Differential Equation of First Order

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First Order Differential Equation

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Here we will start to study some methods which might use to solve first order differential equations.
Consider the equation of order one

$$F(x, y, y') = 0 \tag{1}$$

We suppose that the equation (1) can be written as the form

$$y' = \frac{dy}{dx} = f(x, y). \tag{2}$$

The equation (2) can be written as follows

$$M(x, y)dx + N(x, y)dy = 0, \tag{3}$$

where M and N are two functions of x and y.

Initial Value Problem (IVP)

We are interested in problems in which we seek a solution $y(x)$ of differential equation which satisfies some conditions imposed on the unknown $y(x)$ or its derivatives. On some interval I containing x_0, the problem

$$\text{Solve:} \qquad \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

$$\text{Subject to:} \qquad y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary specified real constants, is clled an *initial-values problem (IVP)* and its $n-1$ derivatives at a single point x_0: $y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$ are called *initial conditions*.

Special cases

First and second-order **IVPs**

$$\text{Solve:} \quad \frac{dy}{dx} = f(x, y)$$
$$\text{Subject to:} \quad y(x_0) = y_0.$$

$$\text{Solve:} \quad \frac{d^2y}{dx^2} = f(x, y, y')$$
$$\text{Subject to:} \quad y(x_0) = y_0, y'(x_0) = y_1.$$

Example (1)

Solve the initial value problem $y' = 10 - x$

Subject to: $y(0) = -1$.

Solution

$$y' = 10 - x \to \frac{dy}{dx} = 10 - x \to dy = (10 - x)dx$$

by integrating both sides with respect to x

$$\Rightarrow y = 10x - \frac{x^2}{2} + c$$

Now by using the initial data, plug it into the general solution and solve for c

$$\Rightarrow -1 = 10(0) - \frac{(0)^2}{2} + c \Rightarrow c = -1$$

$$\therefore \text{solution: } y = 10x - \frac{x^2}{2} - 1$$

Example (2)

Solve the initial value problem

$$\frac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$$

Solution

Step 1:

$\frac{dy}{dx} = 9x^2 - 4x + 5 \Rightarrow dy = (9x^2 - 4x + 5)dx$

$\int dy = \int (9x^2 - 4x + 5)dx \Rightarrow y = 3x^3 - 2x^2 + 5x + c$

Step 2: When $x = -1$, $y = 0$,

$0 = 3(-1)^3 - 2(-1)^2 + 5(-1) + c \Rightarrow 0 = -3 - 2 - 5 + c \Rightarrow c = 10$

The solution is: $y = 3x^3 - 2x^2 + 5x + 10$

Exercises

Exercise 1

Solve the initial value problem $y' + 2xy^2 = 0$
Subject to: $y(0) = -1.$

Exercise 2

Solve the initial value problem $y'' = x$
Subject to: $y(0) = 1, y'(0) = -1.$

Existence of a Unique Solution

Theorem

Consider a first order differential equation

$$\frac{dy}{dx} = f(x, y), \text{with the initial value } y(x_0) = y_0,$$

there exists a unique solution if

- $f(x, y)$ *and* $\frac{\partial f(x,y)}{\partial y}$ *are continuous with in the region* $\mathbb{R}^2$ *of* xy*-plane.*
- (x_0, y_0) *be a point in the region* $\mathbb{R}^2$

Example

Find the largest region of the xy-plane for which the following initial value problems have unique solutions:

(a) $\sqrt{x^2-4}y' = 1 + \sin(x)\ln(y)$, with initial condition $y(3) = 4$.

Solution

$$y' = \frac{1+\sin(x)\ln y}{\sqrt{x^2-4}} = f(x,y)$$

$$y' = \frac{1}{\sqrt{x^2-4}} + \frac{\sin(x)}{\sqrt{x^2-4}}\ln y; \quad y > 0 \text{ and } |x| > 2$$

$$\frac{\partial f}{\partial y} = \frac{\sin x}{\sqrt{x^2-4}}\frac{1}{y}.$$

Then f and $\frac{\partial f}{\partial y}$ are continuous on

$$R = \{(x,y) \in \mathbb{R}^2;\ |x| > 2,\ y > 0\}$$

$$R_1 = \{(x,y) \in \mathbb{R}^2;\ x > 2,\ y > 0\} \cup R_2 = \{(x,y) \in \mathbb{R}^2;\ x < -2,\ y > 0\}$$

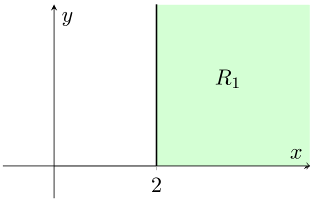


Figure: Largest Region in $xy-$plane for IVP $(3,4)$

As we see the point $(3,4) \in R_1 = \{(x,y);\ x > 2,\ y > 0\}$, so the largest region in xy-plane for which the **IVP** has a unique solution is R_1. If we take any rectangular R_2 with center $(3,4)$ such that $R_2 \subset R_1$, then the **IVP** has also a unique solution, but R_2 is not the largest region.

