Introduction

Definition of a Differential Equation

A differential equation is an equation involving derivatives or differentials.

For example:

$$\frac{d^3y}{dx^3} - 2x\frac{dy}{dx} = 1$$

$$(x + y)dy + (x - y)dx = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Many physical phenomena can be described by differential equations.

- Population dynamics: $\frac{dP}{dt} = kP$
- Falling body: $\frac{d^2x}{dt^2} = -g$
- Flow of a current in an electric circuit:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

- Wave equation: $\frac{\partial^2 U}{dx^2} = c^2\frac{\partial^2 U}{dt^2}$

Classification of differential equations

Differential equations are classified into two types:

- Ordinary differential equations (ODEs) such as:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 5x$$

$$xdy + (x - y)dx = 0$$

$$y''' - y = 2$$

- Partial differential equations (PDEs)

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u$$

The order of a differential equation

is the order of the highest derivative appears in the differential equation.

$$\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^5 - 8y^3 = x^4, \quad \textit{is of order two},$$

$$xdy + (x - \sqrt{y})dx = 0, \quad \textit{is of order one},$$

$$(\Leftrightarrow x\frac{dy}{dx} + x - \sqrt{y} = 0),$$

$$y^{(5)} - y'' = 2, \quad \textit{is of order five}.$$

In general an nth order ODE can be represented as:

$$F(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}) = 0.$$

Differential equations are classified into linear DEs or nonlinear DEs.

An nth order differential equation is said to be linear if it can be written in the form:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x),$$

that is, it satisfies the following two conditions:

(1) the dependent variable (y) and all its derivatives in the equation are of power one.

(2) all the coefficients

$$a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$$

and the function $g(x)$,

are either constants or depend only on the independent variable (x).

If any one of these 2 conditions is not satisfied, then the DE is said to be nonlinear DE.

Examples

The following differential equations are linear:

$$3y^{(4)} - 2y''' + 5x^3 y' + 7y = x - 3e^{2x}$$

$$(x+1)^2 \frac{d^2 y}{dx^2} - 3(x+1)\frac{dy}{dx} + 2y = 3$$

$$(y - x)dx + x^2 dy = 0$$

While the following differential equations are nonlinear:

$$3yy''+5y'+7xy=0$$

$$x\frac{d^3y}{dx^3}-3\left(\frac{dy}{dx}\right)^2+2y=4$$

$$\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-y=\sin y$$

$$\frac{dy}{dx}=\sqrt{3x+2y}$$

$$\frac{dy}{dx}=\frac{1}{2x+y}, \quad (\textit{but it is linear in } x).$$

Solution of a differential equation

A solution of a DE is any function defined on some interval that reduces the equation to an identity.

For example:

$y = xe^x$ is a solution of $y'' - 2y' + y = 0$ on $(-\infty, \infty)$,

while the equation

$$\left(\frac{dy}{dx}\right)^2 + 3y^4 + 1 = 0$$

does not have any real solution.

A solution of a DE: $F(x, y, y', ..., y^{(n)}) = 0$ is said to be explicit if it can be written in the form:

$$y = f(x),$$

and implicit if it is defined by a relation of the form $G(x, y) = 0$.

For example:

$y = \frac{x^4}{16}$ is an explicit solution of: $y' = x\sqrt{y}$ *on* $(-\infty, \infty)$

while $x^2 + y^2 = 4$ is an implicit solution of :

$$\frac{dy}{dx} = \frac{-x}{y}, \quad for \quad -2 < x < 2.$$

Number of solutions

Usually, when a solution exists for a given DE, it has infinite number of solutions.

For example, by direct substitution, we can verify that the one parameter family of curves (functions):

$$y = ce^{x^2}$$

is a solution of the DE

$$\frac{dy}{dx} = 2xy$$

for any real value of the parameter (constant) c.

Similarly, the two parameter family of curves (functions):

$$y = c_1 \cos 2x + c_2 \sin 2x$$

is a solution of the DE $\dfrac{d^2y}{dx^2} + 4y = 0,$

for any real values of the parameters c_1 and c_2.

A solution of a DE which is free of arbitrary constants is called a particular solution.

For example: $y = 3\cos 2x$ is a particular solution of the above DE.

In general, by solving an nth order DE:

$$F(x, y, \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}) = 0$$

we obtain a family of curves:

$$G(x, y, c_1, c_2, ..., c_n) = 0$$

which involves n-parameters.