* Bisection method

<u>Ex:-</u> show that the equation $\cos x = 2x - 1$ has a root in $[0,1]$; How many iteration of Bisection method are required to estimate this root to an accuracy 0.5×10^{-3}, also find the 3-rd approx

<u>Solu</u>

$$\cos x = 2x - 1$$
$$\cos x - 2x + 1 = 0$$

$$f(x) = \cos x - 2x + 1 \qquad [0,1]$$

f is continouse $[0,1]$

and $f(0) = 1 + 1 = 2 \quad +$

$f(1) = \cos 1 - 2 + 1 = -$

$$f(a) \cdot f(b) \leq 0$$

$\Rightarrow$ f has a root in $[0,1]$

$$\boxed{\underset{\text{real root}}{|\alpha} - \underset{\text{approx root}}{x_n|} \leq \frac{b-a}{2^n}}$$

للحفظ

$$\frac{b-a}{2^n} \leq 0.5 \times 10^{-3} \Rightarrow \frac{1-0}{2^n} \leq 0.5 \times 10^{-3}$$

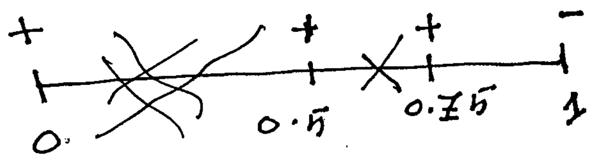
$$\Rightarrow \frac{1}{2^n} \leq 0.5 \times 10^{-3}$$

$$n \cdot \ln 2 \geq \ln(2 \times 10^{3})$$

$$n \geq \frac{\ln(2 \times 10^{3})}{\ln 2} = 10.965$$

$$\Rightarrow n = 11$$

to find 3-rd approx



$$x_1 = \frac{1+0}{2} = 0.5$$

$$f(0.5) = 0.8775 \; +$$

$$x_2 = \frac{0.5+1}{2} = 0.75$$

$$f(0.75) = 0.231 \; +$$

$$\boxed{\cos 0.75 - 2 \times 0.75 + 1}$$

$$x_3 = \frac{1+0.75}{2} = 0.875$$

Ex use bisection method to find the third approx of $\sqrt[3]{2}$ statrting with [1, 2], and find the coresponding absolute error. also find the number of iteration needed to achieve an approx accuracy with 10^{-5}

solu

$$x = \sqrt[3]{2} \Rightarrow x^3 = 2$$

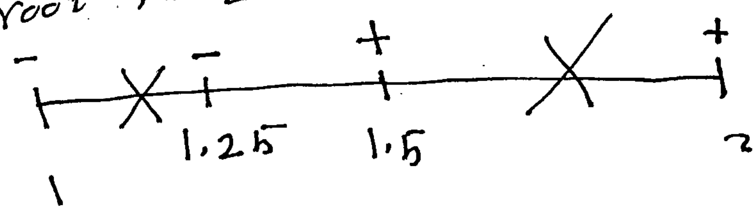
$$x^3 - 2 = 0$$

$f(x) = x^3 - 2$ on $[1,2]$

f is contiouse on $[1,2]$

$f(1) = -1 \; -$; $f(2) = 6 \; +$

f has aroot in $[1,2]$



$$x_1 = \frac{1+2}{2} = 1.5$$

$$f(1.5) = \frac{11}{8} \; +$$

$$x_2 = \frac{1 + 1.5}{2} = 1.25$$

$$f(1.25) = -0.046 \; -$$

$$x_3 = \frac{1.25 + 1.5}{2} = 1.375$$

الآن نجد نسبة الخطأ للتقريب 3

$$|\alpha - x_n| \leqslant \frac{b-a}{2^n}.$$

$$|\alpha - x_3| \leqslant \frac{2-1}{2^3} = \frac{1}{8} = 0.125$$

$$* \quad |\alpha - x_n| \leqslant \frac{b-a}{2^n} \leqslant 10^{-5}$$

$$\frac{2-1}{2^n} \leqslant 10^{-5} \Rightarrow \frac{1}{2^n} \leqslant 10^{-5}$$

