Mathematical Proofs
A Transition to Advanced Mathematics

Chapter 14
Proofs in Calculus

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Proofs in Calculus

The proofs that occur in calculus are considerably different than any of those we have seen thus far. The functions encountered in calculus are real-valued functions defined on sets of real numbers. That is, each function that we study in calculus is of the type

$$f : X \to \mathbf{R}, \text{ where } X \subseteq \mathbf{R}.$$

In the study of limits, we are often interested in such functions having the property that either

(1) $X = \mathbf{N}$ and increasing values in the domain $\mathbf{N}$ result in functional values approaching some real number L or

(2) the function is defined for all real numbers near some specified real number a and domain values approaching a result in functional values approaching some real number L.

We begin with (1), where $X = \mathbf{N}$.

Limits of Sequences

Definition

A **sequence** (of real numbers) is a real-valued function defined on the set of natural numbers; that is, a **sequence** is a function

$$f : \mathbf{N} \to \mathbf{R}.$$

If $f(n) = a_n$ for each $n \in \mathbf{N}$, then

$$f = \{(1, a_1), (2, a_2), (3, a_3), \ldots\}.$$

Since only the numbers $a_1, a_2, a_3, \ldots$ are relevant in f, this sequence is often denoted only by $a_1, a_2, a_3, \ldots$ or by $\{a_n\}$.

Limits of Sequences

Definition

The numbers a_1, a_2, a_3, etc. are called the **terms** of the sequence $\{a_n\}$, with a_1 being the first term, a_2 the second term, etc. Thus, a_n is the nth term of the sequence.

$\left\{\frac{1}{n}\right\}$ is the sequence $1, 1/2, 1/3, \ldots$;
$\left\{\frac{n}{2n+1}\right\}$ is the sequence $1/3, 2/5, 3/7, \cdots$.

In these two examples, the nth term of a sequence is given and, from this, we can easily find the first few terms and, in fact, any particular term. On the other hand, finding the nth term of a sequence whose first few terms are given can be challenging.

For example, the nth term of the sequence

$$\frac{1}{2},\ \frac{1}{4},\ \frac{1}{6},\ \ldots$$

is $1/2n$; the nth term of the sequence

$$1+\frac{1}{2},\ 1+\frac{1}{4},\ 1+\frac{1}{8},\ \cdots$$

is $1+1/2^n$; the nth term of the sequence

$$1,\ \frac{3}{5},\ \frac{1}{2},\ \frac{5}{11},\ \frac{3}{7},\ \frac{7}{17},\ \ldots$$

is $(n+1)/(3n-1)$; the nth term of the sequence

$$1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \ldots$$

is $(-1)^{n+1}$; while the nth term of the sequence $1, 4, 9, 16, \cdots$ is n^2.

Limits of Sequences

For the sequence $\left\{\dfrac{1}{n}\right\}$, the larger the integer n, the closer $1/n$ is to 0; and for the sequence $\left\{\dfrac{n}{2n+1}\right\}$, the larger the integer n, the closer $n/(2n+1)$ is to $1/2$. On the other hand, for the sequence $\{n^2\}$, as the integer n become larger, n^2 becomes increasingly large and does not approach any real number.

Limits of Sequences

For some sequences $\{a_n\}$, there is a real number L (or at least there appears to be a real number L) such that the larger the integer n becomes, the closer a_n is to L.

Definition

A sequence $\{a_n\}$ of real numbers is said to **converge** to the real number L if for every real number $\epsilon > 0$, there exists a positive integer N such that if n is an integer with $n > N$, then $|a_n - L| < \epsilon$.

The number ϵ is a measure of how close the terms a_n are required to be to the number L and N indicates a position in the sequence beyond which the required condition is satisfied.

Limits of Sequences

Definition

If a sequence $\{a_n\}$ converges to L, then $\{a_n\}$ is a **convergent sequence** and L is referred to as the **limit** of $\{a_n\}$ and we write

$$\lim_{n\to\infty} a_n = L.$$

If a sequence does not converge, it is said to **diverge**. Consequently, if a sequence $\{a_n\}$ diverges, then there is *no* real number L such that $\lim_{n\to\infty} a_n = L$.