(b) $\ln(x-2)\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{y-2}$, with initial condition $y\left(\dfrac{5}{2}\right) = 4.$

Solution we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{y-2}}{\ln(x-2)} = f(x,y)$$

by taking the derivative of $f(x,y)$ with respect to y, thus

$$\frac{\partial f}{\partial y} = \frac{1}{\ln(x-2)}\frac{1}{2\sqrt{y-2}}$$

Then f and $\frac{\partial f}{\partial y}$ are continuous on

$$R = \{(x,y) \in \mathbb{R}^2;\ x \neq 2,\ x \neq 3,\ y > 2\}.$$

But

$$R = R_1 = \{(x, y) \in \mathbb{R}^2;\ 2 < x < 3,\ y > 2\}$$
$$\cup\ R_2 = \{(x, y) \in \mathbb{R}^2;\ x > 3,\ y > 2\},$$

As

$$\left(\frac{5}{2}, 4\right) \in R_1 = \{(x, y) \in \mathbb{R}^2;\ 2 < x < 3,\ y > 2\},$$

then the largest region in xy-plane for which the **IVP** has a unique solution is R_1.

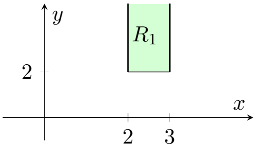


Figure: Largest Region in $xy-$plane for IVP $(\frac{5}{2}, 4)$

(c) $\sqrt{x/y}y' = \cos(x+y);\ y \neq 0,$ with initial condition $y(1) = 1.$

Solution we have

$$y' = \cos(x+y)\left(\frac{x}{y}\right)^{-1/2} = f(x,y),$$

thus,

$$\frac{\partial f}{\partial y} = -\sin(x,y)\left(\frac{x}{y}\right)^{-1/2} - (1/2)\cos(x+y)\left(\frac{x}{y}\right)^{-3/2}\left(\frac{-x}{y^2}\right)'$$

so f and $\frac{\partial f}{\partial y}$ are continuous on

$$R = \{(x,y);\ (x/y) > 0\}.$$

or

$$R = R_1 = \{(x,y) \in \mathbb{R}^2;\ x < 0 \text{ and} y < 0\}$$

$$\cup R_2 = \{(x,y) \in \mathbb{R}^2;\ x > 0 \text{ and} y > 0\}.$$

But

$$(1,1) \in R_2 = \{(x,y);\ x > 0 \text{ and} y > 0\},$$

then the largest region in xy-plane for which the **IVP** has a unique solution is R_2.

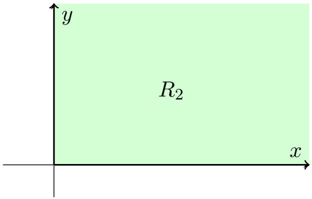


Figure: Largest Region in $xy-$plane for IVP $(1,1)$

Exercise

Determine the largest region of the xy-plane for which the following initial value problem has a unique solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+2x}{y-2x}, \text{with initial condition } \ y(1) = 0.$$

Separable Equations

We begin to study the methods for solving the first-order differential equations. Consider a first-order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0, \tag{4}$$

where M and N are two functions of x and y. Sometimes we can write the equation (4) as

$$F(x)dx + G(y)dy = 0, \tag{5}$$

which is said to be variables separable equation. We solve a variables separable equation by **separating** the variables and integrating.

$$\frac{dy}{G(y)} = f(x)\ dx \Rightarrow \int \frac{dy}{G(y)} = \int f(x)\ dx + c$$

Since we have one arbitrary constant in the solution, we have found the general solution of the variables separable equation.

Separable Equations

Example

Solve the following differential equations:

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$

Solution we can separate the variables of the equation to be

$$dy = 2xdx$$

by integrating the both sides

$$\int dy = \int 2xdx$$

thus,

$$y = x^2 + c.$$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy$

Solution we can separate the variables of the equation to be

$$\frac{dy}{y} = 2xdx$$

by integrating the both sides

$$\int \frac{dy}{y} = \int 2xdx$$

thus,

$$\ln|y| = x^2 + c.$$

(c) $e^x \cos y \; dx + (1 + e^x) \sin y \; dy = 0$

Solution we can separate the variables of the equation to be

$$\frac{\sin y}{\cos y} dy + \frac{e^x}{1 + e^x} dx = 0$$

$$\tan y \; dy + \frac{e^x}{1 + e^x} dx = 0$$

by integrating we have

$$\int \tan y \; dy + \int \frac{e^x}{1 + e^x} dx = c$$

thus,

$$\ln |1 + e^x| + \ln |\sec y| = \ln c_1,$$

$$\ln(|(1 + e^x) \sec y|) = \ln c_1,$$

by taking the Exponential for both side, and from the properties of exponential and logarithmic equations, thus

$$(1+e^x)\sec y = c_1.$$

We have found the general solution of the variables separable equation. Now Find the particular solution at the point $(0,0)$. So we have

$$(1+e^0)\sec 0 = c_1. \quad (\sec\theta = 1/\cos\theta)$$

$$c_1 = 2$$

thus, the particular solution is

$$(1+e^x)\sec y = 2.$$

(d) $2x(y^2+y)dx+(x^2-1)ydy=0, \quad y\neq 0$

Solution we can separate the variables of the equation to be

$$\frac{2x}{x^2-1}dx=\frac{-1}{y+1}dy$$

by integrating the both sides

$$\int \frac{2x}{x^2-1}dx=\int \frac{-1}{y+1}dy$$

thus,

$$\ln|x^2-1|=-\ln|y+1|+c$$
$$\ln|x^2-1|+\ln|y+1|=c$$
$$\ln|(x^2-1)(y+1)|=c$$
$$|(x^2-1)(y+1)|=e^c$$
$$|(x^2-1)(y+1)|=c_1.$$