$$2^n \geqslant 10^5 \Rightarrow n \ln 2 \geqslant \ln 10^5$$

$$n \geqslant \frac{\ln 10^5}{\ln 2} = 16.60$$

$$n = 17$$

5

* fixed point Method

$$f(x) = 0$$
$$\Downarrow$$
$$x = g(x)$$

Ex

show that one of the following iterative schemes

1) $x_{n+1} = \sqrt{2x_n+3}$ 2) $x_{n+1} = 0.5(x_n^2-3)$

will convege to the solution $\underline{\alpha = 3}$ of $\underline{x^2-2x-3=0}$ but the other one willnot

معطى الجذر الحقيقي $\alpha=3$ معلوم ولا يوجد فترة

الحل

$$g_1(x) = \sqrt{2x+3}$$

1) g_1 contrious at 3

2) $g_1(3) = \sqrt{2\times3+3} = \sqrt{9} = 3$ $\boxed{g(\alpha) = \alpha}$

3) $|g_1'(\alpha)| < 1$??

$$g_1'(x) = \frac{\cancel{2}}{\cancel{2}\sqrt{2x+3}}$$

$$|g_1'(3)| = \left|\frac{1}{\sqrt{9}}\right| = \frac{1}{3} = 0.33 < 1$$

will convegent.

$g_2(x) = 0.5(x^2-3)$

1) $g_2(x)$ ~~cts~~ contious at 3 ✓

2) $g(\alpha) = \alpha$??

$g(3) = 0.5(9-3) = 3$ ✓

3) $|g_2'(\alpha)| < 1$??

$g_2'(x) = 0.5(2x) = x$

$|g_2'(3)| = 3 > 1$ ✗

not conv.

7 Ex

which of the following rearrangement

1) $x_{n+1} = \frac{x_n^2 - 3}{2}$, 2) $x_{n+1} = \sqrt{2x_n + 3}$

is stable for solving the $x^2 - 2x = 3$ using fixed point method in $[2,4]$.

Then use it to find the 2-nd approx taking $x_0 = 2.5$ compute the error bound for your approx.

الفترة معطاه , الجذر الحقيقي غير معلوم

solu

$g_1(x) = \frac{x^2-3}{2}$

1) cts on $[2,4]$

2) $g_1(x) \in [2,4]$??

$g_1(2) = \frac{4-3}{2} = 0.5 \notin [2,4] \Rightarrow$?

$g_1(x)$ not stable

$g_2(x) = \sqrt{2x+3}$

1) cts on $[2,4]$

8)

2) $g_2(x) \in [2, 4]$

$g_2(2) = \sqrt{7} = 2.46 \in [2,4]$ ✓

$g_2(4) = \sqrt{11} = 3.13 \in [2,4]$ ✓

$\Rightarrow g_2(x) \in [2,4]$ ✓

$g_2'(x) = \frac{1}{\sqrt{2x+3}} > 0$

g_2 increasing

$[2,4]$

$g_2(x) \in [2,4]$

3) $\max |g_2'(x)| < 1$

$g_2'(x) = \frac{1}{\sqrt{2x+3}}$

$g_2'(2) = \frac{1}{\sqrt{7}} = 0.378$

$g_2'(4) = \frac{1}{\sqrt{11}} = 0.3015$

$\Rightarrow \max |g_2'(x)| = 0.378 < 1$ ✓

will conv. or stable

$g_1'(x) = (2x+3)^{-\frac{1}{2}}$

$g_2''(x) = \frac{}{(2x+3)^{-3/2}}$

$= -\frac{1}{2}(2x+)\cdot 2 <$

$g_2'(x)$ decresing

$\Rightarrow \max|g'(x)| <$

$x_{n+1} = \sqrt{2x_n + 3}$

$x_1 = \sqrt{2x_0+3} = \sqrt{2(2.5)+3} = 2.828$

$x_2 = \sqrt{2x_1+3} = \sqrt{2(2.828)+3} = 2.9422$

9

Dip

$$\boxed{|\alpha - x_n| \leq \frac{K^n}{1 - K^n} |x_1 - x_0|}$$

$$K = \max |g'(x)| = 0.378$$

$$|\alpha - x_2| \leq \frac{(0.378)^2}{1 - (0.378)^2} \left| \overset{2.828 - 2.5}{\cancel{2.942 - 2.8281}} \right|$$

$$\approx 0.0755$$

* Newton Method

$$f(x) = 0 \qquad f(x) = x - g(x)$$

$$\downarrow$$

$$\boxed{x = g(x)} \quad \text{iterative}$$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x_n)}$$

