First order linear differential equation

A first order linear D.E. is on the form

$$a_1(x)\frac{dy}{dx} + a_2(x)y = g(x)$$

or equivalently,

$$\frac{dy}{dx} + p(x)y = q(x) \qquad (1)$$

we seek a solution of equation (1) which is defined on some interval I on which p and q are continuous.

It is easy to see that $\mu(x) = e^{\int p(x)dx}$ is an integrating factor for Equation (1).

Multiplying both sides of Equation (1) by μ(x), we obtain

$$e^{\int p(x)\,dx}\frac{dy}{dx}+e^{\int p(x)\,dx}p(x)\,y=e^{\int p(x)\,dx}q(x)$$

Or

$$\frac{d}{dx}\left\{e^{\int p(x)\,dx}y\right\}=e^{\int p(x)\,dx}q(x)$$

That is

$$\frac{d}{dx}\{\mu(x)\,y\}=\mu(x)\,q(x) \qquad (2)$$

Integrating both sides of (2) we get

$$\mu(x)\,y=c+\int \mu(x)\,q(x)\,dx$$

Or

$$y=\tfrac{1}{\mu(x)}\left[c+\int \mu(x)\,q(x)\,dx\right]$$

Example1: Solve the DE

$$x\frac{dy}{dx} - 4y = x^6 e^x, \; x > 0.$$

Dividing both sides by x we get

$$\frac{dy}{dx} - \frac{4}{x} y = x^5 e^x$$

$$\Rightarrow \; p(x) = -\frac{4}{x} \Rightarrow \; \mu(x) = e^{\int -\frac{4}{x} dx} = x^{-4}$$

Multiply both sides of (1) *by* x^{-4} *to get*

$$\Rightarrow \; x^{-4}\frac{dy}{dx} + (-4x^{-5})y = xe^x$$

$$Or \quad \frac{d}{dx}\left(x^{-4} y\right) = xe^x$$

Which implies

$$x^{-4} y = \int xe^x dx$$

$$= xe^x - e^x + c$$

$$Or \quad y = x^4\left(xe^x - e^x + c\right)$$

Example 2: Solve the DE:

$$(1-\cos x)dy+(2y\sin x-\tan x)dx=0$$

After rearranging, the equation becomes

$$\tfrac{dy}{dx}+\tfrac{2\sin x}{1-\cos x}\,y=\tan x \qquad (1).$$

$$\Rightarrow\ p(x)=\tfrac{2\sin x}{1-\cos x}\Rightarrow\ \mu(x)=e^{\int\frac{2\sin x}{1-\cos x}dx}=(1-\cos x)^2$$

Multiply both sides of (1) *by* $\mu(x)$ *to get*

$$\Rightarrow\ (1-\cos x)^2\frac{dy}{dx}+2\sin x(1-\cos x)y=\tan x(1-\cos x)^2$$

$$Or\quad \frac{d}{dx}\left\{(1-\cos x)^2 y\right\}=\tan x(1-\cos x)^2$$

Which implies

$$(1-\cos x)^2 y=\int\tan x(1-\cos x)^2\,dx$$

$$=-\ln|\cos x|+2\cos x+\tfrac{1}{2}\sin^2 x+c$$

Example 3

Solve the initial value problem

$$xy' - 2y = 5x^2, \quad y(1) = 2,$$

First, put the equation on the standard form:

$$y' - \tfrac{2}{x} y = 5x, \quad \text{for } x \neq 0$$

Then

$$p(x) = \tfrac{-2}{x},\ q(x) = 5x \Rightarrow \mu(x) = e^{\int p(x)dx} = e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = e^{\ln\left(x^{-2}\right)} = \frac{1}{x^2}$$

