Growth and Decay

Let $X(t)$ be a positive quantity depends on the time t , then the rate of change in X with respect to t is $\frac{dX}{dt}$.

Suppose that the rate of change in X is proportional to the amount of X present at time t, then we have

$\frac{dX}{dt} \propto X \Rightarrow \frac{dX}{dt} = kX$, where $k \neq 0$ is a constant

$\Rightarrow \ln X = kt + c_1$

$or \;\; X = ce^{kt}, \; where \;\; c = e^{c_1}.$

If the initial quantity is X_0 , that is $X = X_0 \;\; at \;\; t = 0$, then we have:

$$X(t) = X_0 e^{kt} . \qquad (1)$$

If $(k > 0)$, then Equation (1) represent a growth model, that is X increases in time, and if $k < 0$, then it represents a decay model, that is X decreases in time.

Example 1. The population of a town grows at a rate proportional to the population present at time t. The initial population of 500 increases by 15% in 10 years. What will be the population in 30 years? How fast is the population growing at $t = 30$?

Solution. Let P be the population present at time t. Then we have

$$P = 500 \quad at \quad t = 0, \qquad (1)$$

$$P = 500 + \tfrac{15}{100}(500) = 575 \quad at \quad t = 10. \qquad (2)$$

Since $P = ce^{kt}$, (1) implies $c = 500$, using (2) we obtain

$$575 = 500\, e^{10k} \Rightarrow k = \tfrac{1}{10}\ln(\tfrac{575}{500}) = \tfrac{1}{10}\ln(\tfrac{23}{20})$$

Hence $P = 500\, e^{\frac{t}{10}\ln(\frac{23}{20})}$, therefore at $t = 30$ we have

$$P = 500\, e^{3\ln(\frac{23}{20})} = 760$$

Now, during 30 years the population increases by

$$P - P_0 = 760 - 500 = 260 \ \ person.$$

That is, it increases $\frac{260}{30} \approx 9$ $persons/year.$

Example 2. The population of a community is known to increase at a rate proportional to the number of people present at time t. If an initial population P_0 has doubled in 5 years. How long will it take to triple?

Solution. We have $P = ce^{kt}$.

But, at $t = 0,\ P = P_0$ *and* *at* $t = 5,\ P = 2P_0$, therefore

$$c = P_0 \Rightarrow 2P_0 = P_0 e^{5k} \Rightarrow k = \tfrac{\ln 2}{5}$$

$$\Rightarrow P = P_0 e^{t\frac{\ln 2}{5}}$$

Hence at $P = 3P_0$ *we have* $e^{t\frac{\ln 2}{5}} = 3 \Rightarrow t = \frac{5\ln 3}{\ln 2} = 7.8\ year$.

Example 3. Suppose it is known that the population in example 2 is 10000 after 3 years. What was the initial population?

Solution. Since $P = P_0 e^{t\frac{\ln 2}{5}}$ and $P = 10000\ at\ t = 3\ years$ we get $P_0 = 10000\, e^{-3\frac{\ln 2}{5}} \approx 6598.$

Example 4. Initially 100 mg of a radioactive substance was present. After 6 hours the mass has decreased by 3%. If the rate of decay is proportional to the amount of the substance present at time t, find the amount remaining after 24 hours.

Solution. Assume that the amount present at time t is A. Then we have $A = ce^{kt}$. But

$$A = 100\, mg\ at\ t = 0\ and\ \ A = 100 - \tfrac{3}{100}(100) = 97\ mg\ \ at\ \ t = 6,$$

$$\Rightarrow c = 100 \ \ and \ \ 97 = 100 e^{6k} \Rightarrow k = \frac{\ln(0.97)}{6}$$

$$\Rightarrow A = 100 e^{t \frac{\ln(0..97)}{6}}.$$

Therefore the remaining quantity after $t = 24$ hours is

$$A = 100 e^{24 \frac{\ln(0..97)}{6}} = 100 e^{4 \ln 0.97} \approx 88.53 \ mg.$$

Example 5. Determine the half –life of the material in the above example.

Solution. Half-life is the time for which the remaining amount is half the initial amount. That is

$$t = ? \ \ at \ \ A = \tfrac{1}{2} A_0 = \tfrac{1}{2}(100) = 50.$$

$$\Rightarrow 50 = 100 e^{t \frac{\ln(0..97)}{6}}.$$

$$\Rightarrow t = \frac{6 \ln(0.5)}{\ln 0.97} \approx 136.54 \ hour.$$

Newton's law of cooling

Suppose an object with temperature T at time t is left to loose heat in a surrounding medium with constant temperature T_s, then the rate at which this object cools down is proportional to the difference between the object temperature and the temperature of the surrounding medium, that is: $\frac{dT}{dt} \propto T - T_s \Rightarrow \frac{dT}{dt} = kT - T_s$

where $k \neq 0$ is a constant, hence we have

$$T - T_s = ce^{kt}.$$

Example 1

A thermometer reading is $70^\circ F$ placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads $110^\circ F$ after $\frac{1}{2}$ minute and $145^\circ F$ after 1 minute. How hot is the oven?.