Ex:- successive approximation x_n to the desired root for an equation $f(x) = 0$ are generated by

$$x_{n+1} = \frac{2x_n^3 + 4x_n^2 + 10}{3x_n^2 + 8x_n}$$

use Newton Methode to find the 1-st approx starting with $x_0 = 1.5$

soln

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x = \frac{2x^3 + 4x^2 + 10}{3x^2 + 8x} \quad - \quad \text{نقارن} \quad x = g(x)$$

$$f(x) = x - g(x)$$

$$f(x) = \frac{x}{1} - \frac{2x^3 + 4x^2 + 10}{3x^2 + 8x} = \frac{3x^3 + 8x^2 - 2x^3 - 4x^2 - 10}{3x^2 + 8x}$$

$$f(x) = \frac{x^3 + 4x^2 - 10}{\ldots} = 0$$

3

$$f(x) = x^3 + 4x^2 - 10$$
$$f'(x) = 3x^2 + 8x$$

$$x_{n+1} = x_n - \frac{x_n^3 + 4x_n^2 - 10}{3x_n^2 + 8x_n}$$

$$x_1 = 1.5 - \frac{(1.5)^3 + 4(1.5)^2 - 10}{3(1.5)^2 + 8(1.5)} = 1.3733$$

Ex:- successive approximation x_n to desired root are generted by the scheme

$x_{n+1} = e^{x_n} - 2$, $n \geq 0$ find $f(x_n)$ and it's derivative $f'(x_n)$ and use Newton method to find the first approx. ; $x_0 = 10$

solu: $x = e^x - 2$ $\boxed{x = g(x)}$

$$f(x) = x - g(x)$$
$$f(x) = x - e^x + 2 = 0$$
$$f'(x) = 1 - e^x$$

$$x_{n+1} = x_n - \frac{x - e^x + 2}{1 - e^x}$$

$$x_1 = 10 - \frac{10 - e^{10} + 2}{1 - e^{10}} = 9.0005$$

3

Ex:- The $1-2\cos x+\cos^2 x=0$ has root $\boxed{\alpha=0}$

Develop Newton Methode for computing this root the find the 2-~~nd~~ approx, with $x_0=0.5$

$(\cos x)^2$

solu $f(x)=1-2\cos x+\cos^2 x=0$

$$f'(x)=2\sin x+2\cos x(-\sin x)$$

$$x_{n+1}=x_n-\frac{1-2\cos x+\cos^2 x}{2\sin x-2\cos x\sin x}$$

$$=0.5-\frac{1-2\cos(0.5)+\cos^2(0.5)}{2\sin(0.5)-2\cos(0.5)\sin(0.5)}$$

$$x_1=0.4843$$

$$x_2=0.4701.$$

4

Ex:- successive approx. ~~to~~ x_n to the desired root are generated by

$$x_{n+1} = \frac{e^{x_n}(x_n - 1) + 3x_n^2}{e^{x_n} + 6x_n} \qquad g$$

x

find $f(x)$, $f'(x)$ the use Newton Method to find the 2-nd approx, $x_0 = -0.5$

sol/

$$f(x) = x - \frac{e^x(x-1) + 3x^2}{e^x + 6x}$$

$$= \frac{\cancel{xe^x} + 6x^2 - \cancel{xe^x} + e^x - 3x^2}{e^x + 6x}$$

$$f(x) = \frac{e^x + 3x^2}{e^x + 6x} = 0$$

$$\Rightarrow f(x) = e^x + 3x^2$$

$$f'(x) = e^x + 6x$$

$$x_{n+1} = x_n - \frac{e^x + 3x^2}{e^x + 6x}$$

5 secant Method

Ex:- Use secant Method to find the 2-nd approx of the cubic root of 27 using $x_0 = 2.5$, $x_1 = 3.5$