For a real number x, the *ceiling* $\lceil x \rceil$ of x is the smallest integer greater than or equal to x.

$$\lceil 8/3 \rceil = 3, \left\lceil \sqrt{2} \right\rceil = 2, \lceil -1.6 \rceil = -1 \text{ and } \lceil 5 \rceil = 5.$$

Example 1

Result The sequence $\left\{ \frac{1}{n} \right\}$ converges to 0.

Proof Let $\epsilon > 0$. Choose $N = \lceil 1/\epsilon \rceil$ and let n be any integer such that $n > N$. Thus, $n > 1/\epsilon$ and so $\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \epsilon$. □

Example 2

Result The sequence $\left\{3 + \frac{2}{n^2}\right\}$ converges to 3.

Proof Let $\epsilon > 0$. Choose $N = \left\lceil \sqrt{2/\epsilon} \right\rceil$ and let n be any integer such that $n > N$. Thus, $n > \sqrt{2/\epsilon}$ and $n^2 > 2/\epsilon$. So $\frac{1}{n^2} < \frac{\epsilon}{2}$ and $\frac{2}{n^2} < \epsilon$. Therefore,

$$\left|\left(3 + \frac{2}{n^2}\right) - 3\right| = \left|\frac{2}{n^2}\right| = \frac{2}{n^2} < \epsilon. \qquad \square$$

Example 3

Result The sequence $\left\{ \frac{n}{2n+1} \right\}$ converges to $\frac{1}{2}$.

Proof Let $\epsilon > 0$ be given. Choose $N = \lceil 1/4\epsilon \rceil$ and let $n > N$. Then $n > \frac{1}{4\epsilon} > \frac{1}{4\epsilon} - \frac{1}{2}$ and so $4n > \frac{1}{\epsilon} - 2$ and $4n + 2 > 1/\epsilon$. Hence, $\frac{1}{4n+2} < \epsilon$. Thus

$$\left| \frac{n}{2n+1} - \frac{1}{2} \right| = \left| \frac{2n - 2n - 1}{2(2n+1)} \right| = \left| -\frac{1}{4n+2} \right| = \frac{1}{4n+2} < \epsilon. \qquad \square$$

Example 4

Result The sequence $\left\{(-1)^{n+1}\right\}$ is divergent.

Proof Assume, to the contrary, that the sequence $\left\{(-1)^{n+1}\right\}$ converges. Then $\lim_{n\to\infty}(-1)^{n+1} = L$ for some real number L. Let $\epsilon = 1$. Then there exists a positive integer N such that if $n > N$, then $\left|(-1)^{n+1} - L\right| < \epsilon = 1$. Let k be an odd integer such that $k > N$. Then

$$\left|(-1)^{k+1} - L\right| = |1 - L| = |L - 1| < 1.$$

Therefore, $-1 < L - 1 < 1$ and $0 < L < 2$. Next, let ℓ be an even integer such that $\ell > N$. Then

$$\left|(-1)^{\ell+1} - L\right| = |-1 - L| = |L + 1| = |1 + L| < 1.$$

So $-1 < L + 1 < 1$ and $-2 < L < 0$. Therefore, $L < 0 < L$, which is a contradiction. □

A sequence $\{a_n\}$ may diverge because as n becomes larger, a_n becomes larger and eventually exceeds any given real number. If a sequence has this property, then $\{a_n\}$ is said to diverge to infinity.

Definition

More formally, a sequence $\{a_n\}$ **diverges to infinity**, written $\lim_{n\to\infty} a_n = \infty$, if for every positive number M, there exists a positive integer N such that if n is an integer such that $n > N$, then $a_n > M$.

The sequence $\{(-1)^{n+1}\}$, although divergent, does not diverge to infinity. However, the sequence $\left\{n^2 + \frac{1}{n}\right\}$ *does* diverge to infinity.

Example 5

Result $\lim_{n\to\infty} \left(n^2 + \frac{1}{n}\right) = \infty.$

Proof Let M be a positive number. Choose $N = \left\lceil \sqrt{M} \right\rceil$ and let n be any integer such that $n > N$. Hence, $n > \sqrt{M}$ and so $n^2 > M$. Thus $n^2 + \frac{1}{n} > n^2 > M$. □

Infinite Series

Definition

An important concept in calculus involving sequences is infinite series. For real numbers $a_1, a_2, a_3, \ldots$, we write

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

to denote an **infinite series** (often simply called a **series**).