(e) $(xy + x)dx = (x^2y^2 + x^2 + y^2 + 1)dy = 0$

Solution we have

$$x(y+1)dx = x^2(y^2+1) + (y^2+1)dy$$

$$x(y+1)dx = (x^2+1)(y^2+1)dy$$

we can separate the variables of the equation to be

$$\frac{x}{x^2+1}dx = \frac{y^2+1}{y+1}dy$$

by integrating the both sides

$$\int \frac{x}{x^2+1}dx = \int \frac{y^2+1}{y+1}dy + c$$

thus,

$$(1/2)\ln\left(x^2+1\right) - (1/2)y^2 + y - 2\ln(y+1) = c.$$

Separable Equations

Exercises

Solve the following differential equations:

1. $\frac{dy}{dx} = \frac{y(1-y^2)}{x(1-x^2)}$
2. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x(1-y^2)}{y(1-x^2)}$
3. $(x-1)\frac{\mathrm{d}y}{\mathrm{d}x} = x(y+1)$
4. $y \ln x dx + (1+2y)dy = 0$
5. $e^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x} = e^{2x-y}$

Equations With Homogeneous Coefficients

Definition

A function $F(x, y)$ is called **homogeneous of degree** n if

$$F(tx, ty) = t^n F(x, y), \quad \text{for all } t > 0; t \in \mathbb{R}.$$

A first-order differential equation form

$$M(x, y)dx + N(x, y)dy, \tag{6}$$

is said to be homogeneous if both coefficient functions M and N are homogeneous equations of the **same** degree.
In other words, (6) is homogeneous if

$$M(tx, ty) = t^n M(x, y) \text{ and } N(tx, ty) = t^n N(x, y).$$

Example

1. If $M(x,y)$ and $N(x,y)$ are both homogeneous of the same degree, then $\frac{M(x,y)}{N(x,y)}$ is homogeneous of degree zero. For example $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ is homogeneous of degree zero.
2. The function $f(x,y) = x - 5y + \sqrt{x^2+3y^2}$, is homogeneous of degree one, for

$$\begin{aligned} f(tx,ty) &= tx - 5ty + \sqrt{(tx)^2 + 3(ty)^2} \\ &= t\left[x - 5y + \sqrt{x^2 + ty^2}\right] = tf(x,y) \end{aligned}$$

3. The function $F(x,y) = x^7 \ln(x) - x^7 \ln(y)$, is homogeneous of degree 7, because $f(x,y) = x\ln(x/y)$ and

$$f(tx,ty) = (tx)^7 \ln(tx/ty) = t^7[x\ln(x/y)] = t^7 f(x,y).$$

4. The functions

$$f(x,y) = x^2 + y^2 + \frac{x+y}{x-y} \quad \text{and } g(x,y) = 3x - 2y + e^{x-y},$$

are not homogeneous.

General Method

A first order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ which can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F\left(\frac{y}{x}\right)$$

is called a **homogeneous differential equation**.

To solve the homogeneous differential equation:

by letting $u = \frac{y}{x}$, that is let $y = xu \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$, the equation then becomes

$$x\frac{\mathrm{d}u}{\mathrm{d}x} + u = F(u).$$

Hence

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = F(u) - u.$$

This equation is clearly separable, and can be solved as such.

Or

by letting

$$u = \frac{x}{y}; y \neq 0,$$

that is let $x = yu \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = y\frac{\mathrm{d}u}{\mathrm{d}y} + u,$

the equation then becomes

$$y\frac{\mathrm{d}u}{\mathrm{d}y} + u = F(u).$$

Hence

$$y\frac{\mathrm{d}u}{\mathrm{d}y} = F(u) - u.$$

This equation also is clearly separable, and can be solved as such.

Example

Solve the following differential equations:

1. $(x^2 - xy + y^2)dx - xydy = 0.$
2. $\dfrac{dy}{dx} + \dfrac{3xy + y^2}{x^2 + xy} = 0$; $x \neq 0$ and $y \neq -x$.
3. $ydx + x(\ln\left(\frac{x}{y}\right) - 1)dy = 0$, $y(1) = e$.
4. $x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$; $x > 0$.

Solution

1. $(x^2 - xy + y^2)dx - xydy = 0.$

Solution The coefficients in this equation are both homogeneous and degree two in x and y. Let $u = \frac{y}{x}; x \neq 0$. Thus

$$y = ux \Rightarrow dy = udx + xdu$$

So the equation becomes

$$(x^2 - x(xu) + (xu)^2)dx - x(xu)(udx + xdu) = 0$$
$$(x^2 - x^2u + x^2u^2)dx - x^2u(udx + xdu) = 0$$
$$x^2(1 - u + u^2)dx - x^2u(udx + xdu) = 0$$

by dividing this equation by x^2 we obtain

$$(1 - u + u^2)dx - u(udx + xdu) = 0$$
$$(1 - u + u^2)dx - u^2dx - xudu = 0$$
$$(1 - u + u^2 - u^2)dx - xudu = 0$$

$$(1-u)dx - xudu = 0$$

hence we separate variables to get

$$\frac{dx}{x} - \frac{u}{1-u}du = 0; u \neq 1$$

$$\frac{dx}{x} + \frac{u}{u-1}du = 0$$

$$\frac{dx}{x} + \left[1 + \frac{1}{u-1}\right] du = 0$$

a family of solutions is seen to be

$$\ln x + u + \ln(u-1) = \ln c; c \neq 0$$

$$\ln x(u-1) + u = \ln c$$

$$x(u-1)e^u = c$$

$$x\left(\frac{y}{x} - 1\right) e^{\frac{y}{x}} = c.$$

2.