Solu

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$x = \sqrt[3]{27} \Rightarrow x^3 = 27 \Rightarrow x^3 - 27 = 0$

$f(x) = x^3 - 27$

$f(2.5) = \cancel{-0.0113\times 10}$ -11.375

$f(3.5) = 15.875$

~~$x_{n+1} =$~~

$n = 1$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{2.5 f(3.5) - 3.5 f(2.5)}{f(3.5) - f(2.5)}$$

$x_2 = 2.8648$

$f(2.8648) = -3.4883$

$n = 2$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 2.987$$

6

Ex show that the iterative method for evaluating p-th root of positive numbe N by using secant Methode is

$$x_{n+1} = \frac{\left(x_n^{p-1} - x_{n-1}^{p-1}\right) x_n x_{n-1} + N(x_n - x_{n-1})}{x_n^{p} - x_{n-1}^{p}}$$

Then use it to find the first approx of ~~of~~ fifth root of 64 with $x_0 = 1$, $x_1 = 1.5$

Sol

$$x = \sqrt[p]{N} \Rightarrow x^p = N \Rightarrow x^p - N = 0$$

$$f(x) = x^p - N$$

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$x_{n+1} = \frac{x_{n-1}(x_n^p - N) - x_n(x_{n-1}^p - N)}{x_n^p \cancel{- N} - (x_{n-1}^p \cancel{- N})}$$

$$x_{n+1} = \frac{x_{n-1} x_n^p - N x_{n-1} - x_n x_{n-1}^p + N x_n}{x_n^p - x_{n-1}^p}$$

$x_n x_n^{p-1}$ $\quad$ $x_{n-1} x_{n-1}^{p-1}$

$$x_{n+1} = x_{n-1} x_n^p - x_n x_{n-1}^p + N(x_n - x_{n-1})$$

7

$$x_{n+1} = \frac{x_{n-1}\, x_n\, x_n^{p-1} - x_n x_{n-1}\, x_{n-1}^{p-1} + N(x_n - x_{n-1})}{x_n^{p} - x_{n-1}^{p}}$$

$$x_{n+1} = \frac{x_n x_{n-1}\left(x_n^{p-1} - x_{n-1}^{p-1}\right) + N(x_n - x_{n-1})}{x_n^{p} - x_{n-1}^{p}} \quad \#$$

$N = 64, \quad p = 5 \quad , x_0 = 1 , \ x_1 = 1.5$

$$x_2 = \frac{1 \times 1.5\left((1.5)^4 - 1^4\right) + 64(1.5 - 1)}{(1.5)^5 - 1^5} = 5.777^{2}$$

multiplicity root

α root of multiplicity $\underline{m}$ of $f(x)=0$

(1) $f(x) = (x-\alpha)^m h(x); \quad h(\alpha) \neq 0$

or

(2) $f(\alpha) = f'(\alpha) = f''(\alpha) \dots, f^{(m-1)}(\alpha) = 0$

and $f^{(m)}(\alpha) \neq 0$

ex

$m = 4$

$f(\alpha) = 0, \quad f'(\alpha) = 0, \quad f''(\alpha) = 0, \quad f'''(\alpha) = 0$

$f^{(4)}(\alpha) \neq 0$

* Newton Modified :-

multiplicity $\alpha = m$

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$

rate of conv. quadrati

Best Method

* rate of conv.

α root $f(x)$

and iterative scheme

$x = g(x) \qquad g(\alpha) = \alpha$

~~$g(\alpha) \neq$~~ 1) $g'(\alpha) \neq 0$ linear conv.

