Definition

A function f in two variables x and y, is said to be homogeneous of degree n, if

$$f(tx, ty) = t^n f(x, y)$$

for any real number t.

For example:

$$f(x,y) = \sqrt[3]{x^2 + y^2} \quad \textit{is} \text{ hom}\textit{ogenous of } \text{deg}\textit{ree } n = \tfrac{2}{3}$$

$$f(x,y) = \frac{5xy^4}{x^2 + 7y^2} \quad \textit{is} \text{ hom}\textit{ogenous of } \text{deg}\textit{ree } n = 3$$

$$f(x,y)=\frac{1}{x^2+7y^2} \quad is \;\; homogenous \;\; of \;\; degree \;\; n=-2$$

$$f(x,y)=3-\frac{2x}{y}+e^{\frac{x^2}{3y^2}} \quad is \;\; homogenous \;\; of \;\; degree \;\; n=0$$

$$f(x,y)=-8x\sin(xy^{-1}) \;\; is \;\; homogenous \;\; of \;\; degree \;\; n=1$$

While $g(x,y)=1+x+xy^3$ is not homogeneous.

Here the function g is a polynomial.

Remark: A polynomial in two variables is homogeneous if all it's terms are of the same degree.

For example $g(x,y)=x^3-5xy^2-4x^2y$

Homogeneous Differential Equation

A first order DE

$$\frac{dy}{dx} = f(x, y)$$

is said to be homogeneous if the function f is homogeneous of degree zero.

While the DE $M(x, y)dx + N(x, y)dy = 0$

is homogeneous if both $M \ and \ N$ are homogeneous functions of the same degree.

A homogeneous DE is solved by reducing it to a separable DE using one of the substitutions

$$y = ux \Rightarrow dy = udx + xdu$$

$$or \quad x = vy \Rightarrow dx = vdy + ydv$$

Example 1

Solve the DE $xydy + (x^2 - y^2)dx = 0$.

The DE is homogeneous.

Let $y = ux \Rightarrow dy = udx + xdu$

$$\Rightarrow ux^2(udx + xdu) + (x^2 - u^2x^2)dx = 0$$

$$\Rightarrow ux^3du = -x^2dx$$

$$\Rightarrow udu = -\frac{1}{x}dx$$

$$\Rightarrow \frac{1}{2}u^2 = -\ln x + c_1$$

Thus, the solution of the DE is given by

$$y^2 = -2x^2 \ln x + cx^2$$

Example 2

Solve the following differential equation.

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}\,,\ x > 0.$$

Solution.

The DE is homogeneous.

Thus, let $y = ux \Rightarrow dy = udx + xdu$

Hence, we have

$$xudx + x^2du = \left(xu + x\sqrt{1+u^2}\right)dx$$

$$\Rightarrow \frac{1}{\sqrt{1+u^2}}du = \frac{1}{x}\,dx \ \Rightarrow \int \frac{1}{\sqrt{1+u^2}}du = \int \frac{1}{x}dx$$

$$\Rightarrow \sinh^{-1} u = \ln x + c \ \Rightarrow \sinh^{-1}(\tfrac{y}{x}) = \ln x + c\,.$$

Example 3

Solve the initial value problem

$$x\frac{dy}{dx} = y + xe^{\frac{y}{x}}, \quad y(1) = 1.$$

Solution

The DE is homogeneous, thus, let

$$y = ux \Rightarrow dy = udx + xdu.$$

Hence, we have

$$xudx + x^2du = \left(xu + xe^u\right)dx$$

$$\Rightarrow e^{-u}du = \frac{1}{x}\,dx$$

$$\Rightarrow -e^{-u} = \ln|x| + c$$

$$\Rightarrow -e^{\frac{-y}{x}} = \ln|x| + c$$

Since $y(1) = 1 \Rightarrow c = -e^{-1}$

Hence , the solution is

$$\Rightarrow -e^{\frac{-y}{x}} = \ln|x| - e^{-1}$$

Homework

Solve the following DE

$$\frac{dy}{dx}+\frac{x+y}{x-y}=0$$

Differential Equations with linear coefficients

Consider the first order DE

$$\frac{dy}{dx}=\frac{ax+by+c}{hx+ky+l},$$

where a,b,c,h,k,l are constants and $h,\ k\neq 0.$

If $\dfrac{a}{h}=\dfrac{b}{k}$, then the above differential equation can be reduced to a

separable DE using the substitution $u=ax+by\ \ or\ \ v=hx+ky.$

If $\dfrac{a}{h}\neq\dfrac{b}{k}$, then it can be converted to a homogeneous DE as

follows:

Put $\quad ax+by+c=0$

$and \quad hx+ky+l=0$

then, solve these two equations simultaneously, assume the solution is $x=x_0 \quad and \quad y=y_0$.

Now, let

$$x = X + x_0 \quad and \quad y = Y + y_0$$

$$\Rightarrow dy = dY, \; dx = dX.$$

This substitution will reduce the equation to a homogeneous DE, then it can be solved by reducing it to a separable DE.

Example 1

Solve the DE: $\dfrac{dy}{dx} = \dfrac{2+x+y}{1-2x-2y}$

Here $a=b=1,\ h=k=-2.$ Hence $\dfrac{a}{h} = \dfrac{b}{k} = \dfrac{1}{-2}$, therefore put

$$u = x + y \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1.$$

Hence the DE becomes

$$\frac{du}{dx} - 1 = \frac{2+u}{1-2u} \Rightarrow \frac{du}{dx} = \frac{3-u}{1-2u}$$

Which is a separable DE.

Example 2

Solve the DE: $\dfrac{dy}{dx}=\dfrac{3+x+y}{1-x+y}$

Here $a=b=1,\ \ h=-1,\ k=1.$ Hence $\dfrac{a}{h}\neq\dfrac{b}{k}$, therefore, solve the two equations $3+x+y=0 \ \ and \ \ 1-x+y=0$

to obtain $x=-1,\ \ y=-2.$

Now let $x=X+(-1) \ \ and \ \ y=Y+(-2)$.

Hence the DE becomes

$$\frac{dY}{dX}=\frac{X+Y}{-X+Y}$$

Which is a homogeneous DE.