$$\frac{dy}{dx} + \frac{3xy + y^2}{x^2 + xy} = 0 \ ; \ \ x \neq 0 \ \text{ and } \ y \neq -x.$$

Solution The Coefficients of the differential equation are homogeneous, and it can be written in the form

$$(x^2 + xy)dy + \left(3xy + y^2\right) dx = 0 \ .$$

Let

$$u = \frac{x}{y}; y \neq 0,$$

hence

$$x = yu \implies dx = ydu + udy.$$

Then

$$(u^2y^2 + y^2u)dy + (3y^2u + y^2)(ydu + udy) = 0 \ ,$$

or

$$2y^2udy + 4y^2u^2dy = -y^3(3u+1)du \ ,$$

$$-\frac{dy}{y} = \frac{3u+1}{2u(2u+1)}du \quad ; y \neq 0 \ , \quad u \neq 0 \ \text{ and } \ u \neq \frac{-1}{2} \ ,$$

but

$$\frac{3u+1}{2u(2u+1)} = \frac{1}{2u} + \frac{1}{2(2u+1)} \ ,$$

then

$$\ln|y| + \frac{1}{2}\ln|u| + \frac{1}{4}\ln|2u+1| = \ln|c| \quad ; c \neq 0$$

or

$$\ln\left[y^4u^2\left|2u+1\right|\right] = \ln c^4,$$

$$\ln\left[y^2x^2\left|\frac{2x}{y}+1\right|\right] = \ln c^4,$$

$$x^2\left|2xy+y^2\right| = c^4,$$

hence

$$yx^2(2x+y) = c_1.$$

is the family of curves defines the solutions of the DE, where $c_1 = c^4$ is an arbitrary constant.

3.

$$ydx + x\left(\ln\frac{x}{y} - 1\right)dy = 0 \quad , \quad y(1) = e$$

Solution The Coefficients of the differential equation are homogeneous with degree one .So we can put $u = \frac{x}{y}$ then

$$x = yu \implies dx = ydu + udy \ ,$$

we can suppose that $y > 0$ because the initial condition $y(1) > 0$. We obtain

$$y(ydu + udy) + yu(\ln u - 1)dy = 0,$$

$$y^2du + yu\ln u \ dy = 0 \ ,$$

hence

$$\frac{du}{u\ln u} + \frac{dy}{y} = 0 \quad ; \quad u \neq 1,$$

$$\ln|y\ln u| \quad = c \quad \Longrightarrow \quad |y\ln u| = e^c,$$

or

$$y\ln\left|\frac{x}{y}\right| = \mp e^c = c_1,$$

is the solution of differential equation. Now we use the initial condition $x = 1, y = e \implies c_1 = -e,$ then the solution of the IVP for the DE is given by

$$y\ln\left(\frac{x}{y}\right) = -e \text{ , where } x > 0 \text{ and } y > 0$$

4.

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2} \quad ; \; x > 0 \; .$$

Solution The differential equation is also homogeneous . Let $u = \frac{y}{x}$ then

$$y = ux \Longrightarrow \frac{dy}{dx} = u + xu',$$

hence

$$u + xu' - u = \sqrt{1 + u^2} \; ,$$

or

$$\frac{du}{\sqrt{1 + u^2}} = \frac{dx}{x} \quad \Longrightarrow \quad \sinh^{-1}(u) - \ln x = c \; ,$$

So the solution of the DE is given by

$$\sinh^{-1}(\frac{y}{x}) - \ln x = c, \text{ where } c \text{ is an arbitrary constant .}$$

Summary

Let us summarize the steps to follow

1. Recognize that your equation is an homogeneous equation; that is, you need to check that $f(tx, ty) = f(x, y)$, meaning that f(tx,ty) is independent of the variable t;
2. Write out the substitution $u = y/x$;
3. Through easy differentiation, find the new equation satisfied by the new function u. You may want to remember the form of the new equation:
$$x\frac{\mathrm{d}u}{\mathrm{d}x} = F(u) - u \text{ or } y\frac{\mathrm{d}u}{\mathrm{d}y} = F(u) - u$$
4. Solve the new equation (which is always separable) to find u;

5. Go back to the old function y through the substitution $y = xu$;
6. If you have an **IVP**, use the initial condition to find the particular solution.