2) $g'(\alpha) = 0$ at least quadratic

2

Ex: show that the Newton Method for finding resiprocals (مقلوب) by soluing $\frac{1}{x} - c = 0$ reaslute by the iterative $x_{n+1} = x_n(2 - c x_n)$, $c \neq 0$

Also show that this iterative conv. quadraticly

$x = g(x)$

$x = x(2 - cx)$

solu $f(x) = \frac{1}{x} - c$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(x_n) = \frac{-1}{x^2}$$

$$x_{n+1} = x_n - \frac{\frac{1}{x} - c}{-\frac{1}{x_n^2}} = .$$

$$= x_n + x_n^2\left(\frac{1}{x_n} - c\right)$$

$$= x_n + x_n - c x_n^2 = 2x_n - c x_n^2$$

$$= x_n(2 - c x_n) \quad \#$$

$g(x) = x(2 - cx)$

$= 2x - cx^2$

α root $\frac{1}{x} - c = 0$

$\frac{1}{x} = c \Rightarrow x = \frac{1}{c}$

$\alpha = \frac{1}{c}$

$g'(x) = 2 - 2cx$

$g'(\alpha) = g'\left(\frac{1}{c}\right) = 2 - 2 \cdot c\frac{1}{c} = 0$

$g''(x) = -2c$

$g''(\alpha) = -2c \neq 0$

3

Ex what is the multiplicity of root $\alpha = 0$ of the equation $x^2 = 2 - 2\cos x$. Use a numerical method that conv. quadraticly to compute 1-st approx., when $x_0 = 0.5$

Solu

$$f(x) = x^2 + 2\cos x - 2 = 0$$

$$\alpha = 0$$

$$f(0) = 0$$

$$f'(x) = 2x - 2\sin x$$

$$f'(0) = 0 - 0 = 0$$

$$f''(x) = 2 - 2\cos x$$

$$f''(0) = 2 - 2 = 0$$

$$f^{(3)}(x) = +2\sin x$$

$$f^{(3)}(0) = 0$$

$$f^{(4)}(x) = 2\cos x$$

$$f^{(4)}(0) = 2 \neq 0.$$

$$m = 4$$

Newton Modifed (quadratic conv.)

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \quad ; \quad x_0 = 0.5$$

$$x_n = x_n - 4\,\frac{x^2 + 2\cos x - 2}{2x - 2\sin x}$$

4

Ex: show that the Newton Method conv. linearly to the root $\alpha = 1$ of equation $x^3 - 3x^2 + 3x - 1 = 0$, use quadratic conv. method to find the first approx of root using $x_0 = 0.5$. Also compute the absolute error.

sol:- $\alpha = 1$, $f(x) = x^3 - 3x^2 + 3x - 1 = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad g(x)$$

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n^2 + 3x_n - 1}{3x_n^2 - 6x_n + 3}$$

$$g(x) = x - \frac{x^3 - 3x^2 + 3x - 1}{3x^2 - 6x + 3}$$

$$g'(\alpha) = 1 - \frac{(3x^2 - 6x + 3)^2 - (x^3 - 3x^2 + 3x - 1)(6x - 6)}{(3x^2 - 6x + 3)^2}$$

مقام x مشتقة البسط − البسط x مشتقة المقام / المقام ^{2}

$$g'(\alpha) = g(1) = 1 - \frac{0 - 0}{0}$$

بسبب ان α هو جذر مكرر

$$f(x) = (x - \alpha)^m h(x)$$

يجب اعادة الحل

5

$f'(x) = 3x^2 - 6x + 3$

$f'(1) = 0$

$f''(x) = 6x - 6$

$f''(1) = 0$

$f'''(x) = 6 \;, \quad f'''(1) \neq 0 \qquad m = 3$

$x^3 - 3x^2 + 3x - 1 = (x-1)^3 \, \cancel{h(x)}$ 1

$f(x) = (x-1)^3 \;, \quad f'(x) = 3(x-1)^2$

$$x_{n+1} = x_n - \frac{(x_n - 1)^3}{3(x_n-1)^2}$$

$$= x_n - \frac{1}{3}(x_n - 1)$$

$$g(x) = x - \frac{1}{3}(x-1) = x - \frac{1}{3}x + \frac{1}{3}$$

$$= \frac{2}{3}x + \frac{1}{3}$$

$g'(x) = \frac{2}{3}$

$g'(\alpha) = g'(1) = \frac{2}{3} \neq 0$ linear conv.

