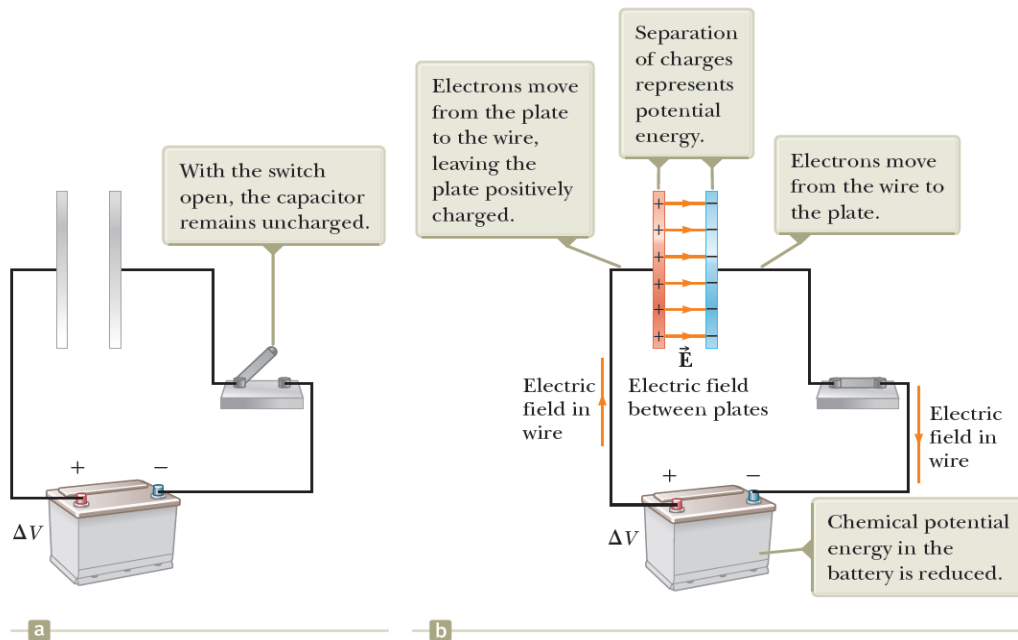


25.4 Energy stored in a charged capacitor

Capacitors are devices that store electrical energy by accumulating charge on their plates. The energy stored in a capacitor originates from the work required to move charges against the electric field.



You can watch the following video:

<https://www.youtube.com/watch?v=pOerI4cPAFA>

1- Calculation of Energy Stored

Consider a capacitor with charge Q and potential difference V . The small amount of work dW required to move an infinitesimal charge dq onto the capacitor is:

$$dW = Vdq$$

$$dW = \left(\frac{q}{C}\right)dq$$

To find the total work required to charge the capacitor from $q=0$ to $q=Q$, we integrate:

$$W = \int_0^q \left(\frac{q}{C}\right) dq$$

Solving the integral:

$$W = \frac{1}{2} \frac{q^2}{C} \quad \text{or} \quad W = \frac{1}{2} CV^2$$

This is equal to the electrical potential energy of the system:

$$U = \frac{1}{2} CV^2 = \frac{1}{2} qV$$

Where:

U is the **stored energy**
 Q is the **charge** on the capacitor
 V is the **potential difference**
 C is the **capacitance**

2- Energy Density in a Capacitor

The energy stored in a parallel-plate capacitor can be related to the electric field E.

Since: $V=Ed$ and $C = \epsilon_0 \frac{A}{d}$

The energy density u (energy per unit volume) can be calculated as follows:

$$U = \frac{1}{2} \left(\epsilon_0 \frac{A}{d}\right) (Ed)^2 = \frac{1}{2} \epsilon_0 (Ad) E^2$$

Since the quantity (Ad) represents the volume between the plates, we can define the electric energy density u as:

$$u = \frac{U}{Ad} = \frac{1}{2} \epsilon_0 E^2$$

Where:

ϵ_0 is the **permittivity of free space**
 E is the **electric field** between the plates.

$$u \propto E^2$$

- **That is, the energy density in any electric field is proportional to the square of the magnitude of the electric field at a given point.**

- **Think about it:**

You have three capacitors and a battery. In which of the following combinations of the three capacitors is the maximum possible energy stored when the combination is attached to the battery?

- (a) series
- (b) parallel
- (c) no difference because both combinations store the same amount of energy.

🔍 **The relationship between the energy stored in a capacitor and its capacitance depends on the conditions under which it changes.** Let's analyze different cases using the energy formula:

$$U = \frac{1}{2} CV^2$$

Where:

- U is the stored energy,
- C is the capacitance,
- V is the voltage applied across the capacitor.

Case 1: Constant Voltage (V is fixed)

If the voltage across the capacitor remains constant while increasing capacitance, the stored energy **increases** because:

$$U \propto C$$

Example:

If C doubles while V remains the same, then U also doubles.

Thus, a larger capacitance at the same voltage results in more stored energy.

Case 2: Constant Charge (Q is fixed)

Since capacitance is related to charge and voltage by: $Q=CV$

We can rewrite energy as: $U = \frac{Q^2}{2C}$

Here, **energy is inversely proportional to capacitance**. If capacitance increases while the charge remains constant, the stored energy **decreases**.

Example:

If C doubles while Q is fixed, then U is **halved**.

Conclusion

- **If voltage is held constant:** Increasing capacitance **increases** stored energy.
- **If a charge is held constant:** Increasing capacitance **decreases** stored energy.

So, whether energy increases or decreases depends on whether voltage or charge is controlled.