Exercise

Solve the following differential equations:

1. $(x^2 + y^2)dx - 2xydy = 0.$
2. $(x - y)dx + (2x + y)dy = 0.$
3. $2x^2y' - y(2x + y) = 0.$
4. $xdx + \sin^2\left(\frac{x}{y}\right)[ydx - xdy] = 0.$

Homogeneous Equations Requiring a Change of Variables

Solving Some Differential Equations by Using Appropriate Substitution

If we have a differential equation of the form

$$\frac{dy}{dx} = f(ax + by),$$

we use the substitution $u = ax + by$, then we get

$$\frac{du}{dx} = a + b\frac{dy}{dx}.$$

Example

Solve the following differential equations by using appropriate substitution:

1. $\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0.$
2. $\frac{dy}{dx} = \frac{1-4x-4y}{x+y}$; $y \neq -x$
3. $\frac{dy}{dx} = \frac{x-y-3}{x+y-1}; \quad x + y - 1 \neq 0.$
4. $\frac{dy}{dx} = \frac{y(1+xy)}{x(1-xy)}; x > 0, y > 0, xy \neq 1.$ (Use the substitution $u = xy$)

Solution

1. $\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0.$

If we let $u = -2x + y$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = -2 + \frac{\mathrm{d}y}{\mathrm{d}x}$, so the equation is transformed into

$$\frac{\mathrm{d}u}{\mathrm{d}x} + 2 = u^2 - 7 \quad \text{or} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = u^2 - 9.$$

The last equation is separable, thus

$$\frac{du}{u^2 - 9} = dx.$$

Using partial fractions

$$\frac{du}{(u-3)(u+3)} = dx \qquad \text{or} \quad \frac{1}{6}\left[\frac{1}{u-3} - \frac{1}{u+3}\right]$$

and then integrating yields

$$\frac{1}{6}\left[\ln\frac{u-3}{u+3}\right] = x + c_1 \qquad \text{or} \quad \frac{u-3}{u+3} = e^{6x+6c_1} = ce^{6x}.$$

Finally, applying the initial condition $y(0) = 0$ to get the particular solution

$$y = 2x + \frac{3(1 - e^{6x})}{(1 + e^{6x})}.$$

2. $\frac{dy}{dx} = \frac{1-4x-4y}{x+y}$; $y + x \neq 0$.

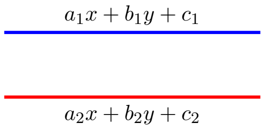
We see that the two straight lines $1 - 4x - 4y$ and $x + y$ are parallels, i.e if we have equation in form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$$

and

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

The figure below shows the nature of the two lines



In this case we let $u = x + y$. Hence

$$y' = u' - 1,$$

and we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-4u}{u} = \frac{\mathrm{d}u}{\mathrm{d}x} - 1 \quad \text{or} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1-3u}{u}$$

The last equation is separable, thus

$$\frac{u}{1-3u}du = dx.$$

and then integrating yields

$$\frac{x+y}{3} + \frac{1}{9}\ln|1-3u| + x = c.$$

3. $\frac{dy}{dx} = \frac{x-y-3}{x+y-1}; \quad x+y-1 \neq 0.$

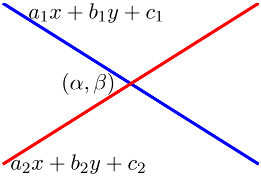
We see that the two straight lines $x-y-3$ and $x+y-1$ are not parallels, i.e if we have equation in form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

and

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}.$$

The figure below shows the nature of the two lines



In this case we need first to find the **intersection point** (α, β), then we use the substitutions

$$x = u + \alpha \quad \text{and} \quad y = v + \beta$$

Thus, in this example we need to solve the two equations to find the **intersection point** which is $(2, -1)$.

Now we will use the substitutions

$$x = u + 2 \quad \text{and} \quad y = v - 1$$

thus,

$$dx = du \quad \text{and} \quad dy = dv.$$

Then

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{u + 2 - (v - 1) - 3}{u + 2 + (v - 1) - 1} = \frac{u - v}{u + v}.$$

So now we have this homogeneous differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{u - v}{u + v},$$

so we let $t = \frac{v}{u}$, where $u \neq 0$. Then $v = ut$ and

$$\frac{\mathrm{d}v}{\mathrm{d}u} = t + u\frac{\mathrm{d}t}{\mathrm{d}u},$$

thus,

$$t + u\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{u - ut}{u + ut} \quad \text{or} \quad u\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{1 - t}{1 + t} - t = \frac{1 - 2t - t^2}{1 + t}$$

by integrating

$$\int \frac{du}{u} = \int \frac{1+t}{1-2t-t^2} dt$$

$$\int \frac{du}{u} = -\frac{1}{2} \int \frac{-2-2t}{1-2t-t^2} dt$$

$$\ln u + \frac{1}{2} \ln\left(1-2t-t^2\right) = c$$

$$\ln \left[u^2 (1 - 2\frac{v}{u} - \frac{v^2}{u^2} \right] = 2c$$

$$\ln \left[u^2 - 2vu - v^2 \right] = 2c$$

$$e^{\ln\left[u^2-2vu-v^2\right]} = e^{2c}$$

$$u^2 - 2vu - v^2 = c_1$$

thus, the solution is $(x-2)^2 - 2(x-2)(y+1) - (y+1)^2 = c_1$.

Exact Differential Equations

A differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0,$$

is called **exact**, if there is a function F of x and y such that

$$dF(x, y) = M(x, y)dx + N(x, y)dy = 0.$$

Recall that the total differential of a function $F(x, y)$ is

$$dF(x, y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

provided that the partial derivatives of the function F is exists.

Theorem (Criterion for an Exact Differential)

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b, c < y < d$. Then a necessary and sufficient condition that $M(x, y)dx + N(x, y)dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example

Prove that the following differential equations are exact and find their solutions

1. $(6x^2 + 4xy + y^2)dx + (2x^2 + 2xy - 3y^2)dy = 0$
2. $\left[\cos x \ln(2y-8) + \frac{1}{x}\right] dx + \frac{\sin x}{y-4} dy$; $x \neq 0$ and $y > 4$.
3. $(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$

To prove that we need to check for the differential equation

$$M(x,y)dx + N(x,y)dy = 0,$$

if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

then the differential equation is exact.