quadratic conv. method

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} \;, \quad x_0 = 0.5$$

$$= x_n - \cancel{3} \cdot \frac{1}{\cancel{3}}(x_n - 1)$$

6

Ex If $x=\alpha$ root of multiplicity $m=5$ of $f(x)=0$ then show that the rate of conv. of modified Newton Method is at least quadratic

Sol

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$

$$= x_n - 5\frac{f(x_n)}{f'(x_n)}$$

$$f(x) = (x-\alpha)^m h(x); \quad h(\alpha) \neq 0$$

$$f(x) = (x-\alpha)^5 h(x)$$

$$f'(x) = (x-\alpha)^5 . h'(x) + 5(x-\alpha)^4 h(x)$$

$$x_{n+1} = x_n - 5\frac{(x-\alpha)^5 h(x)}{\underset{(x-\alpha)^4}{(x-\alpha)^5} h'(x) + 5\underset{(x-\alpha)^4}{\underline{(x-\alpha)^4}} h(x)}$$

$$x_{n+1} = x - 5\frac{(x-\alpha)h(x)}{(x-\alpha)h'(x)+5h(x)}$$

$$g(x) = x - 5\frac{(x-\alpha)h(x)}{(x-\alpha)h'(x)+5h(x)}$$

$\underline{\underline{7}}$

$$g'(x) = 1 - 5\,\frac{\big((x-\alpha)h'(x) + 5h(x)\big)\big((x-\alpha)h'(x) + h(x)\big) - \big((x-\alpha)h(x)\big)\big((x-\alpha)h''(x) + h'(x) + 5h'(x)\big)}{\big((x-\alpha)h'(x) + 5h(x)\big)^2}$$

$$g'(\alpha) = 1 - 5\,\frac{(0 + 5h(\alpha))(0 + h(\alpha)) - 0}{(0 + 5h(\alpha))^2}$$

$$g'(\alpha) = 1 - 5\,\frac{5h^2(\alpha)}{25h^2(\alpha)} = 1 - \frac{25\,h^2(\alpha)}{25\,h^2(\alpha)} = 1 - 1 = 0$$

at least quadratic

8

Ex| find the values of $\underline{a}$ and $\underline{b}$ such that the iterative scheme

$$x_{n+1} = x_n^3 - \frac{12a}{x_n^2} + \frac{8b}{x_n} + 6$$

conv. g quadraticly to $\alpha = 2$

sol:-

$$g(x) = x^3 - \frac{12a}{x^2} + \frac{8b}{x} + 6$$

conv. quad: $g'(\alpha) = 0$ $\quad \alpha = 2$

fixed point $g(\alpha) = \alpha$

$g(2) = 2$

$8 - 3a + 4b + 6 = 2$

$-3a + 4b = -12$ —— ①

$g'(2) = 0$

$$g'(x) = 3x^2 + \frac{24a}{x^3} - \frac{8b}{x^2}$$

g

$12 + 3a - 2b = 0$

$3a - 2b = -12$ —— ①

9)

$$-3\cancel{a} + 4b = -12$$

(+)

$$\cancel{3a} - 2b = -12$$

$$2b = -24 \Rightarrow b = -12$$

$$3a + 24 = -12$$

$$3a = -36$$

$$a = \frac{-36}{3} = -12$$

* non linear sys.

Ex:- find the first approximation for non linear system

$$x^3 + 3y^2 = 21 \quad — ①$$

$$x^2 + 2y = -2 \quad — ②$$

using Newton method starting with $(x_0\ y_0)^T = (1,-1)$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - J^{-1} \begin{bmatrix} f_1(x_n, y_n) \\ f_2(x_n, y_n) \end{bmatrix}$$

$$f_1 = x^3 + 3y^2 - 21 \quad , \quad f_2 = x^2 + 2y + 2$$

$$J = \begin{bmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{bmatrix} = \begin{bmatrix} 3x^2 & 6y \\ 2x & 2 \end{bmatrix}$$