For example,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

and

$$\sum_{k=1}^{\infty} \frac{k}{2k^2+1} = \frac{1}{3} + \frac{2}{9} + \frac{3}{19} + \cdots$$

are infinite series.

Infinite Series

Definition

The numbers $a_1, a_2, a_3, \ldots$ are called the **terms** of the series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots.$$

The notation certainly seems to suggest that we are adding the terms $a_1, a_2, a_3, \ldots$. But what does it mean to add infinitely many numbers? A meaning must be given to this.

Definition

For this reason, we construct a sequence $\{s_n\}$, called the **sequence of partial sums** of the series. Here $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$ and, in general, for $n \in \mathbf{N}$,

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

Because s_n is determined by adding a finite number of terms, there is no confusion in understanding the terms of the sequence $\{s_n\}$.

Definition

If the sequence $\{s_n\}$ converges, say to the number L, then the series $\sum_{k=1}^{\infty} a_k$ is said to **converge** to L and we write $\sum_{k=1}^{\infty} a_k = L$.

This number L is called the **sum** of $\sum_{k=1}^{\infty} a_k$.

If $\{s_n\}$ diverges, then $\sum_{k=1}^{\infty} a_k$ is said to **diverge**.

Infinite Series

We consider an example of a convergent series.

Result to Prove

The infinite series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges to 1.

Lemma

For every positive integer n,

$$s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Proof. We proceed by induction. For $n = 1$, we have

$$s_1 = \frac{1}{1 \cdot 2} = \frac{1}{1+1}$$

and the result holds. Assume that

$$s_k = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1},$$

where k is a positive integer. We show that

$$s_{k+1} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

Proof (continued)

Observe that

$$
\begin{aligned}
s_{k+1} &= \left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}\\
&= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}\\
&= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.
\end{aligned}
$$

By the Principle of Mathematical Induction,

$$s_n = \frac{n}{n+1}$$

for every positive integer n. □

Lemma

$\lim_{n\to\infty} \dfrac{n}{n+1} = 1.$

Proof. Let $\epsilon > 0$ be given. Choose $N = \lceil 1/\epsilon \rceil$ and let $n > N$. Then $n > \dfrac{1}{\epsilon} > \dfrac{1}{\epsilon} - 1$. So $n > \dfrac{1}{\epsilon} - 1$. Thus, $n + 1 > \dfrac{1}{\epsilon}$ and $\dfrac{1}{n+1} < \epsilon$. Hence,

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \epsilon.$$

□

We are now prepared to give a proof of the result.

Example 6

Result The infinite series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges to 1.

Proof. The nth term of the sequence $\{s_n\}$ of partial sums of the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ is

$$s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}.$$

Example 6 (continued)

By the first lemma,

$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

and so $s_n = \frac{n}{n+1}$. By the second lemma,

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Since $\lim_{n\to\infty} s_n = 1$, it follows that $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$. □

Infinite Series

Definition

The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is a famous series, called the **harmonic series**.

Example 7

Result The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Proof. Assume, to the contrary, that $\sum_{k=1}^{\infty} \frac{1}{k}$ converges, say to the number L. For each positive integer n, let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Hence, the sequence $\{s_n\}$ of partial sums converges to L.

Example 7 (continued)

Therefore, for each $\epsilon > 0$, there exists a positive integer N such that if $n > N$, then $|s_n - L| < \epsilon$. Let's consider $\epsilon = 1/4$ and let n be an integer with $n > N$. Then

$$-\frac{1}{4} < s_n - L < \frac{1}{4}.$$

Since $2n > N$, it is also the case that $|s_{2n} - L| < \dfrac{1}{4}$ and so

$$-\frac{1}{4} < s_{2n} - L < \frac{1}{4}.$$

Example 7 (continued)

Observe that

$$s_{2n} = s_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > s_n + n\left(\frac{1}{2n}\right) = s_n + \frac{1}{2}.$$

Hence,

$$\frac{1}{4} > s_{2n} - L > s_n + \frac{1}{2} - L = (s_n - L) + \frac{1}{2} > -\frac{1}{4} + \frac{1}{2} = \frac{1}{4},$$

which is impossible. □

Limits of Functions

Definition

Let f be a real-valued function defined on a set X of real numbers and let $a \in \mathbf{R}$ such that f is defined in some deleted neighborhood of a.