Solution

1. $(6x^2 + 4xy + y^2)dx + (2x^2 + 2xy - 3y^2)dy = 0$

$M(x, y) = 6x^2 + 4xy + y^2 \Rightarrow \frac{\partial M}{\partial y} = 4x + 2y$ $N(x, y) = 2x^2 + 2xy - 3y^2$ $\Rightarrow \frac{\partial N}{\partial x} = 4x + 2y$ i.e. $\frac{\partial M}{\partial y} = 4x + 2y = \frac{\partial N}{\partial x}$. Thus, the differential

equation is exact.

Now to find the solution

$$\int (6x^2 + 4xy + y^2)dx = 2x^3 + 2x^2y + y^2x.$$

$$\int (2x^2 + 2xy - 3y^2)dy = 2x^2y + xy - y^3.$$

Thus, the family solution is

$$2x^3 + 2x^2y + y^2x - y^3 = c$$

.

2. $\left[\cos x \ln(2y-8) + \frac{1}{x}\right] dx + \frac{\sin x}{y-4} dy$; $x \neq 0$ and $y > 4$.

$M(x,y) = \left[\cos x \ln(2y-8) + \frac{1}{x}\right] \Rightarrow \frac{\partial M}{\partial y} = 2\cos x \frac{1}{2y-8} = \cos x \frac{1}{y-4}$
$N(x,y) = \frac{\sin x}{y-4} \Rightarrow \frac{\partial N}{\partial x} = \cos x \frac{1}{y-4}$

i.e. $\frac{\partial M}{\partial y} = \cos x \frac{1}{y-4} = \frac{\partial N}{\partial x}$.

Thus, the differential equation is exact.

Now to find the solution

$$\begin{aligned} \int \left[\cos x \ln(2y-8) + \frac{1}{x}\right] dx &= \sin x \ln(2y-8) + \ln x \\ &= \sin x \ln[2(y-4)] + \ln x \end{aligned}$$

$$\int \frac{\sin x}{y-4} dy = \sin x \ln(y-4),$$

Thus, the family solution is

$$\sin x \ln(y-4) + \ln x + c = 0$$

3. $(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$

$M(x,y) = e^{2y} - y\cos xy \Rightarrow \frac{\partial M}{\partial y} = 2e^{2y} - \cos xy + xy\sin xy$
$N(x,y) = 2xe^{2y} - x\cos xy + 2y \Rightarrow \frac{\partial N}{\partial x} = 2e^{2y} - \cos xy + xy\sin xy$
i.e. $\frac{\partial M}{\partial y} = 2e^{2y} - \cos xy + xy\sin xy = \frac{\partial N}{\partial x}$.
Thus, the differential equation is exact.
Now to find the solution

$$\int (e^{2y} - y\cos xy) \; dx = xe^{2y} - \sin xy$$

$$\int (2xe^{2y} - x\cos xy + 2y) \; dy = xe^{2y} - \sin xy + y^2,$$

Thus, the family solution is

$$xe^{2y} - \sin xy + y^2 + c = 0$$

.

Integrating Factor

Consider a first order differential equation

$$M(x,y)dx + N(x,y)dy = 0, \tag{7}$$

where M, N and $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous on a certain region R in xy-plane. Suppose that the equation (7) is **not exact**, i.e

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Definition

A function $\mu(x,y)$ is called an **integrating factor** of (7) if the differential equation

$$(\mu M)dx + (\mu N)dy = 0, \tag{8}$$

is **exact**, i.e

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}. \tag{9}$$

In other words, if the equation (7) is **not exact**, we can often make it so by multiplying throughout by an **integrating factor** $\mu(x,y)$ and the finding $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. The integrating factors are able to be determined by solving

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

for μ.

The integrating factor will be in one of the following forms

1. $\mu = \mu(x)$
2. $\mu = \mu(y)$
3. $\mu = \mu(x, y) = x^m y^n$

We can rewrite the equation (9) as follows:

$$N_{\mu_x} - M_{\mu_y} = (M_y - N_x)_\mu \tag{10}$$

In general, it is very difficult to solve the equation (10). In this section we will only consider that μ is a one variable function (x or y, not both). There are two cases:

1. If μ depends on x ($\mu = \mu(x)$). Then $\mu_y = 0$, so the equation (10) becomes
$$\frac{1}{\mu}\mu_x = \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{M_y - N_x}{N},$$
so
$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}.$$

2. If μ depends on y ($\mu = \mu(y)$). Then $\mu_x = 0$, so the equation (10) becomes
$$\frac{1}{\mu}\mu_y = \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}y} = \frac{N_x - M_y}{M},$$
so
$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}.$$

We summarize that for the differential equation

$$M(x,y)dx + N(x,y)dy = 0,$$

as following

1. If $(M_y - N_x)/N$ is a function of x only, then the integrating factor for the differential equation is
$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}.$$
2. If $(N_x - m_y)/M$ is a function of y only, then the integrating factor for the differential equation is
$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}.$$

Example

Solve the following differential equations:

1. $xydx + (2x^2 + 3y^2 - 20)dy = 0; x \neq 0, y > 0.$
2. $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0, x(x + 2y) \neq 0.$

Solution

1. $xydx + (2x^2 + 3y^2 - 20)dy = 0; x \neq 0, y > 0.$

$M(x, y) = xy \Rightarrow \frac{\partial M}{\partial y} = x$
$N(x, y) = 2x^2 + 3y^2 - 20 \Rightarrow \frac{\partial N}{\partial x} = 4x$ so, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Thus, the differential equation is not exact.