$$J^{-1} = \frac{1}{6x^2 - 12xy} \begin{bmatrix} 2 & -6y \\ -2x & 3x^2 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \frac{1}{6x^2 - 12xy} \begin{bmatrix} 2 & -6y \\ -2x & 3x^2 \end{bmatrix} \begin{bmatrix} x_n^3 + 3y_n^2 - 21 \\ x_n^2 + 2y_n + 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{1}{18} \begin{bmatrix} 2 & 6 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -17 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{1}{18} \begin{bmatrix} -28 \\ 37 \end{bmatrix} = \begin{bmatrix} 1 + \frac{28}{18} \\ -1 - \frac{37}{18} \end{bmatrix}$$

Ex consider the non linear system

$$x^3 + 3y^2 = 21$$

$$x^2 + 2y + a = 0$$

suppose that Appling Newton method to this system starting with initial approx $(x_0, y_0)^T = (1, -1)$ gives $(x_1, y_1)^T = (2.5556, -3.0556)^T$

find the value of a

$f_1 = x^3 + 3y^2 - 21$, $f_2 = x^2 + 2y + a$

solu

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - J^{-1} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$f_1(1,-1) = -17$, $f_2(1,-1) = 1 - 2 + a = -1 + a$

$$J = \begin{bmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{bmatrix} = \begin{bmatrix} 3x^2 & 6y \\ 2x & 2 \end{bmatrix} \Rightarrow J_{1\,(1,-1)} = \begin{bmatrix} 3 & -6 \\ 2 & 2 \end{bmatrix}$$

$$J^{-1} = \frac{1}{18} \begin{bmatrix} 2 & 6 \\ -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{1}{18} \begin{bmatrix} 2 & 6 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -17 \\ a-1 \end{bmatrix} = \begin{bmatrix} 2.5556 \\ -3.0556 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{-34-6+6a}{18} \\ \frac{34+3a-3}{18} \end{bmatrix} = \begin{bmatrix} 2.5556 \\ -3.0556 \end{bmatrix}$$

$\underline{\underline{3}}$

$\Rightarrow 1 - \frac{-\cancel{40} + 6a}{18} = 2.5556 \quad — (1)$

$-1 - \frac{31 + 3a}{18} = -3.0556 \quad — (2)$

1.5556.

(1) $-\frac{-40 + 6a}{18} = 2.5556 - 1$

$\frac{-40 + 6a}{18} = \frac{-1.5556}{1}$

$-40 + 6a = -1.5556 \times 18$

$\cancel{6}a = \frac{-1.5556 \times 18 + 40}{6} = 1.9998$

* Revive

Ex The equation $1-2\cos x+\cos^2 x=0$ has a root $\alpha=0$; Develop Newton's Method formula for computing this root, then use it to find the 2-nd approx. with $x_0=0.5$; also find the order of conv. of your formula

Sol

$$f(x) = 1-2\cos x+\cos^2 x$$
$$f'(x) = 2\sin x - 2\cos x\sin x$$

$$x_{n+1} = x_n - \frac{f}{f'}$$

$$= x_n - \frac{1-2\cos x_n+\cos^2 x_n}{2\sin x_n - 2\sin x_n\cos x_n}$$

$$x_1 = 0.5 - \frac{1-\cos 0.5+\cos^2 0.5}{2\sin 0.5-2\sin 0.5\cos 0.5} = 0.3723$$

x_2

$\Rightarrow$

$f'(0) = 2\sin 0 - 2\cos 0 \sin 0 = 0$

$\Rightarrow$ 0 root of multiplicity $\underline{m}$; $m \geq 2$

$f(x) = 1 - 2\cos x + \cos^2 x = (x-0)^m h(x)$

$= x^m h(x)$; $h(0) \neq 0$

$f'(x) = x^m h'(x) + m x^{m-1} h(x)$

Newton

$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^m h(x)}{x^m h'(x) + m x^{m-1} h(x)}$

x^{m-1}

$g(x) = x - \frac{x' h(x)}{x h'(x) + m h(x)}$

$$g'(x) = 1 - \frac{(x h'(x) + m h(x))(x h'(x) + h(x)) - x h(x)(x h''(x) + h'(x) + m h'(x))}{(x h'(x) + m h(x))^2}$$

$$g'(0) = 1 - \frac{(0 + m h(0))(h(0)) - 0}{(m h(0))^2}$$

$$= 1 - \frac{m \cancel{h^2(0)}}{m^2 \cancel{h^2(0)}} = 1 - \frac{1}{m} \neq 0 \quad m \geq 2$$

linearly conv.