hence $\mu(x)y = \left[c + \int \mu(x)q(x)dx\right]$

$$\Rightarrow \tfrac{1}{x^2} y = \left[c + \int \frac{5}{x} dx\right]$$

$$\Rightarrow y = 5x^2 \ln|x| + c\,x^2$$

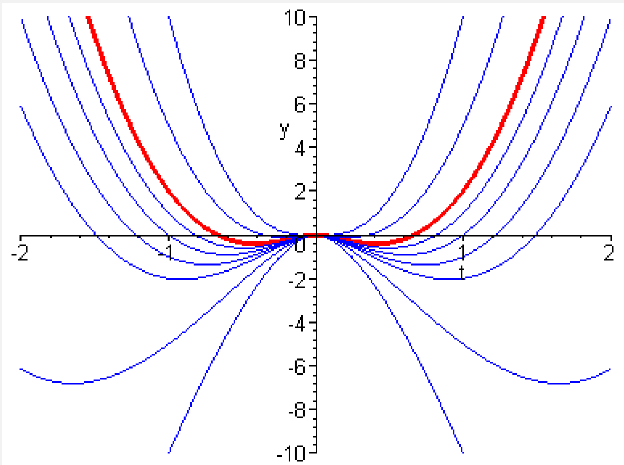
Using the initial condition $y\ (1) = 2$ in the general solution

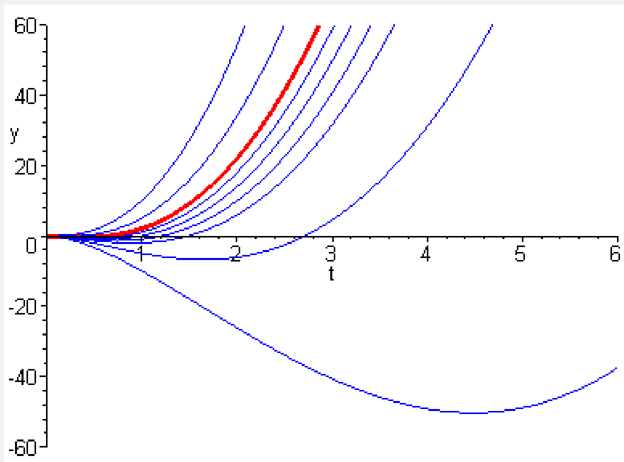
$$y = 5x^2 \ln|x| + cx^2,$$

it follows that

$$c = 2 \;\Rightarrow\; y = 5x^2 \ln|x| + 2x^2$$

The graphs below show several curves for different values of c, and a particular solution (in red) whose graph passes through the initial point (1,2).





Bernoulli's D. Equation

A first order DE on the form

$$\frac{dy}{dx} + p(x)y = f(x)\,y^n \qquad (1)$$

Where n is a real number different than 0 or 1 is called Bernoulli's DE, which can be reduced to a first order linear DE using a suitable substitution.

Indeed, divide both sides of (1) by y^n to get

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = f(x) \qquad (2)$$

$$Let \quad w = y^{1-n} \Rightarrow \frac{dw}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dw}{dx}$$

Therefore (2) becomes

$$\frac{1}{(1-n)}\frac{dw}{dx} + p(x)w = f(x)$$

Which is linear DE in w

Example1

Solve the DE

$$x\frac{dy}{dx}+y(1-x^2y)=0 \qquad (1)$$

Solution. Rewrite Equation (1) on the standard form

$$\frac{dy}{dx}+\frac{1}{x}y=xy^2 \qquad (2)$$

Now, (2) is a Bernoulli's equation. Dividing both sides of (2) by y^2 we get

$$y^{-2}\frac{dy}{dx}+\frac{1}{x}y^{-1}=x \qquad (3)$$

$$Let\ \ w=y^{-1}\Rightarrow\frac{dw}{dx}=-y^{-2}\frac{dy}{dx}$$

$$\Rightarrow y^{-2}\frac{dy}{dx}=-\frac{dw}{dx}$$

Using these values in (3) we obtain

$$-\frac{dw}{dx}+\frac{1}{x}w=x \qquad (4)$$

Multiplying (4) by (-1) we obtain

$$\frac{du}{dx}+\left(\frac{-1}{x}\right)w=-x \qquad (5)$$

which is LDE.

$$\Rightarrow\ p(x)=\left(\frac{-1}{x}\right)\Rightarrow\ \mu(x)=e^{\int\frac{-1}{x}dx}=\frac{1}{x}$$