Now let's us find the solution

$$\frac{M_y - N_x}{N} = \frac{4x - x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20},$$

we note that the quotient is depended on x and y.

So we need to find

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3}{y} = g(y),$$

we note that the quotient is depended only on y, thus the integrating factor

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int g(y) dy} = e^{\int \frac{3}{y} dy} = e^{3 \ln y} = e^{\ln y^3} = y^3.$$

Then we multiply the equation by $\mu(y) = y^3$,
thus, the equation becomes

$$xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3) dy = 0$$

this equation is exact now, that is because

$$\frac{\partial M}{\partial y} = 4xy^3 = \frac{\partial N}{\partial x}.$$

So

$$\int xy^4 \, dx = \frac{1}{2}x^2y^4$$

$$\int (2x^2y^3 + 3y^5 - 20y^3) \, dy = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4,$$

Thus, the family solution is

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 + c = 0$$

.

2. $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0, x(x + 2y) \neq 0.$

$M(x, y) = 4xy + 3y^2 - x \Rightarrow \frac{\partial M}{\partial y} = 4x + 6y$
$N(x, y) = x(x + 2y) \Rightarrow \frac{\partial N}{\partial x} = 2x + 2y,$
so, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.
Thus, the differential equation is not exact.
Now let's us find the solution

$$\frac{M_y - N_x}{N} = \frac{4x + 6y - 2x - 2y}{x(x + 2y)} = \frac{2(x + 2y)}{x(x + 2y)} = \frac{2}{x} = f(x),$$

we note that the quotient is depended on x, thus the integrating factor

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int f(x) dx} = e^{\int \frac{2}{x} dy} = e^{2 \ln x} = e^{\ln x^2} = x^2.$$

Then we multiply the equation by $\mu(x) = x^2$,
thus, the equation becomes

$$(4x^3y + 3x^2y^2 - x^3)dx + (x^4 + 2x^3y)dy = 0$$

this equation is exact now, that is because

$$\frac{\partial M}{\partial y} = 4x^3 + 6x^3y = \frac{\partial N}{\partial x}.$$

So

$$\int (4x^3y + 3x^2y^2 - x^3)\ dx = x^4y + x^3y^2 - \frac{1}{4}x^4,$$

$$\int (x^4 + 2x^3y)\ dy = x^4y + x^3y^2.$$

Thus, the family solution is

$$x^4y + x^3y^2 - \frac{1}{4}x^4 + c = 0$$

Example

Find m, n such that

$$\mu(x,y) = x^m y^n,$$

is an integrating factor of the differential equation

$$(2y^2 + 4x^2 y)dx + (4xy + 3x^3)dy = 0.$$

Solution

$$(2y^2 + 4x^2 y)dx + (4xy + 3x^3)dy = 0,$$

we need to find m and n such that the equation

$$(2x^m y^{n+2} + 4x^{m+2} y^{n+1})dx + (4x^{m+1} y^{n+1} + 3x^{m+3} y^n)dy = 0,$$

thus,

$$\frac{\partial M}{\partial y} = 2(n+2)x^m y^{n+1} + 4(n+1)x^{m+2} y^n,$$

$$\frac{\partial N}{\partial x} = 4(m+1)x^m y^{n+1} + 3(m+3)x^{m+2} y^n.$$

For the exactness we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

by equating coefficients we have that

$$2(n+2) = 4(m+1) \Rightarrow 2n - 4m = 0 \Rightarrow n = 2m,$$

and

$$4(n+1) = 3(m+3) \Rightarrow 4n - 3m - 5 = 0.$$

Therefor,

$$m = 1 \quad \text{and} \quad n = 2.$$

Thus, integrating factor of the differential equation

$$\mu(x,y) = xy^2.$$

Therefor the solution for the given differential equation is

$$x^2y^4 + x^4y^3 = c.$$

Exercises

Solve the following differential equations:

1. $(x^2 + y^2 + 1)dx + x(x - 2y)dy = 0.$
2. $y(x + y + 1)dx + x(x + 3y + 2)dy = 0; y(x + y + 1) \neq 0$

The General Solution of a Linear Differential Equations

Consider the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{11}$$

where P and Q are continuous function on the interval (a, b).
The integrating factor of the differential equation (11) is

$$\mu(x) = e^{\int P(x)dx}.$$

The general solution of equation (11) is given by

$$y\mu(x) = \int \mu(x)Q(x)dx + C.$$

Since $\mu(x) \neq 0$, for $x \in (a, b)$, then we can write

$$y\mu(x) = \int \mu(x)Q(x)dx + C,$$

$$y(x) = e^{-\int P(x)dx} \int \mu(x)Q(x)dx + Ce^{-\int P(x)dx}.$$

Example

Solve the following differential equations:

1. $x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x^3.$
2. $(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + xy + x^3 + x = 0.$
3. $(y - x + xy\cot x)dx + xy + xdy = 0; 0 < y < \pi$ with initial value problem $y(\pi/2) = 0.$

Solution

1. $x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x^3.$

The equation can be written in the form $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2y}{x} = x^2$, we can see that the equation is in a Linear Differential Equation Form. Where $P(x) = \frac{2}{x}$, and $Q(x) = x^2$.

