Definition

Let f be a function in two variables x and y, then the differential of f,

denoted by (df) is defined by

$$df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy$$

or

$$df = (f_x)\, dx + (f_y)\, dy$$

Exact Equations

A first order DE on the form

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be **exact** if there is a function $f(x,y)$ satisfies:

$$df = M(x,y)dx + N(x,y)dy$$

That is $f_x(x,y) = M(x,y), \;\; f_y(x,y) = N(x,y)$

hence $f(x,y) = \int M(x,y)\,dx$

or $f(x,y) = \int N(x,y)\,dy$

then the solution of this DE is given implicitly by $f(x,y) = c$

Theorem

Suppose M, N, $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ are continuous on an open

region R in the xy-plane. Then, the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad for\ all\ \ (x, y)\, in\ R$$

Example 1

The DE $(x^2+5y)dx+(y^3+5x)dy=0$

is exact, since $M(x,y)=x^2+5y,\ N(x,y)=y^3+5x$

and $\frac{\partial M}{\partial y}=5,\ \frac{\partial N}{\partial x}=5$

While, the DE $(x^2+y^2)dx+(3y+x)dy=0$

is not exact, because

$$M(x,y)=x^2+y^2, N(x,y)=3y+x$$

and $\frac{\partial M}{\partial y}=2y\neq\frac{\partial N}{\partial x}=1$

Example 2

Solve the differential equation

$$(y\cos x + 2xe^{y})dx + (\sin x + x^{2}e^{y} - 1)dy = 0$$

Here

$$M(x,y) = y\cos x + 2xe^{y},\ N(x,y) = \sin x + x^{2}e^{y} - 1$$

hence

$$M_{y}(x,y) = \cos x + 2xe^{y} = N_{x}(x,y) \Rightarrow \text{D.E. is exact}$$

Thus

$$f(x,y) = \int M(x,y)dx = \int \left(y\cos x + 2xe^{y}\right)dx$$

$$= y\sin x + x^{2}e^{y} + g(y)$$

But

$$\frac{\partial f}{\partial y} = N \Rightarrow \frac{\partial}{\partial y}\left(y\sin x + x^2 e^y + g(y)\right) = \sin x + x^2 e^y - 1$$

$$\Rightarrow \sin x + x^2 y + g'(y) = \sin x + x^2 e^y - 1$$

$$\Rightarrow g'(y) = -1$$

$$\Rightarrow g(y) = -y + c_1$$

Therefore, $f(x, y) = y\sin x + x^2 e^y - y + c_1$

hence the solution is given implicitly by

$$y\sin x + x^2 e^y - y + c_1 = c$$

or

$$y\sin x + x^2 e^y - y = k$$

Example 3

Solve the following differential equation.

$$\frac{dy}{dx}+\frac{x+4y}{4x-y}=0 \;\Leftrightarrow\; (x+4y)dx+(4x-y)dy=0$$

Here we have

$$M(x,y)=x+4y,\; N(x,y)=4x-y$$

hence $\quad M_y(x,y)=4=N_x(x,y) \;\Rightarrow\; \textit{the}$ D.E. is exact

(Also, it is homogeneous D.E.)

Thus, the solution is given by $f(x,y)=c$ where

$$f_x(x,y)=M(x,y),\;\; f_y(x,y)=N(x,y)$$

Hence

$$f(x,y)=\int M(x,y)dx=\int(x+4y)dx=\frac{1}{2}x^2+4xy+g(y)$$

But

$$\frac{\partial f}{\partial y} = N \Rightarrow \frac{\partial}{\partial y}\left(\frac{1}{2}x^2 + 4xy + g(y)\right) = 4x - y$$

