First order ODEs

In this chapter we will consider first order ODEs , $F(x, y, \frac{dy}{dx}) = 0$, and we assume that the equation can be written in the form $\frac{dy}{dx} = f(x, y).$

Three questions may be raised:

Does a solution of a first order DE exist?

If yes, is it unique?

And how can we find this solution?

Initial Value Problem

Some times we are interested in solving a differential equation subject to some given conditions.

The problem

Solve the DE: $\frac{dy}{dx} = f(x, y)$

Subject to the condition: $y(x_0) = y_0$

is called a first order initial value problem.

Example 1

The DE $\frac{dy}{dx} = 2xy$ has the one-parameter family of solutions $y = ce^{x^2}$ on $(-\infty, \infty)$. Exactly one member of this family satisfies the condition $y(0) = 2$. Namely, $y = 2e^{x^2}$, which is the unique member of this family whose curve passes through the point (0,2).

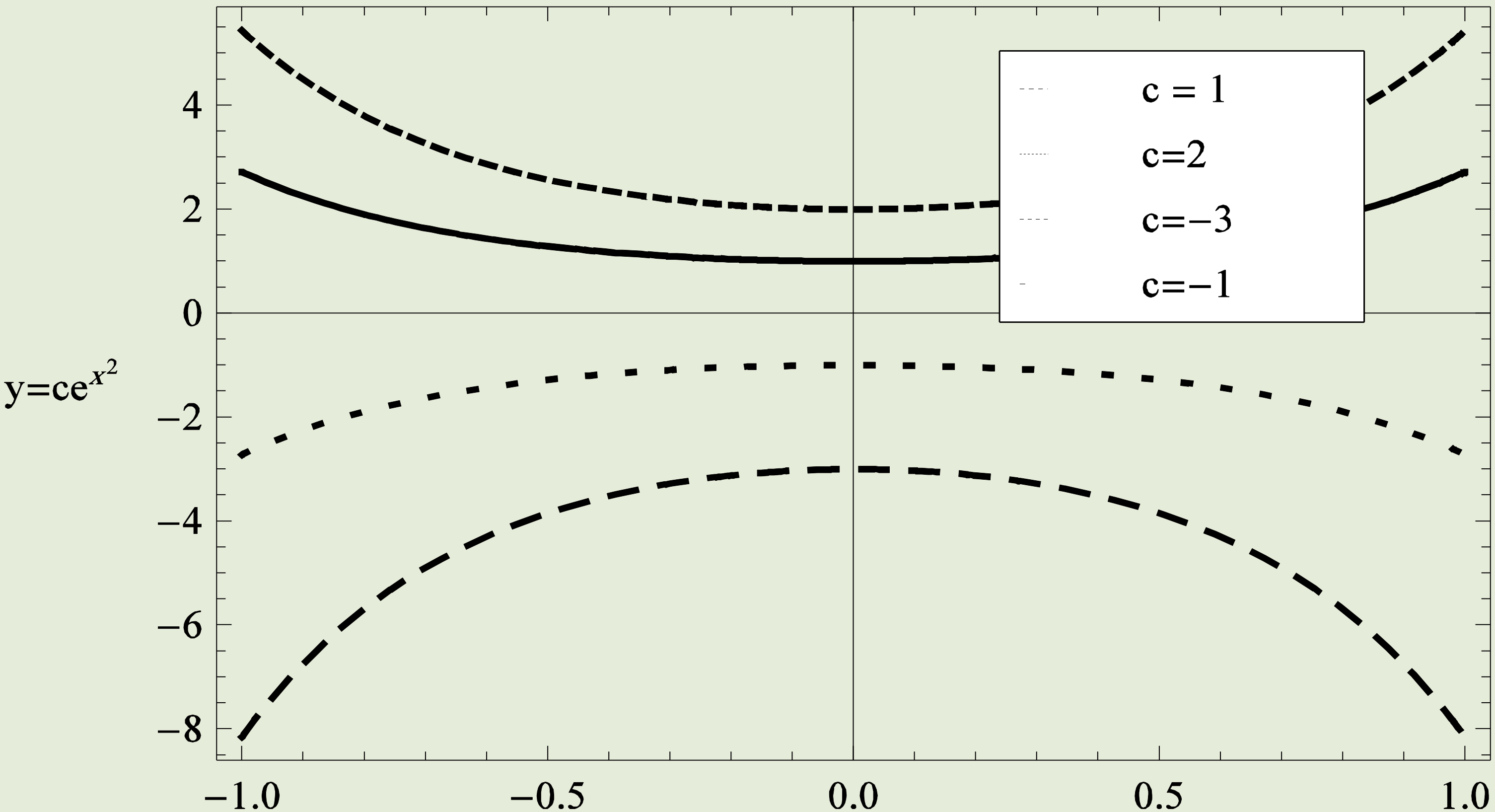
Thus the IVP:

$$\begin{cases} \frac{dy}{dx} = 2xy, \\ y(0) = 2, \end{cases}$$

has the unique solution $y = 2e^{x^2}$.

Figure 1.

$$y = c e^{x^2}$$



Example 2

The DE $\frac{dy}{dx} = x\sqrt{y}$ has the one-parameter family of solutions $y = \left(\frac{x^2+c}{4}\right)^2$ on $(-\infty, \infty)$.

In fact the DE has 2 solutions satisfying the condition $y(0) = 0$, namely, $y = \frac{x^4}{16}$, $y = 0$, since their graphs pass through the point (0,0), Thus the IVP:

$$\begin{cases} y' = x\sqrt{y}, \\ y(0) = 0, \end{cases}$$

has two solutions .

But the initial value problem

$$\begin{cases} (y')^2 + y^4 + 5 = 0, \\ y(0) = 0, \end{cases}$$

Does not have any real solution.

When does a solution of a given IVP exist and it is unique?

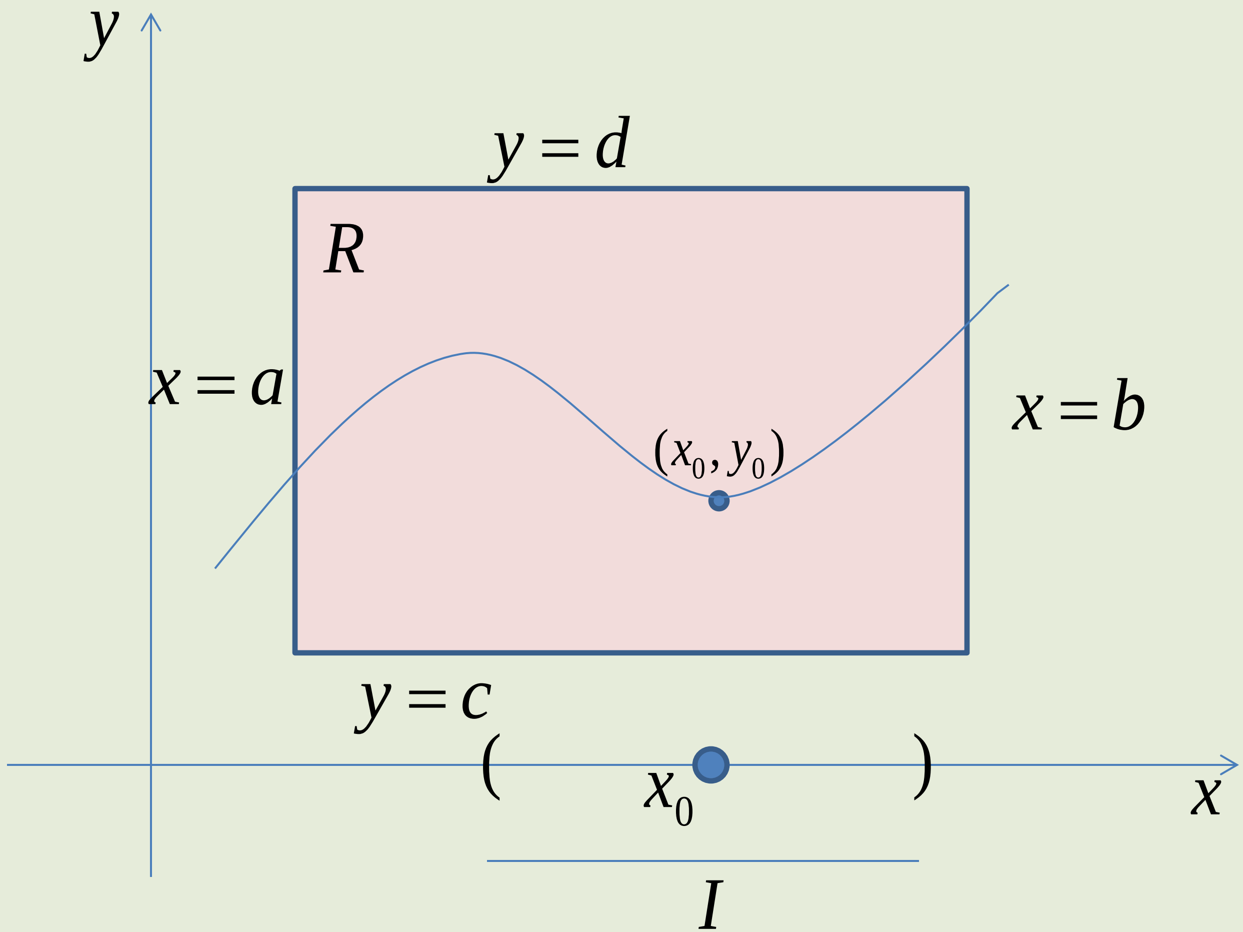
Theorem (Picard) (Existence and uniqueness)

