all \ \ x \ in \ \ I,$$

which implies

$$1 * f_1(x) - 1 * f_2(x) - 2 * f_3(x) = 0 \ \textit{for all} \ x \ \textit{in} \ I,$$

that is, there are constants $c_1 = 1, c_2 = -1, \ \textit{and} \ c_3 = -2$

not all zero such that $c_1 f(x)_1 + c_2 f_2(x) + c_3 f_3(x) = 0$

for all x in I.

Hence $f_1, \ f_2 \ \textit{and} \ f_3$ are linearly dependent on the interval $I = (-\infty, \infty)$.

Remark. If $f_1, f_2, \ldots, f_n$ are linearly dependent functions on some interval I, then one of them can be written as a linear combination of the other ones.

Linear independence

A set of functions $f_1, f_2, ..., f_n$ is said to be linearly independent on an interval I, if they are not linearly dependent on I. That is if

$$c_1\ f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0 \ \ for\ all\ \ x\ in\ I$$

then all the constants $c_1, c_2, ..., c_n$ must be zero.

Example.

The functions: $f_1(x) = x^2 \ \ and \ \ f_2(x) = x$

are linearly independent on $I = (-\infty,\ \infty)$.

Because, if $c_1 f(x)_1 + c_2 f_2(x) = 0 \ \ for \ all \ \ x \ in \ I,$

then, $c_1 x^2 + c_2 x = 0$ for all x in $(-\infty, \infty).$

In particular for $x = 1 \ and \ \ x = -1$ we get

$$c_1 + c_2 = 0, \ \ and$$

$$c_1 - c_2 = 0$$

hence $c_1 = c_2 = 0.$

Example 2. The functions: $f_1(x) = x \ \ and \quad f_2(x) = |\, x \,|$

are linearly independent on $[-1, 1]$, but they are linearly dependent on $[0, 1]$.

Example 2. Show that the functions:

$$f_1(x) = x, \quad f_2(x) = \sin x \;\; and \;\; f_3(x) = \cos x$$

are linearly independent on the interval $I = (-\infty, \infty)$.

Solution. Assume that

$$c_1\, f(x)_1 + c_2 f_2(x) + c_3 f_3(x) = 0 \;\; for \;\; all \;\; x \;\; in \;\; I.$$

In particular for $x = 0, \;\; x = \pi \;\; and \;\; x = \frac{\pi}{2}$ we have

$$x = 0 \Rightarrow c_1(0) + c_2(0)x + c_3(1) = 0 \;\Rightarrow c_3 = 0,$$

$$x = \pi \Rightarrow c_1(\pi) + c_2(0)x + c_3(-1) = 0 \;\Rightarrow c_1 = 0,$$

$$x = \tfrac{\pi}{2} \Rightarrow c_1(\tfrac{\pi}{2}) + c_2(1)x + c_3(0) = 0 \;\Rightarrow c_2 = 0.$$

Hence the functions are linearly independent on *I*.

Definition

Assume that the functions $f_1, f_2, \ldots, f_n$ possess at least $n-1$ derivatives on an interval I. Then the determinant

$$W(x, f_1, \ldots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ & & \ldots & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix},$$

is called the Wronskian of $f_1, f_2, \ldots, f_n$.

Theorem. (Criterion for linear independence)

Assume that the functions $f_1, f_2, \dots, f_n$ possess at least $n-1$ derivatives on an interval I.

If $W(x, f_1, \dots, f_n) \neq 0$ for at least one value x_0 in I, then $f_1, f_2, \dots, f_n$ are linearly independent on I.

Example 3. Verify that the functions

$$f_1(x) = x,\ f_2(x) = e^x,\ and\ \ f_3(x) = e^{-x}$$

are linearly independent on $I = (-\infty, \infty)$.

Solution. Since,

$$W(x, f_1, \dots, f_n) = \begin{vmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2x \neq 0 \ \ for\ all\ \ x \neq 0\ in\ I,$$

hence the function are linearly independent on I.