- Applications

- **Capacitors in Circuits:** Used in **power supplies, signal processing, and energy storage.**
- **Energy Storage in Electric Vehicles:** Supercapacitors store large amounts of energy for quick charge/discharge cycles.
- **Medical Devices:** Defibrillators use capacitors to deliver sudden bursts of electrical energy.

Exercise-1:

(a) A $3.00\ \mu\text{F}$ capacitor is connected to a $12.0\ \text{V}$ battery. How much energy is stored in the capacitor?

(b) How much energy would have been stored if the capacitor had been connected to a $6.00\ \text{V}$ battery?

Solution:

Given: $C = 3\ \mu\text{F}$, $V = 12\ \text{V}$

Formula: $U = \frac{1}{2} C V^2$

$$(a) U = \frac{1}{2} (3 \times 10^{-6}\ \text{F})(12)^2 = 2.16 \times 10^{-4}\ \text{J} = 216 \times 10^{-6}\ \text{J} = 216\ \mu\text{J}$$

$$(b) U = \frac{1}{2} (3 \times 10^{-6}\ \text{F})(6)^2 = 54 \times 10^{-6}\ \text{J} = 54\ \mu\text{J}$$

Please note the difference between the two results when reducing the potential from $12\ \text{V}$ to $6\ \text{V}$.

Exercise-2:

A uniform electric field $E = 3000\ \text{V/m}$ exists within a particular region. What volume of space contains an energy equal to $1.00 \times 10^{-7}\ \text{J}$? Express your answer in cubic meters and in liters.

Solution:

Given: $E = 3000 \text{ V/m}$, $U = 1 \times 10^{-7} \text{ J}$

Formula: $u = 1/2 \epsilon_0 E^2$ (please pay attention to the u , here, represents the energy density)

Volume $V = 2U / (\epsilon_0 E^2)$

$$V = 2 \times 1 \times 10^{-7} / (8.85 \times 10^{-12} \times (3000)^2) = 2.51 \times 10^{-3} \text{ m}^3 = 2.51 \text{ L}$$

Exercise-3:

Two capacitors, $C_1 = 4.0 \text{ } \mu\text{F}$ and $C_2 = 6.0 \text{ } \mu\text{F}$, are connected in series and charged to 100 V. Find the total energy stored.

Solution:

Given: $C_1 = 4 \text{ } \mu\text{F}$, $C_2 = 6 \text{ } \mu\text{F}$, $V = 100 \text{ V}$ (series connection)

Formula: $1/C_{eq} = 1/C_1 + 1/C_2$

$$C_{eq} = (4 \times 6) / (4 + 6) = 2.4 \text{ } \mu\text{F}$$

$$\text{Energy: } U = 1/2 C_{eq} V^2 = 1/2 (2.4 \times 10^{-6})(100)^2 = 1.2 \times 10^{-2} \text{ J}$$

Exercise-4:

A parallel-plate capacitor has a plate area of 0.02 m^2 , a plate separation of 2 mm , and is air-filled ($\epsilon_0 = 8.85 \times 10^{-12} \text{ F/m}$). Find the energy density if the electric field between the plates is 5000 V/m .

Solution:

Given: $A = 0.02 \text{ m}^2$, $d = 2 \times 10^{-3} \text{ m}$, $E = 5000 \text{ V/m}$

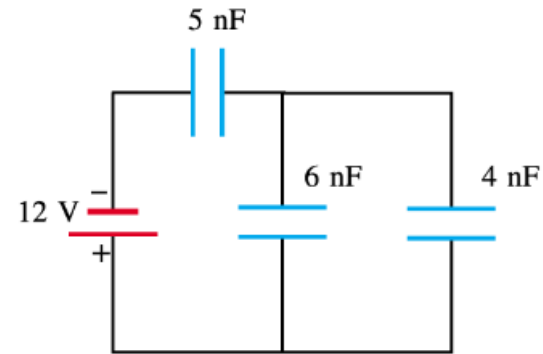
Formula: $u = 1/2 \epsilon_0 E^2$

$$u = 1/2 (8.85 \times 10^{-12})(5000)^2 = 1.11 \times 10^{-4} \text{ J/m}^3$$

Exercise-5:

For the system of capacitors shown in the figure.

- What is the magnitude of the electric potential difference across the 4 nF capacitor in (V) units? (answer: 4)
- The total energy stored by the system of capacitors in (nJ) units? (answer: 240)



Solution:

Equivalent capacitance calculation:

Capacitors (6) and (4) are connected in parallel due to the charge distributed through them:

$$C_{eq} = 6 + 4 = 10 \text{ nF}$$

Now, this resultant capacitor is connected in series with (5)

$$\text{Series: } \frac{1}{10} + \frac{1}{5} = \frac{3}{10} \rightarrow C_{eq} = \frac{10}{3} \text{ nF}$$

$$\text{Charge: } Q = C_{eq} \times V = \left(\frac{10}{3}\right) \times 12 = 40 \text{ nC}$$

We know that the capacitors (5) and (10) are connected in series; then we conclude that the potential difference on both capacitors is different, but the charges are equal:

$$(a) \text{ Voltage across } 10 \text{ } \mu\text{F: } V = Q/C = 40/10 = 4 \text{ V}$$

$$(b) \text{ Energy: } U = \frac{1}{2} C_{eq} V^2 = \frac{1}{2} \left(\frac{10}{3} \times 10^{-6}\right) (12)^2 = 2.40 \times 10^{-4} \text{ J}$$