A number L is the **limit** of a function $f(x)$ as x approaches a, written $\lim_{x \to a} f(x) = L$, if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every real number x with $0 < |x - a| < \delta$, it follows that $|f(x) - L| < \epsilon$. This implies that if $0 < |x - a| < \delta$, then certainly $f(x)$ is defined.

If there exists a number L such that $\lim_{x \to a} f(x) = L$, then we say that the limit $\lim_{x \to a} f(x)$ exists and is equal to L; otherwise, this limit does not exist.

Thus, to show that $\lim_{x \to a} f(x) = L$, it is necessary to specify $\epsilon > 0$ first and then show the existence of a real number $\delta > 0$. Ordinarily, the smaller the value of ϵ, the smaller the value of δ. However, we must be certain that the number δ selected satisfies the requirement regardless of how small (or large) ϵ may be.

Even though our choice of δ depends on ϵ, it should not depend on which real number x with $0 < |x - a| < \delta$ is being considered.

Example 8

Result $\lim_{x\to 4}(3x-7)=5$.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/3$. Let $x \in \mathbf{R}$ such that $0 < |x-4| < \delta = \epsilon/3$. Then

$$|(3x-7)-5| = |3x-12| = |3(x-4)| = 3|x-4| < 3(\epsilon/3) = \epsilon. \quad \square$$

Example 9

Result $\lim_{x \to \frac{3}{2}} \dfrac{4x^2 - 9}{2x - 3} = 6.$

Proof. Let $\epsilon > 0$ be given and choose $\delta = \epsilon/2$. Let $x \in \mathbf{R}$ such that $0 < |x - 3/2| < \delta = \epsilon/2$. So $2|x - 3/2| < \epsilon$ and $|2x - 3| < \epsilon$. Hence, $|(2x + 3) - 6| < \epsilon$. Since $2x - 3 \neq 0$, it follows that

$$\left| \frac{(2x+3)(2x-3)}{2x-3} - 6 \right| < \epsilon$$

and so $\left| \dfrac{4x^2 - 9}{2x - 3} - 6 \right| < \epsilon.$ □

Example 10

Result $\lim_{x \to 3} x^2 = 9$.

Proof. Let $\epsilon > 0$ be given and choose $\delta = \min(1, \epsilon/7)$. Let $x \in \mathbf{R}$ such that $0 < |x - 3| < \delta = \min(1, \epsilon/7)$. Since $|x - 3| < 1$, it follows that $-1 < x - 3 < 1$ and so $5 < x + 3 < 7$. In particular, $|x + 3| < 7$. Because $|x - 3| < \epsilon/7$, it follows that

$$|x^2 - 9| = |x - 3||x + 3| < |x - 3| \cdot 7 < (\epsilon/7) \cdot 7 = \epsilon. \qquad \square$$

Example 11

Result $\lim_{x \to 2} (x^5 - 2x^3 - 3x - 7) = 3.$

Proof. Let $\epsilon > 0$ be given and choose $\delta = \min(1, \epsilon/170)$. Let $x \in \mathbf{R}$ such that $0 < |x - 2| < \delta = \min(1, \epsilon/170)$. Since $|x - 2| < 1$, it follows that $1 < x < 3$ and so

$$|x^4 + 2x^3 + 2x^2 + 4x + 5| \leq |x^4| + |2x^3| + |2x^2| + |4x| + |5| < 170.$$

Since $|x - 2| < \epsilon/170$, we have

$$\begin{aligned} |(x^5 - 2x^3 - 3x - 7) - 3| &= |x^5 - 2x^3 - 3x - 10| \\ &= |x - 2| \cdot |x^4 + 2x^3 + 2x^2 + 4x + 5| \\ &< (\epsilon/170) \cdot 170 = \epsilon. \end{aligned}$$

□

Example 12

Result $\lim_{x \to 1} \dfrac{x^2 - 1}{2x - 1} = 0.$

Proof. Let $\epsilon > 0$ be given and choose $\delta = \min(1/4, 2\epsilon/9)$. Let $x \in \mathbf{R}$ such that $0 < |x - 1| < \delta$. Since $\delta \le 1/4$, it follows that $|x - 1| < 1/4$ and so $3/4 < x < 5/4$. Hence, $|x + 1| < 5/4 + 1 = 9/4$. Also, $|2x - 1| > 2\left(\frac{3}{4}\right) - 1 = 1/2$ and so $\dfrac{1}{|2x - 1|} < 2.$