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = x^2.$$

The general solution will be in form

$$y\mu(x) = \int \mu(x)Q(x)dx + C,$$

so,

$$yx^2 = \int x^2x^2dx \Rightarrow yx^2 = \int x^4dx.$$

Thus, the general solution is

$$yx^2 = \frac{1}{5}x^5 + c.$$

2. $(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + xy + x^3 + x = 0.$

The equation can be written in the form $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{1+x^2}y = -x$, we can see that the equation is in a Linear Differential Equation Form. Where $P(x) = \frac{x}{1+x^2}$, and $Q(x) = -x$.

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{x}{1+x^2}dx} = e^{\frac{1}{2}\ln\left(1+x^2\right)} = (1+x^2)^{\frac{1}{2}}.$$

The general solution will be in form

$$y\mu(x) = \int \mu(x)Q(x)dx + C,$$

so,

$$y\sqrt{1+x^2} = -\int x\sqrt{1+x^2}dx,$$

thus, the general solution is

$$y\sqrt{1+x^2} = \frac{-1}{3}(1+x^2)^{\frac{3}{2}} + c.$$

Exercise

Find the initial value problem **(IVP)**

$$(y - x + xy\cot x)dx + xdy = 0; 0 < y < \pi$$

and

$$y(\pi/2) = 0.$$

(Hint: $P(x) = \frac{1 - x\cot x}{x}$ and $Q(x) = 1$)

Bernoulli's Equation

The Bernoulli's equation is a first order differential equation, which can be written in the form

$$y' + P(x)y = Q(x)y^n, \tag{12}$$

where $n \in \mathbb{R}$.

1. If $n = 0$ then the equation (12) is a linear first order differential equation and we can solve it as we saw before.
2. If $n = 1$ then the equation (12) is becomes a differential equation with separable variables, and we can solve it by by separating the variables.

3. If $n \neq 0$ and $n \neq 1$ then the equation (12) can be written in the form

$$y^{-n}y' + P(x)y^{-n+1} = Q(x).$$

Now we let $u = y^{-n+1}$, then we have

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (-n+1)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$$

or

$$u' = (-n+1)y^{-n}y'.$$

$$\frac{1}{-n+1}u' + P(x)u = Q(x)$$

or

$$u' + (-n+1)P(x)u = (-n+1)Q(x),$$

which is a linear first order differential equation and we can solve it.

Bernoulli's Equation

Example

Solve the following differential equations:

1. $\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = xe^{-x^2}y^3$.
2. $y(6y^2 - x - 1)dx + 2xdy = 0; x \neq 0$.

Solution

1. $\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = xe^{-x^2}y^3.$

We can see that; the equation is in the Bernoulli's Equation Form. The equation can be written in the form

$$y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy^{-2} = xe^{-x^2}.$$

Now we let $u = y^{-2}$, thus we have

$$u' = -2y^{-3}y'.$$

Thus, the equation becomes

$$\frac{-1}{2}\frac{\mathrm{d}u}{\mathrm{d}x} + 2xu = xe^{-x^2}.$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} - 4xu = -2xe^{-x^2}, \tag{13}$$

thus, the equation (13) is in linear first order differential equation and we can solve it. Where $P(x) = -4x$, and $Q(x) = -2xe^{-x^2}$.

$$\mu(x) = e^{\int P(x)dx} = e^{\int -4x \ dx} = e^{-2x^2}.$$

The general solution will be in form

$$u\mu(x) = \int \mu(x)Q(x)dx + C,$$

so,

$$ue^{-2x^2} = \int e^{-2x^2}(-2xe^{-x^2}) \ dx$$

$$ue^{-2x^2} = \frac{-2}{-6} \int -6xe^{-3x^2} \ dx$$

$$ue^{-2x^2} = \frac{1}{3}e^{-3x^2} + c,$$

$$u = \frac{1}{3}e^{-x^2} + ce^{-2x^2},$$

thus, the general solution is

$$\frac{1}{y^2} = \frac{1}{3}e^{-x^2} + ce^{-2x^2}.$$

2. $y(6y^2 - x - 1)dx + 2xdy = 0; x \neq 0.$

The equation can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{x+1}{2x}y = \frac{-3}{x}y^3.$$

So we have the Bernoulli's Equation, and it might be written in the form

$$y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{x+1}{2x}y^{-2} = \frac{-3}{x}.$$

Now we let $u = y^{-2}$, thus we have

$$u' = -2y^{-3}y'.$$

Thus, this equation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{x+1}{2x}u = \frac{6}{x}. \tag{14}$$

Thus, the equation (14) is in linear first order differential equation and we can solve it. Where $P(x) = \frac{x+1}{2x}$, and $Q(x) = \frac{6}{x}$.

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{x+1}{2x} \; dx} = xe^x.$$

The general solution will be in form

$$u\mu(x) = \int \mu(x)Q(x)dx + C,$$

$$uxe^x = 6e^x + C$$

thus, the general solution is

$$y^2(6 + Ce^{-x}) = x.$$

Exercises

Solve the following differential equations:

1. $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}y = -2e^{x}y^{2}.$
2. $(2y^3 - x^3)dx + 2xy^2dy = 0; x \neq 0$ with **IV** $y(1) = 1.$