Integral Calculus

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Main Contents

1. Antiderivatives.
2. Indefinite Integrals.
   - Basic Integration Rules.
   - Properties of Indefinite Integrals.

Definition

A function F is called an antiderivative of f on an interval I if

$$F^{'}(x) = f(x) \text{ for every } x \in I.$$

Example

1. *Let* $F(x) = x^2 + 3x + 1$ *and* $f(x) = 2x + 3$. *Since* $F^{'}(x) = f(x)$, *then the function* $F(x)$ *is an antiderivative of* $f(x)$.

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2. *Let* $G(x) = x + \sin\ x$ *and* $g(x) = 1 + \cos\ x$. *Since* $G'(x) = 1 + \cos\ x$, *then the function* $G(x)$ *is an antiderivative of* $g(x)$.

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Question: Let $f(x) = 2x$,
Is the function $F(x) = x^2$ antiderivative of the function f?
Is the function $G(x) = x^2 + 2$ antiderivative of the function f?
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Theorem

If functions F and G are antiderivatives of f on an interval I, there exists a constant c such that $G(x) = F(x) + c$ for every $x \in I$.

The function $F(x) = x^2 + c$ is the general form of the antiderivatives (the family) of the function $f(x) = 2x$.

Basic Integration Rules

Definition

Let f be a continuous function on an interval I. The indefinite integral of f is the general antiderivative of f on I:

$$\int f(x)\ dx = F(x) + c.$$

The function f is called the integrand, the symbol $\int$ is the integral sign, x is called the variable of the integration and c is the constant of the integration.

Basic Integration Rules

- **Rule 1:** Power of x.

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, \quad \text{so} \quad \int x^n\ dx = \frac{x^{n+1}}{n+1} + c \quad \text{for} \quad n \neq -1$$

In words, to integrate the function x^n, we add 1 to the power (i.e., $n+1$) and divide the function by $n+1$. If $n = 1$, we have a special case

$$\int 1\ dx = x + c.$$

Basic Integration Rules

Rule 2: Trigonometric functions.

The list of basic integration rules

Derivative	Indefinite Integral
$\frac{d}{dx}(x) = 1$	$\int 1 \ dx = x + c$
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, \ n \neq -1$	$\int x^n \ dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}(\sin \ x) = \cos \ x$	$\int \cos \ x \ dx = \sin \ x + c$
$\frac{d}{dx}(\cos \ x) = -\sin \ x$	$\int (-\sin \ x) \ dx = \cos \ x + c$
$\frac{d}{dx}(\tan \ x) = \sec^2 \ x$	$\int \sec^2 \ x \ dx = \tan \ x + c$
$\frac{d}{dx}(\cot \ x) = -\csc^2 \ x$	$\int (-\csc^2 \ x) \ dx = \cot \ x + c$
$\frac{d}{dx}(\sec \ x) = \sec \ x \ \tan \ x$	$\int \sec \ x \ \tan \ x \ dx = \sec \ x + c$
$\frac{d}{dx}(\csc \ x) = -\csc \ x \ \cot \ x$	$\int (-\csc \ x \ \cot \ x) \ dx = \csc \ x + c$

Example

Evaluate the integral.

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Solution:

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1. $\int x^{-3}\, dx$
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Solution:

1. $\int x^{-3}\, dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c.$
2. $\int \frac{1}{\cos^2 x}\, dx = \int \sec^2 x\, dx = \tan x + c.$

$(\sec x = \frac{1}{\cos x} \Rightarrow \sec^2 x = \frac{1}{\cos^2 x})$

Exercise: Evaluate the integral.

1. $\displaystyle\int x^5 \, dx$
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Solution:

1. $\displaystyle\int x^5 \, dx = \frac{x^6}{6} + c$
2. $\displaystyle\int \frac{1}{\sqrt{x}} \, dx = \int x^{-\frac{1}{2}} \, dx = \frac{x^{\frac{1}{2}}}{1/2} + c$

   $(-\frac{1}{2} + 1 = \frac{-1+2}{2} = \frac{1}{2})$

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3. $\int \frac{\tan x}{\cos \; x} \; dx = \int \tan x \;\; \sec x \; dx = secx + c$

Properties of Indefinite Integrals

Theorem

Assume f and g are continuous on an interval I, then

1. $\dfrac{d}{dx}\displaystyle\int f(x)\, dx = f(x).$
2. $\displaystyle\int \frac{d}{dx}(F(x))\, dx = F(x) + c.$
3. $\displaystyle\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx.$
4. $\displaystyle\int kf(x)\, dx = k\int f(x)\, dx$, *where k is a constant.*

Example

Evaluate the integral.

1. $\displaystyle\int (4x+3)\ dx$
2. $\displaystyle\int (2\sin\ x + 3\cos\ x)\ dx$
3. $\displaystyle\int (\sqrt{x} + \sec^2\ x)\ dx$
4. $\displaystyle\int \frac{d}{dx}(\sin\ x)\ dx$
5. $\displaystyle\frac{d}{dx}\int \sqrt{x+1}\ dx$

Solution:

1. $\int (4x+3)\ dx = 4\int x\ dx + \int 3\ dx = \frac{4x^2}{2} + 3x + c = 2x^2 + 3x + c.$

   $\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx.$ and $\int kf(x)\ dx = k\int f(x)\ dx$

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2. $\int (2\sin\ x + 3\cos\ x)\ dx = 2\int \sin\ x\ dx + 3\int \cos\ x\ dx = -2\cos\ x + 3\sin\ x + c.$

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