Example 12 (continued)

Therefore,

$$\frac{|x+1|}{|2x-1|} < \frac{9}{4} \cdot 2 = \frac{9}{2}.$$

Since $|x-1| < \delta \leq 2\epsilon/9$, it follows that

$$\left|\frac{x^2-1}{2x-1} - 0\right| = \left|\frac{x^2-1}{2x-1}\right| = \frac{|x+1|}{|2x-1|}|x-1| < \frac{2\epsilon}{9} \cdot \frac{9}{2} = \epsilon. \qquad \square$$

Example 13

Result $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Proof. Assume, to the contrary, that $\lim_{x \to 0} \frac{1}{x}$ exists. Then there exists a real number L such that $\lim_{x \to 0} \frac{1}{x} = L$. Let $\epsilon = 1$. Then there exists $\delta > 0$ such that if x is a real number for which $0 < |x| < \delta$, then $\left|\frac{1}{x} - L\right| < \epsilon = 1$. Choose an integer n such that $n > \lceil 1/\delta \rceil \geq 1$. Since $n > 1/\delta$, it follows that $0 < 1/n < \delta$. We consider two cases.

Example 13 (continued)

Case 1. $L \leq 0$. Let $x = 1/n$. So $0 < |x| < \delta$. Since $-L \geq 0$, it follows that

$$\left|\frac{1}{x} - L\right| = |n - L| = n - L \geq n > 1 = \epsilon,$$

which is a contradiction.

Case 2. $L > 0$. Let $x = -1/n$. So, $0 < |x| < \delta$. Thus,

$$\left|\frac{1}{x} - L\right| = |-n - L| = |-(n + L)| = n + L > n > 1 = \epsilon,$$

producing a contradiction in this case as well. □

Fundamental Properties of Limits of Functions

Theorem

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

$$\lim_{x \to a} (f(x) + g(x)) = L + M.$$

Proof. Let $\epsilon > 0$. Since $\epsilon/2 > 0$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \epsilon/2$. Also, there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \epsilon/2$. Choose $\delta = \min(\delta_1, \delta_2)$ and let $x \in \mathbf{R}$ such that $0 < |x - a| < \delta$. Since $0 < |x - a| < \delta$, it follows that both $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$. Therefore,

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

□

Fundamental Properties of Limits of Functions

Theorem

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

$$\lim_{x \to a} f(x) \cdot g(x) = LM.$$

Theorem

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Theorem

Let $n \in \mathbf{N}$ and let $f(x) = x^n$ for all $x \in \mathbf{R}$. Then for each $a \in \mathbf{R}$,

$$\lim_{x \to a} f(x) = a^n.$$

Fundamental Properties of Limits of Functions

Theorem

Let $f_1, f_2, \cdots, f_n$ be functions ($n \in \mathbf{N}$) such that

$$\lim_{x \to a} f_i(x) = L_i$$

for $1 \le i \le n$. Then

$$\lim_{x \to a} (f_1(x) + f_2(x) + \cdots + f_n(x)) = L_1 + L_2 + \cdots + L_n.$$

$$\lim_{x \to 2} (x^5 - 2x^3 - 3x - 7) = 2^5 - 2 \cdot 2^3 - 3 \cdot 2 - 7 = 3.$$

Continuity

Definition

Let $f : X \to \mathbf{R}$ be a function, where $X \subseteq \mathbf{R}$, and let a bea real number such that f is defined in some deleted neighborhood of a.

If f is defined at a, then f is said to **continuous** at a if $\lim_{x \to a} f(x) = f(a)$.

Therefore, a function f is continuous at a if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. (Notice that in this instance, $0 < |x - a| < \delta$ is being replaced by $|x - a| < \delta$.) Thus, for f to be continuous at a, three conditions must be satisfied:

(1) f is defined at a;

(2) $\lim_{x \to a} f(x)$ exists;

(3) $\lim_{x \to a} f(x) = f(a)$.

Example 14

A function f is defined by

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 1}$$

for all $x \in \mathbf{R} - \{-1, 1\}$. Is f continuous at 1 under any of the following circumstances?

(a) f is not defined at 1;

(b) $f(1) = 0$;

(c) $f(1) = -1/2$?