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b,\ c \leq y \leq d$ and contains the point (x_0, y_0) in its interior.

If both $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on R , then there exists an interval I centered at x_0 and a unique function $y(x)$ defined on I which satisfies the IVP:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Figure 3



Example 1

Find and sketch the largest region in the xy-plane through which the IVP:

$$\frac{dy}{dx} = \sqrt{xy}, \quad y(1) = 1,$$

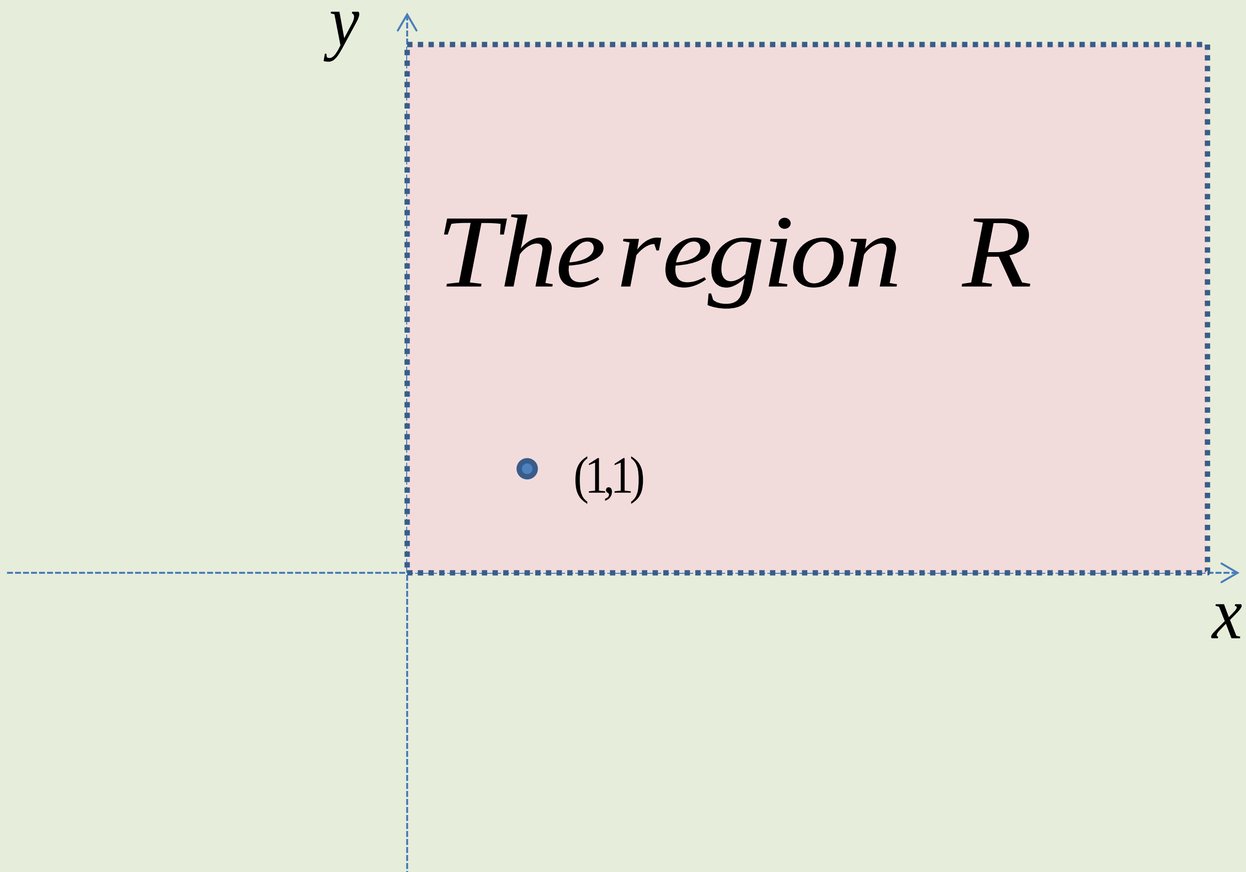
has a unique solution.

Solution: $f(x, y) = \sqrt{xy}$ *and* $\dfrac{\partial f}{\partial y} = \dfrac{x}{2\sqrt{xy}}$

both functions are continuous provided that

$xy > 0$. Since $(x_0, y_0) = (1,1)$ lies in the first quadrant we have $R = \{(x, y) \in R^2 : x > 0, y > 0\}$

- Figure 4



Example 2

Find and sketch the largest region in the xy-plane through which the IVP:

$$\frac{dy}{dx} = \sqrt{(x+y)^2 - 9}, \quad y(1) = 4,$$

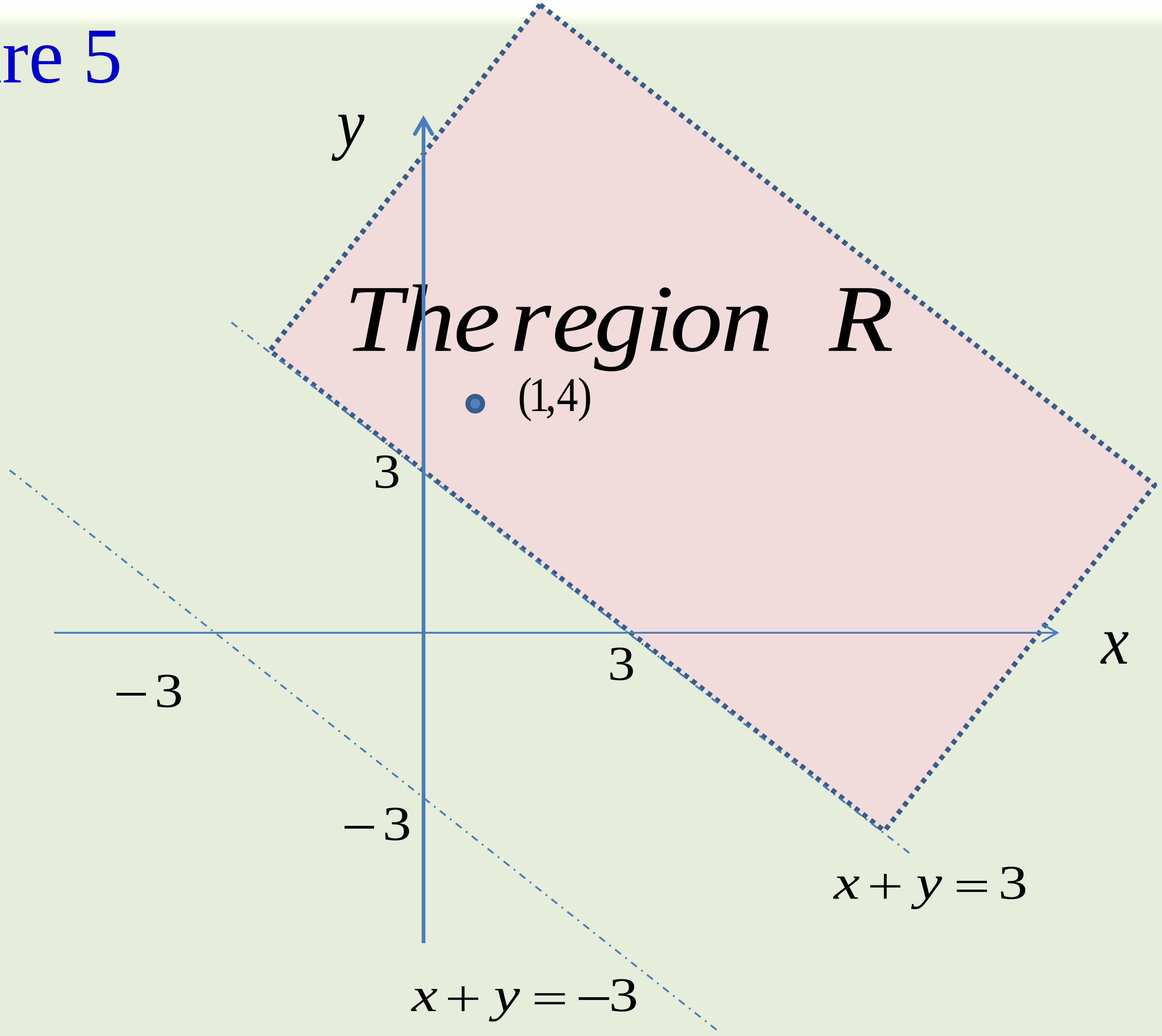
has a unique solution.

Solution:

$$f(x,y) = \sqrt{(x+y)^2 - 9} \ \ and \ \ \frac{\partial f}{\partial y} = \frac{x+y}{\sqrt{(x+y)^2 - 9}}$$

both are continuous provided that $|x+y| > 3$ that is $x+y>3$ or $x+y<-3$. Thus the region R is given by $R = \{(x,y) \in R^2 : x+y>3\}$.

- Figure 5



Example 3

Find and sketch the largest region in the xy plane through which the IVP:

$$(y^2 + 2y + 1)\frac{dy}{dx} - \ln(4 - x^2) = 0, \quad y(-1) = 3,$$

has a unique solution.