$$\Rightarrow 4x + g'(y) = 4x - y$$

$$\Rightarrow g'(y) = -y$$

$$\Rightarrow g(y) = -\tfrac{1}{2}y^2 + c_1$$

Hence $$f(x,y) = \frac{1}{2}x^2 + 4xy - \frac{1}{2}y^2 + c_1$$

It follows that the general solution is given by

$$\frac{1}{2}x^2 + 4xy - \frac{1}{2}y^2 + c_1 = c$$

or

$$\frac{1}{2}x^2 + 4xy - \frac{1}{2}y^2 = k$$

Example 4

Solve the IVP

$$(1+\ln x+\frac{y}{x})dx=(1-\ln x)dy,\;\; y(1)=2$$

First, put the DE on the form

$$(1+\ln x+\frac{y}{x})dx+(\ln x-1)dy=0$$

Hence

$$M=(1+\ln x+\frac{y}{x}),\;\; N=(\ln x-1)$$

$$\Rightarrow M_y=\frac{1}{x}=N_x \Rightarrow the\, DE\, is\, Exact$$

$$\Rightarrow there\; is\;\; a\;\; function\; f(x,y)\; such\;\; that$$

$$f_x=M\;\; and\;\;\; f_y=N$$

Therefore

$$f(x,y)=\int N\,dy=\int(\ln x-1)\,dy$$

$$=y\ln x-y+h(x)$$

$$But\ \ f_x=M\Rightarrow\frac{y}{x}+h'(x)=1+\ln x+\frac{y}{x}$$

$$\Rightarrow h'(x)=1+\ln x$$

$$\Rightarrow h(x)=\int(1+\ln x)dx$$

$$=x\ln x+c_1$$

$$\Rightarrow\ f(x,y)=y\ln x-y+x\ln x+c_1$$

thus, the general solution is given by

$$y\ln x-y+x\ln x=k$$

$$Since\ \ y(1)=2\Rightarrow k=-2$$

$$\Rightarrow the\ solution\ of\ the\ ivp\ is:\quad y\ln x-y+x\ln x+2=0$$

Example 5

Solve the differential equation.

$$(2x-\sqrt{y}+6x^2y)dx+(2x^3-\frac{x}{2\sqrt{y}})dy=0.$$

Here we have

$$M=2x-\sqrt{y}+6x^2y,\ \ N=2x^3-\frac{x}{2\sqrt{y}},$$

$$\Rightarrow M_y=-\frac{1}{2\sqrt{y}}+6x^2,\ \ N_x=6x^2-\frac{1}{2\sqrt{y}}\Rightarrow the\, DE\, is\, Exact$$

$$\Rightarrow the\ solution\ is\ \ f(x,y)=c,\ where\ \ f\ \ is\ a\ function\ \ satisfies$$

$$f_x=M\ \ and\ \ \ f_y=N.$$

Hence

$$f(x,y)=\int M(x,y)\,dx=\int(2x-\sqrt{y}+6x^2y)\,dx=x^2-x\sqrt{y}+2x^3y+g(y)\ \ ....(1),$$

Also, we have

$$f(x,y)=\int N(x,y)\,dy=\int (2x^3-\frac{x}{2\sqrt{y}})dy=2x^3y-x\sqrt{y}+h(x) \quad ....(2),$$

Comparing (1) and (2), we find that

$$f(x,y)=x^2-x\sqrt{y}+2x^3y+c_1,$$

Hence, the solution is given by

$$f(x,y)=x^2-x\sqrt{y}+2x^3y+c_1=c_2,$$

or

$$x^2-x\sqrt{y}+2x^3y=k,$$

Where $k=c_2-c_1.$

Integrating Factors

Sometimes, it is possible to convert a non-exact DE to an exact equation by multiplying it by a suitable function $\mu(x, y)$ (the function μ is called an integrating factor) :

Consider a non-exact D.E. $M(x, y)\,dx + N(x, y)\,dy = 0$

Case 1: If $\frac{1}{N}(M_y - N_x) = f(x)$, that is it does not depend on y.

Then $\mu(x) = e^{\int f(x)\,dx}$.

Case 2: If $\frac{1}{M}(N_x - M_y) = g(y)$, that is it does not depend on x.

Then $\mu(y) = e^{\int g(y)\,dy}$.

