Orthogonal Trajectories

Recall, two non-vertical lines l_1 *and* l_2 with slopes m_1 *and* m_2, respectively, are orthogonal if and only if $m_1 = \frac{-1}{m_2}$.

Let $F(x, y, c_1) = 0$ and $G(x, y, c_2) = 0$ be two families of curves in a plane such that their tangent lines are orthogonal at each point of their intersection, then the two families are said to be orthogonal trajectories of each other.

Hence, if the D. E. of one family is $\frac{dy}{dx} = f(x, y)$,
then the D. E. of the second one is $\frac{dy}{dx} = \frac{-1}{f(x,y)}$.

Example 1

Find the orthogonal trajectories for the family of circles

$$x^2 + (y-c)^2 = c^2. \qquad (1)$$

Solution. Let us rewrite equation (1) on the form

$$x^2 + y^2 - 2cy = 0. \qquad (2)$$

Differentiating equation (2) with respect to x we get

$$2x + 2yy' - 2cy' = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{c-y}$$

From (2) we have $c = \frac{x^2+y^2}{2y}$, hence the differential equation of the given family is

$$\frac{dy}{dx} = \frac{2xy}{x^2-y^2} = f(x, y)$$

Therefore the differential equation of the orthogonal family is $\frac{dy}{dx} = \frac{-1}{f(x,y)} \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

$$or \quad (x^2 - y^2)dx + 2xydy = 0 \qquad (4)$$

which is homogeneous D.E.

To solve equation (4)

let $y = ux \Rightarrow dy = udx + xdu$ hence equation (4) becomes

$$(x^2 - x^2u^2)dx + 2x^2u(udx + xdu) = 0$$

$$\Rightarrow x^2(1+u^2)dx + 2x^3udu = 0$$

$$\Rightarrow x^2(1+u^2)dx = -2x^3udu$$

$$\Rightarrow -\frac{1}{x}dx = \frac{2u}{1+u^2}du \quad \textit{which is separable D.E.}$$

Integrating both sides we obtain

$$\ln|x| + \ln(1+u^2) = c$$

$$or \quad \ln[x(1+\tfrac{y^2}{x^2})] = c \;\Rightarrow\; x^2 + y^2 = c_1 x,$$

which represents also a family of circles, where $c_1 = e^c$.

Example 2.

Find the member of the orthogonal trajectories of the family of curves $y^2 = c\,x^3$ which passes through the point A(2,0).

Solution. Since $y^2 = c\,x^3 \Rightarrow 2yy' = 3cx^2$.

But $c = \frac{y^2}{x^3} \Rightarrow \frac{dy}{dx} = \frac{3y}{2x} = f(x,y)$, this is the D. E. of the given family, hence the D. E. of the orthogonal family is

$$\frac{dy}{dx} = \frac{-1}{f(x,y)} \quad or \quad \frac{dy}{dx} = \frac{-2x}{3y} \quad \textit{which is separable D. E.}$$

Integrating both sides we obtain

$$x^2 + \tfrac{3}{2} y^2 = c.$$

Since the curve passes through A(2,0), substituting this point in the last equation implies $c = 4$, hence the member is

$$x^2 + \tfrac{3}{2} y^2 = 4.$$

Example 3

Find the orthogonal trajectories for the family of curves

$$y = e^{cx}, \quad x \neq 0.$$

Solution. Since $y = e^{cx} \Rightarrow \ln y = cx$

$$\Rightarrow \frac{y'}{y} = c \quad or \quad \frac{y'}{y} = \frac{\ln y}{x}$$

$$\Rightarrow y' = \frac{y \ln y}{x} = f(x, y)$$

Therefore, the DE of the orthogonal family is

$$\frac{dy}{dx} = \frac{-1}{f(x,y)} = \frac{-x}{y\ln y} \quad (separable\, DE)$$

$$\Rightarrow y\ln y\, dy = -xdx$$

$$\Rightarrow \int y\ln y\, dy = \int - xdx$$

$$\Rightarrow \tfrac{1}{2}y^2 \ln y - \tfrac{1}{4}y^2 = -\tfrac{1}{2}x^2 + c.$$

Example 4

Find the orthogonal trajectories for the family of the hyperbolas

$$x^2 + 2xy - y^2 + 4x - 4y = c.$$

Solution

Differentiating both sides with respect to x we get

$$2x + 2y + 2xy' - 2yy' + 4 - 4y' = 0,$$

$$\Rightarrow y'(2x - 2y - 4) = -2x - 2y - 4,$$

$$\Rightarrow y' = \frac{-2(x + y + 2)}{2(x - y - 2)} = \frac{-(x + y + 2)}{x - y - 2}$$

That is the D.E of the given family (first family) is

$$\frac{dy}{dx} = \frac{-(x + y + 2)}{x - y - 2},$$

hence the D. E. for the new family (the second family, or the orthogonal trajectories) is

$$\frac{dy}{dx} = \frac{x - y - 2}{x + y + 2} \quad or \quad (x - y - 2)\,dx - (x + y + 2)dy = 0 \quad ....(1).$$

Which is a first order D. E. with linear coefficients.

These linear coefficients represent two intersected lines given by

$$x - y - 2 = 0 \qquad and \qquad x + y + 2 = 0.$$

Solving these equations for x and y we get: $x = 0 \quad and \quad y = -2.$

Now let $x = X \quad and \quad y = -2 + Y \Rightarrow \dfrac{dy}{dx} = \dfrac{dY}{dX},$

using these values in the D. E. (1) we get

$$(X - Y)\,dX - (X + Y)\,dY = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

which is homogeneous differential equation. To solve it let

$$Y = UX \quad \Rightarrow dY = UdX + XdU$$

Using these values in (2) we obtain

$$(X - UX)\,dX - (X + UX)(U\,dX + XdU) = 0$$

$$\Rightarrow (X - 2UX - U^2 X)dX = (X^2 + UX^2)dU,$$

$$\Rightarrow X\,(1 - 2U - U^2)dX = X^2(1 + U)\,dU, \qquad \div (X^2 * (1 - 2U - U^2)),$$

which is Separable D.E

$$\frac{1}{X}dX = \frac{1+U}{1-2U-U^2}dU,$$

and by integrating both sides we obtain

$$-2\ln|X| = \ln|1-2U-U^2| + c,$$

$$\Rightarrow \frac{c_1}{X^2} = 1 - \frac{2Y}{X} - \frac{Y^2}{X^2}, \quad where\ c_1 = e^{-c},$$

$$\Rightarrow \frac{c_1}{x^2} = 1 - \frac{2(y+2)}{x} - \frac{(y+2)^2}{x^2} \quad \Rightarrow \quad x^2 - 2x(y+2) - (y+2)^2 = c_1.$$

Notice, that D.E. (1) is exact. Since

$$M(x,y) = (x-y-2) \quad \Rightarrow \frac{\partial M}{\partial y} = -1,$$

$$N(x,y) = -(x+y+2) \Rightarrow \frac{\partial N}{\partial x} = -1.$$

Hence the general solution is on the form: $f(x,y) = c, \quad where$

$$f(x,y) = \int M(x,y)\,dx = \tfrac{1}{2}x^2 - yx - 2x + g(y),\ and\ also$$

$$f(x,y) = \int N(x,y)\,dy = -xy - \tfrac{1}{2}y^2 - 2y + h(x),$$

Hence we have

$$f(x,y) = \tfrac{1}{2}x^2 - yx - 2x - \tfrac{1}{2}y^2 - 2y = c.$$

Homework

Find the orthogonal trajectories for the family of curves

$$x^3 + 3xy^2 = c, \quad c \neq 0.$$