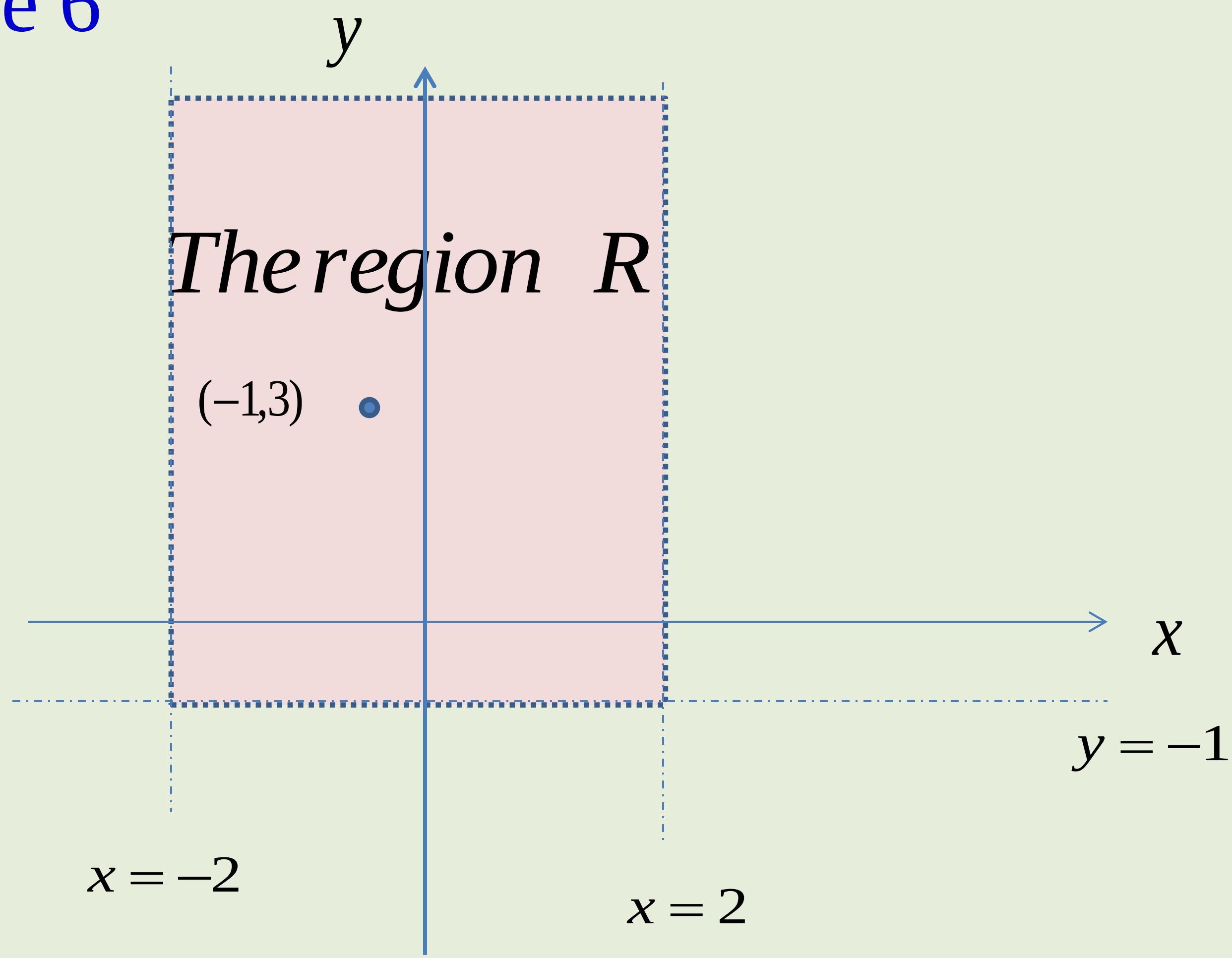
Solution: $f(x, y) = \frac{\ln(4-x^2)}{(y+1)^2}$ *and* $\frac{\partial f}{\partial y} = \frac{-2\ln(4-x^2)}{(y+1)^3}$

both are continuous provided that $-2 < x < 2$ and $y \neq -1$.

Hence the region R is given by

$$R = \{(x, y) \in R^2 : -2 < x < 2, y > -1\}$$

- Figure 6



Homework

Determine whether the existence and uniqueness theorem guarantees that the differential equation:

$$\frac{dy}{dx} = \sqrt{y^2 - 9}$$

has a unique solution at any of the following points:

$(i)\ (5,3)$ $\qquad$ $(ii)\ (2,-3)$

$(iii)\ (1,-1)$ $\qquad$ $(iv)\ (3,4)$