Separable Differential Equations

A first order ordinary differential equation

$$\frac{dy}{dx} = F(x, y)$$

is said to be Separable if it can be written in the form

$$\frac{dy}{dx} = p(x)\ q(y)$$

Or equivalently: $f(x)dx = g(y)dy$.

In this case the solution is obtained by integrating both sides:

$$\int f(x)\,dx = \int g(y)\,dy$$

Example 1

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{x^2+1}{y^2-1}$$

Separating variables, and integrating both sides we get

$$(y^2-1)dy = (x^2+1)dx$$

$$\int (y^2-1)dy = \int (x^2+1)dx$$

$$\frac{1}{3}y^3 - y = \frac{1}{3}x^3 + x + c$$

$$y^3 - 3y = x^3 + 3x + c.$$

Example 2:

Solve the differential equation

$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2(y-1)}$$

Separating variables, then integrating both sides we obtain:

$$2(y-1)dy=\left(3x^2+4x+2\right)dx$$

$$2\int(y-1)dy=\int\left(3x^2+4x+2\right)dx$$

$$y^2-2y=x^3+2x^2+2x+C$$

Example 3

Solve the initial value problem:

$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0)=1$$

Separating variables and integrating both sides, we obtain

$$\frac{1+3y^3}{y}dy = \cos x\,dx$$

$$\int\left(\frac{1}{y}+3y^2\right)dy = \int\cos x\,dx$$

$$\ln|y| + y^3 = \sin x + c$$

Using the initial condition we get c =1 , hence

$$\ln y + y^3 = \sin x + 1$$

Example 4:

Solve the differential equation

$$\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$$

First, let us factorize the numerator and denominator by grouping to obtain

$$\frac{dy}{dx} = \frac{x(y+3) - (y+3)}{x(y-2) + 4(y+8)} = \frac{(y+3)[x-1]}{(y-2)[x+4]}$$

Now, separating variables, then integrating both sides we get:

$$\frac{y-2}{y+3}dy = \frac{x-1}{x+4}dx$$

$$\Rightarrow \int \frac{y-2}{y+3}\,dy = \int \frac{x-1}{x+4}\,dx$$

$$\Rightarrow \int \left[1-\frac{5}{y+3}\right]dy = \int \left[1-\frac{5}{x+4}\right]dx$$

$$\Rightarrow y-5\ln|y+3| = x-5\ln|x+4|+c.$$

Homework

Solve the DE

$$\frac{dy}{dx} = \sin x\left(\cos(2y)-\cos^2 y\right).$$

Example 5:

Find the equation of the curve passing through the point $(2,1)$ and has a slope $\left(\frac{x-1}{y+2}\right)^3$.

Solution. Since the derivative represents the slope at any point we have the DE $\frac{dy}{dx}=\left(\frac{x-1}{y+2}\right)^3$.

Separating variables, then integrating both sides we obtain:

$$(y+2)^3\,dy=(x-1)^3\,dx$$

$$\int (y+2)^3\,dy=\int (x-1)^3\,dx$$

$$(y+2)^4=(x-1)^4+4C$$

Since the curve passing through the point $(2,1)$ we get $C=20$.

Hence the equation of the curve is

$$(y+2)^4=(x-1)^4+20.$$

Example 6

Solve the initial value problem

$$e^{y}\ \frac{dy}{dx} = \cos(2x) + 2e^{y}\sin^{2}(x) - 1,\quad y(\tfrac{\pi}{2}) = \ln 2.$$

Solution. By separating variables, we obtain:

$$\begin{aligned} e^{y}\ \frac{dy}{dx} &= \cos(2x) + 2e^{y}\sin^{2}(x) - 1 \\ &= e^{y}(1-\cos(2x)) - (1-\cos(2x)) \\ &= (e^{y}-1)(1-\cos(2x)), \end{aligned}$$

hence

$$\int \tfrac{e^{y}}{e^{y}-1}\,dy = \int (1-\cos(2x))\,dx,$$

therefore $\ln|e^{y}-1| = x - \tfrac{\sin(2x)}{2} + c,$

and using the given condition we get

$$\ln|e^{y}-1| = x - \tfrac{\sin(2x)}{2} - \tfrac{\pi}{2}.$$

Separation by substitution

First order ordinary differential equations of the form

$$\frac{dy}{dx} = f(ax+by+c), \quad a,b,c \in R$$

can be reduced to a separable DE by the substitution:

$$u = ax+by+c \Rightarrow \frac{du}{dx} = a + b\frac{dy}{dx}$$

Example 1

Solve $\frac{dy}{dx} = \tan^2(x+y)$.

Solution: Let $u = x+y \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$

Thus we have $\frac{du}{dx} = 1 + \tan^2(u) \Rightarrow \frac{du}{dx} = \sec^2(u)$

Which is a separable differential equation.

Separating variables and integrating both sides we obtain:

$$\cos^2(u)du = dx$$

$$\Rightarrow \int \cos^2(u)du = \int dx$$

$$\Rightarrow \int \tfrac{1}{2}(1+\cos(2u))du = \int dx$$

$$\Rightarrow \tfrac{1}{2}(u + \tfrac{1}{2}\sin(2u) = x + c$$

$$\Rightarrow \tfrac{1}{2}(x + y + \tfrac{1}{2}\sin(2(x+y)) = x + c.$$

Example 2

Solve the DE

$$\frac{dy}{dx} - 2 = \sqrt{y - 2x + 3}.$$

Solution: Let $u = y - 2x + 3 \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 2$

Separating variables and integrating both sides we obtain:

$$\frac{du}{dx} = \sqrt{u}$$

$$\Rightarrow u^{\frac{-1}{2}} du = dx$$

$$\Rightarrow 2u^{\frac{1}{2}} = x + c$$

$$\Rightarrow 2\sqrt{y - 2x + 3} = x + c.$$

Example 3

Solve the DE

$$\frac{dy}{dx}=\frac{1-x-y}{x+y} \Leftrightarrow \frac{dy}{dx}=\frac{1-(x+y)}{x+y}.$$

Solution: Let $u=x+y \Rightarrow \frac{dy}{dx}=\frac{du}{dx}-1$

Using these values in the DE we obtain:

$$\frac{du}{dx}-1=\frac{1-u}{u} \Rightarrow \frac{du}{dx}=1+\frac{1-u}{u}$$

$$\Rightarrow \frac{du}{dx}=\frac{1}{u} \Rightarrow udu=dx$$

$$\Rightarrow \frac{1}{2}u^2=x+c$$

$$\Rightarrow \frac{1}{2}(x+y)^2=x+c.$$