Solution. We have

$$T = 70^\circ F \quad at \quad t = 0$$

$$T = 110^\circ F \quad at \quad t = \tfrac{1}{2}$$

$$T = 145^\circ F \quad at \quad t = 1$$

Using these conditions in $T - T_s = ce^{kt}$ we get 3 equations:

$$70 - T_s = c \quad \Rightarrow \quad T_s = 70 - c \qquad (1)$$

$$110 - T_s = ce^{0.5k} \qquad (2)$$

$$145 - T_s = ce^{k} \qquad (3)$$

Using (1) in (2) and (3) we obtain

$$40 = c(e^{0.5k} - 1) \qquad (4)$$

$$75 = c(e^{k} - 1) \qquad (5)$$

Dividing (5) by (4) we get

$$\frac{e^k - 1}{e^{0.5k} - 1} = \frac{15}{8} \Rightarrow 8e^k - 15e^{0.5k} + 7 = 0$$

$$\Rightarrow 8(e^{0.5k})^2 - 15e^{0.5k} + 7 = 0 \ (a\ quadratic\quad equation\quad in\quad e^{0.5k})$$

$$\Rightarrow (8e^{0.5k} - 7)(e^{0.5k} - 1) = 0$$

$$\Rightarrow e^{0.5k} = \tfrac{7}{8} \Rightarrow c = \frac{75}{e^k - 1} = \frac{75}{\frac{49}{64} - 1} = -320$$

$$\Rightarrow T_s = 70 - c = 390^\circ F.$$

Example 2

A thermometer is removed from a room where the temperature is $70^\circ F$ and is taken outside where the air temperature is $10^\circ F$. After one-half minute the thermometer reads $50^\circ F$. What is the reading of the thermometer at $t=1$? How long will it take the thermometer to reach $15^\circ F$?

Solution. We have $T - T_s = ce^{kt}$ and

$$T = 70^\circ F \quad at \quad t = 0,\ T_s = 10^\circ F \ and\ T = 50^\circ F \quad at \quad t = 0.5$$

Using these conditions we get

$$c = 60 \ and \quad k = 2\ln\tfrac{2}{3}$$

$$\Rightarrow T = T_s + ce^{kt} = 10 + 60\,e^{2t\ln\frac{2}{3}}.$$

Hence we have

$$at\ t=1\ we\ have\ \ T=10+60e^{2\ln\frac{2}{3}}\approx 36.7,$$

$$and\, at\ T=15\ we\ have\, 15=10+60e^{2t\ln\frac{2}{3}}$$

$$\Rightarrow t=\frac{\ln\frac{1}{12}}{2\ln\frac{2}{3}}\approx 3.1.$$

Example 3. A thermometer is taken from an inside room To the outside where the air temperature is $5^\circ F$. After one minute the thermometer reads $55^\circ F$, and after 5 minutes it reads $30^\circ F$. What is the initial temperature of the inside room?

Solution. We have

$$T_s=5,\ T=55^\circ F\quad at\quad t=1\ and\quad T=30^\circ F\quad at\quad t=5.$$

Using these conditions in $T - T_s = ce^{kt}$ we get:

$$T - 5 = ce^{kt}$$

$$55 - 5 = ce^{k} \Rightarrow ce^{k} = 50 \qquad (1)$$

$$30 - 5 = ce^{5k} \Rightarrow ce^{5k} = 25 \qquad (2)$$

Dividing (2) by (1) we get

$$e^{4k} = 0.5 \Rightarrow k = \frac{\ln 0.5}{4} \Rightarrow c = 50\, e^{-\frac{\ln 0.5}{4}}$$

Hence, we have

$$T = T_s + ce^{kt} = 5 + 50\, e^{-\frac{\ln 0.5}{4}} e^{\frac{t \ln 0.5}{4}}$$

Now, at $t = 0$ we have

$$T = 5 + 50\, e^{-\frac{\ln 0.5}{4}} \approx 64.46^{\circ} F.$$

Which is the initial temperature of the room.

Example 4. A small metal bar, whose initial temperature was $20^\circ C$, is dropped into a large container of boiling water. How long will it take the bar to reach $90^\circ C$ if it is known that its temperature increases 2° in 1 second?

Solution. We have

$$T = 20^\circ C \ at \ \ t = 0, \ T_s = 100 \ (Bioling\ water) \ and \ T = 20 + 2 = 22^\circ C \ \ at \ \ t = 1$$

Using these conditions in $T - T_s = ce^{kt}$ we obtain:

$$T - 100 = ce^{kt}$$

$$20 - 100 = ce^{0} \Rightarrow c = -80$$

$$22 - 100 = -80e^{k} \Rightarrow e^{k} = \tfrac{-78}{-80} = \tfrac{39}{40} \Rightarrow k = \ln \tfrac{39}{40}$$

Hence,

$$T = 100 - 80e^{t \ln \frac{39}{40}}$$

Therefore, at $T = 90^\circ C$ we have

$$90 = 100 - 80e^{t\ln\frac{39}{40}} \Rightarrow -80e^{t\ln\frac{39}{40}} = -10$$

$$\Rightarrow e^{t\ln\frac{39}{40}} = \ln\tfrac{1}{8}$$

$$\Rightarrow t = \frac{\ln\frac{1}{8}}{\ln\frac{39}{40}} \Rightarrow t \approx 82 \text{ seconds.}$$