Multiplying both sides of (5) by $\mu(x)=\frac{1}{x}$ we obtain

$$\frac{1}{x}\frac{du}{dx}+\left(\frac{-1}{x^2}\right)w=-1$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}w\right)=-1\Rightarrow \frac{1}{x}w=c-x$$

$$\Rightarrow w=cx-x^2\ \Rightarrow\ y^{-1}=cx-x^2$$

Or

$$y(cx-x^2)=1$$

Example 2

Solve the DE

$$y' + xy - xe^{-x^2} y^{-3} = 0 \qquad (1)$$

Rewrite Equation (1) on the standard form

$$y' + xy = xe^{-x^2} y^{-3} \qquad (2)$$

which is Bernoulli's DE. Multiplying both sides of (2) by y^3 we get

$$y^3 y' + xy^4 = xe^{-x^2} \qquad (3)$$

$$Let \quad w = y^4 \Rightarrow \frac{dw}{dx} = 4y^3 \frac{dy}{dx}$$

$$\Rightarrow y^3 \frac{dy}{dx} = \frac{1}{4} \frac{dw}{dx}$$

Using these values in (3) we obtain

$$\frac{1}{4} \frac{dw}{dx} + xw = xe^{-x^2} \qquad (4)$$

Multiplying (4) by 4 we get

$$\frac{dw}{dx} + 4xw = 4xe^{-x^2} \qquad (5)$$

which is LDE.

$$\Rightarrow p(x) = 4x \Rightarrow \mu(x) = e^{\int 4x dx} = e^{2x^2}$$

Multiplying both sides of (5) by $\mu(x)$ we obtain

$$e^{2x^2} \frac{dw}{dx} + 4xe^{2x^2} w = 4xe^{x^2}$$

$$\Rightarrow \frac{d}{dx}(e^{2x^2} w) = 4xe^{x^2} \Rightarrow e^{2x^2} w = c + \int 4xe^{x^2} dx$$

$$\Rightarrow w = e^{-2x^2}(c + 2e^{x^2})$$

$$Or \quad y^4 = ce^{-2x^2} + 2e^{-x^2}.$$

Example 3

Solve the DE

$$3(1+x^2)y' = 2xy(y^3-1) \qquad (1)$$

Rewrite Equation (1) on the standard form

$$y' + \frac{2x}{3(1+x^2)}y = \frac{2x}{3(1+x^2)}y^4 \qquad (2)$$

which is Bernoulli's DE. Dividing both sides of (2) by y^4 we get

$$y^{-4}y' + \frac{2x}{3(1+x^2)}y^{-3} = \frac{2x}{3(1+x^2)} \qquad (3)$$

$$Let \quad w = y^{-3} \Rightarrow \frac{dw}{dx} = -3y^{-4}\frac{dy}{dx} \Rightarrow y^{-4}\frac{dy}{dx} = \frac{1}{-3}\frac{dw}{dx}$$

Using these values in (3) we obtain

$$-\frac{1}{3}\frac{dw}{dx} + \frac{2x}{3(1+x^2)}w = \frac{2x}{3(1+x^2)} \Rightarrow \frac{dw}{dx} - \frac{2x}{(1+x^2)}w = \frac{-2x}{(1+x^2)} \qquad (4)$$

which is LDE with $\mu(x) = e^{\int \frac{-2x}{1+x^2}dx} = \frac{1}{1+x^2}$

Hence $\mu\, w = c - \int \frac{2x}{(1+x^2)^2}dx \Rightarrow \frac{y^{-3}}{1+x^2} = c + \frac{1}{1+x^2}$

Notice that equation (1) is separable D.E.

$$\int \frac{3\,dy}{y(y^3-1)}\,dy = \int \frac{2x}{1+x^2}\,dx$$

But the left hand integral is lengthy, in fact it needs partial fraction and after that completing the square for one of the resulting fractions.

Homework.

Solve the DE $\frac{x}{y}\frac{dy}{dx} + xy = 1$