3

show that ~~order~~ of conv. of scheme

$$x_{n+1} = \frac{x_n(x_n^2+21)}{3x_n^2+7}$$

at least qudratic to $\alpha = \sqrt{7}$

$$g(x) = \frac{x(x^2+21)}{3x^2+7} = \frac{x^3+21x}{3x^2+7}$$

$$g'(x) = \frac{(3x^2+7)(3x^2+21) - (x^3+21x)(6x)}{(3x^2+7)^2}$$

$$g'(\sqrt{7}) = \frac{\cancel{\quad}}{\quad} = \frac{0}{(3\times 7+7)^2} = 0$$

at least quadratic

4

Ex

Show that the secant method for finding approx. of the square root of N

$$x_{n+1} = \frac{x_n x_{n-1} + N}{x_n + x_{n-1}}$$

$x = \sqrt{N}$

$x^2 = N$

$f(x) = x^2 - N$

sol

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$x_{n+1} = \frac{x_{n-1}(x_n^2 - N) - x_n(x_{n-1}^2 - N)}{x_n^2 - \cancel{N} - (x_{n-1}^2 - \cancel{N})}$$

$$= \frac{x_{n-1} x_n^2 - x_{n-1} N - x_n x_{n-1}^2 + x_n N}{x_n^2 - x_{n-1}^2}$$

$$= \frac{x_{n-1} \overset{x_n x_n}{x_n^2} - x_n \overset{x_{n-1} x_{n-1}}{x_{n-1}^2} + N(x_n - x_{n-1})}{x_n^2 - x_{n-1}^2}$$

$$= \frac{x_n x_{n-1}(x_n - x_{n-1}) + N(x_n - x_{n-1})}{x_n^2 - x_{n-1}^2}$$

$$= \frac{\cancel{(x_n - x_{n-1})}(x_n x_{n-1} + N)}{\ldots} = \frac{x_n x_{n-1} + N}{x_n + x_{n-1}}$$

5

Ex] Find the value of a and b so that the rate of conv. of scheme

$$x_{n+1} = a x_n + \frac{bN}{x_n^2} \quad ; \quad n \geq 0$$

for computin the third root of N becomes quadratic

Soln

$$g(x) = a x + \frac{bN}{x^2}$$ quadrtic

$\alpha = \sqrt[3]{N}$, $g'(\alpha) = 0$

$g(\alpha) = \alpha$

$$g(\sqrt[3]{N}) = a\sqrt[3]{N} + \frac{bN}{(\sqrt[3]{N})^2} = \sqrt[3]{N} \quad — ①$$

$$a\sqrt[3]{N} + \frac{bN^1}{N^{2/3}} = \sqrt[3]{N}$$

bNx^{-2}

$-2bNx^{-3}$

$$a\sqrt[3]{N} + bN^{1/3} = \sqrt[3]{N}$$

$$a\sqrt[3]{N} + b\sqrt[3]{N} = \sqrt[3]{N} \quad — ①$$

$$g'(\alpha) = 0 \Rightarrow g'(x) = a - \frac{2bN}{x^3}$$

$$g'(\sqrt[3]{N}) = a - \frac{2bN}{(\sqrt[3]{N})^3} = 0$$

$\therefore 6$

$a - 2b = 0 \quad —— ②$

① $\left(a\sqrt[3]{N} + b\sqrt[3]{N} = \sqrt[3]{N}\right) \div \sqrt[3]{N}$

$a + b = 1 \quad —— ①$

① − ② $\quad 3b = 1 \quad \Rightarrow b = \frac{1}{3}$

$a = 1 - \frac{1}{3} = \frac{2}{3}$