Example 6

The following DE is not exact

$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$

Here, $M = 3xy + y^2 + 1,\ N = x^2 + xy$

$$\Rightarrow M_y = 3x + 2y,\ N_x = 2x + y$$

$$\Rightarrow \frac{1}{N}\left(M_y - N_x\right) = \frac{x+y}{x^2+xy} = \frac{1}{x} = f(x),\ (\mathit{free\ of}\ y)$$

$$\Rightarrow I.F. is\ \mu(x) = e^{\int \frac{1}{x} dx} = x$$

Multiplying both sides of the DE by $\mu(x) = x$, it becomes

$$(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$$

Which is exact DE.

Example 7

The following DE is not exact

$$6xydx + (4y + 9x^2)dy = 0$$

Here, $M = 6xy,\ N = 4y + 9x^2$

$$\Rightarrow M_y = 6x,\ N_x = 18x$$

$$\Rightarrow \frac{1}{M}\left(N_x - M_y\right) = \frac{12x}{6xy} = \frac{2}{y} = g(y),\ \ (it\ is\ \ free\ of\ \ x)$$

$$\Rightarrow I.F.is\ \ \mu(y) = e^{\int \frac{2}{y}dy} = y^2$$

Multiplying the DE by $\mu(y)$ it becomes

$$6xy^3dx + (4y^3 + 9x^2y^2)dy = 0$$

Which is exact DE.

Example 8

The D.E. $(3x^2+y)dx+(2x^2y-x)dy=0$ is not exact

Find an appropriate integrating factor and solve it.

$$M=3x^2+y,\quad N=2x^2y-x$$

$$\Rightarrow M_y=1,\ N_x=4xy$$

$$\Rightarrow \frac{1}{N}\left(M_y-N_x\right)=\frac{1-(4xy-1)}{2x^2y-x}=\frac{2(1-2xy)}{x(2xy-1)}$$

$$=\frac{-2}{x}=f(x)$$

$$\Rightarrow I.F.is\ \mu(x)=e^{\int\frac{-2}{x}dx}=x^{-2}=\frac{1}{x^2}$$

Multiplying the DE by $\mu(x)$ it becomes

$$(3+\tfrac{y}{x^2})dx+(2y-\tfrac{1}{x})dy=0$$

Now, $M=3+\frac{y}{x^2},\quad N=2y-\frac{1}{x}\Rightarrow M_y=N_x=\frac{1}{x^2}$

Hence the equation is exact and the general solution is $f(x,y)=c,$

where, $f_x = M$ & $f_y = N$

Hence

$$f(x,y)=\int (3+\tfrac{y}{x^2})dx = 3x-\tfrac{y}{x}+g(y).$$

$$Since \quad f_y = N \Rightarrow \tfrac{-1}{x}+g'(y)=2y-\tfrac{1}{x}$$

$$\Rightarrow g(y)=y^2+c_1$$

$$Therefore,\ f(x,y)=3x-\tfrac{y}{x}+y^2+c_1,$$

$$and \ \ the\, solution\ is\ 3x-\tfrac{y}{x}+y^2=c.$$

Example 9

Find the values of m & n so that the function $\mu(x,y)=x^m y^n$ is an integrating factor for the DE $y(x^3-y)dx-(x^4+xy)dy=0$

Solution. Multiplying both sides by $x^m y^n$ we get

$$(x^{m+3}y^{n+1}-x^m y^{n+2})dx-(x^{m+4}y^n+x^{m+1}y^{n+1})dy=0$$

Now,

$$M=x^{m+3}y^{n+1}-x^m y^{n+2}\Rightarrow \frac{\partial M}{\partial y}=(n+1)x^{m+3}y^n-(n+2)x^m y^{n+1}$$

$$N=-x^{m+4}y^n-x^{m+1}y^{n+1}\Rightarrow \frac{\partial N}{\partial x}=-(m+4)x^{m+3}y^n-(m+1)x^m y^{n+1}$$

But the last DE is exact, because $\mu(x,y)$ is an I.F., hence

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\Rightarrow (n+1)x^{m+3}y^n-(n+2)x^m y^{n+1}=-(m+4)x^{m+3}y^n-(m+1)x^m y^{n+1}$$

$$\Rightarrow\ (n+1)=-(m+4)\ \ and\ \ (n+2)=(m+1)$$

$$or\ \ n+m=-5\ \ and\ \ n-m=-1$$

$$\Rightarrow n=-3,\ m=-2$$