Corollary.

If the functions $f_1, f_2, \ldots, f_n$ are linearly dependent on an interval I, then $W(x, f_1, \ldots, f_n) = 0$ for all x in I.

Remark. if $W(x, f_1, \ldots, f_n) = 0$ for all x in the interval I, then it does not imply that $f_1, f_2, \ldots, f_n$ are linearly dependent on I.

Example. The functions $f(x) = x^2$, *and* $g(x) = x\,|\,x\,|$ are linearly independent on $I = [-1, 1]$, (check), but

$$W(x, f, g) = \begin{vmatrix} x^2 & x\,|\,x\,| \\ 2x & 2\,|\,x\,| \end{vmatrix} = 0, \ \ for\ all\ \ x\ in\ \ I.$$

Theorem. Let $y_1, y_2, \ldots, y_k$ be solutions of the Hom. L.D.E.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0,$$

on an interval I, then for any constants $c_1, c_2, \ldots, c_k$ the function $y = c_1 y_1 + c_2 y_2 + \ldots + c_k y_k$ is also a solution on the interval I.

Definition. Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the n^{th} order Hom. L.D.E.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (1)$$

on an interval I, is called a Fundamental Set of Solutions on this interval.

Theorem. Let $y_1, y_2, \ldots, y_n$ be n solutions of the Hom. L.D.E.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0,$$

on an interval I. Then, these solutions are linearly independent on I if and only if

$$W(x, y_1, \ldots, y_n) \neq 0$$

for every x in I.

Definition. Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the Hom. L.D.E.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0,$$

on an interval I. Then, the general solution on I is defined by

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example. Verify that $y_1 = 1, y_2 = e^x, \; and \;\; y_3 = e^{-x}$ form a fundamental set of solutions of the H.D.E.

$$y''' - y' = 0,$$

on the interval $I = (-\infty, \infty)$, then write down the general solution.

Solution. It is easy to check that $y_1, y_2 \; and \;\; y_3$ are solutions of Eq.(1). On the other hand we have

$$W(x, y_1, y_2, y_3) = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2 \neq 0 \;\; for \; all \;\; x \; in \; I,$$

hence they are linearly independent on I.

Therefore the general solution is $y = c_1 + c_2 e^x + c_3 e^{-x}$.

Definition.

Let y_p be any particular solution of the nonhomogeneous L.D.E.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (1)$$

on an interval I and let

$$y_c = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n,$$

be the general solution of the associated Hom. D.E.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0,$$

on this interval, then the general solution of Eq.(1) is

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n + y_p.$$

Example. Verify that $y = c_1 + c_2 e^{x} + c_3 e^{-x} + x^3 - x$

is the general solution of the Nonhom. D.E.

$$y''' - y' = 7 - 3x^2,$$

on the interval $I = (-\infty, \infty)$.

Solution. It is easy to see that $y_1 = 1, y_2 = e^{x}$ *and* $y_3 = e^{-x}$

are solutions of the Hom. D.E. $y''' - y' = 0,$

and

$$W(x, y_1, y_2, y_3) = \begin{vmatrix} 1 & e^{x} & e^{-x} \\ 0 & e^{x} & -e^{-x} \\ 0 & e^{x} & e^{-x} \end{vmatrix} = 2 \neq 0 \;\; \textit{for all} \;\; x \textit{ in } I,$$

hence they are linearly independent on I .

Hence $y_c = c_1 + c_2 e^{x} + c_3 e^{-x}.$

On the other hand the function $y = x^3 - x$

satisfies the Nonhom. D.E. $y''' - y' = 7 - 3x^2$,

i.e. $y_p = x^3 - x$ is a particular solution.

Hence $y = y_c + y_p = c_1 + c_2 e^x + c_3 e^{-x} + x^3 - x$,

is the general solution of the above Nonhom. D.E.