Example 14 (continued)

Solution. For f to be continuous at 1, the function f must be defined at 1. So, we can answer question (a) immediately. The answer is no. In order to answer questions (b) and (c), we must first determine whether $\lim_{x \to 1} f(x)$ exists. Observe that

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 1} = \frac{(x-1)(x-2)}{(x-1)(x+1)} = \frac{x-2}{x+1}$$

since $x \neq 1$.

Example 14 (continued)

Because $f(x) = \dfrac{x-2}{x+1}$ is a rational function, it follows that

$$\lim_{x \to 1} \frac{x-2}{x+1} = \frac{\lim_{x \to 1}(x-2)}{\lim_{x \to 1}(x+1)} = \frac{-1}{2} = -\frac{1}{2}.$$

Hence, if $f(1) = -1/2$, then f is continuous at 1. Therefore, the answer to question (b) is no and the answer to (c) is yes. ♦

Example 15

Result The function f defined by

$$f(x) = \sqrt{x}$$

for $x \geq 0$ is continuous at 4.

Proof. Let $\epsilon > 0$ be given and choose $\delta = \min(1, 3\epsilon)$. Let $x \in \mathbf{R}$ such that $|x - 4| < \delta$. Since $|x - 4| < 1$, it follows that $3 < x < 5$ and so $\sqrt{x} + 2 > 3$. Therefore, $1/(\sqrt{x} + 2) < 1/3$. Hence,

$$|\sqrt{x} - 2| = \left|\frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{\sqrt{x} + 2}\right| = \frac{|x - 4|}{\sqrt{x} + 2} < \frac{1}{3}(3\epsilon) = \epsilon. \quad \square$$

Differentiability

Definition

A function $f : X \to \mathbf{R}$, where $X \subseteq \mathbf{R}$, that is defined in a neighborhood of a real number a is said to be **differentiable** at a if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. This limit is called the **derivative** of f at a and is denoted by $f'(a)$. Therefore,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Example 16

Show that the function f defined by $f(x) = 1/x^2$ for $x \neq 0$ is differentiable at 1 and determine $f'(1)$.

Solution. Thus, we need to show that

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{x^2} - 1}{x - 1}$$

exists. In a deleted neighborhood of 1,

$$\frac{\frac{1}{x^2} - 1}{x - 1} = \frac{\frac{1-x^2}{x^2}}{x - 1} = \frac{1 - x^2}{x^2(x - 1)} = \frac{(1 - x)(1 + x)}{x^2(x - 1)} = -\frac{1 + x}{x^2}.$$

Since $\dfrac{1 + x}{-x^2}$ is a rational function, we see that

$$\lim_{x \to 1} \frac{1 + x}{-x^2} = \frac{\lim_{x \to 1}(1 + x)}{\lim_{x \to 1}(-x^2)} = \frac{2}{-1} = -2$$

and so $f'(1) = -2$. ♦

An $\epsilon - \delta$ proof of the previous example:

Example 17

Result Let f be the function defined by $f(x) = 1/x^2$ for $x \neq 0$. Then $f'(1) = -2$.

Proof. Let $\epsilon > 0$ be given and choose $\delta = \min(1/2, \epsilon/16)$. Let $x \in \mathbf{R}$ such that $0 < |x - 1| < \delta$. Since $|x - 1| < 1/2$, it follows that $1/2 < x < 3/2$. Thus, $x^2 > 1/4$ and so $1/x^2 < 4$. Also, $|2x + 1| < 4$. Since $|x - 1| < \epsilon/16$, it follows that

$$\left|\frac{f(x) - f(1)}{x - 1} - (-2)\right| = \left|\frac{\frac{1}{x^2} - 1}{x - 1} - (-2)\right| = \left|-\frac{1 + x}{x^2} + 2\right|$$

$$= \left|\frac{2x^2 - x - 1}{x^2}\right| = \frac{|2x + 1|}{x^2} \cdot |x - 1|$$

$$< 4 \cdot 4 \cdot \frac{\epsilon}{16} = \epsilon.$$

□

Differentiability

Theorem

If a function f is differentiable at a, then f is continuous at a.

Proof. Since f is differentiable at a, it follows that $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists and equals the real number $f'(a)$. To show that f is continuous at a, we need to show that $\lim_{x \to a} f(x) = f(a)$. We write $f(x)$ as

$$f(x) = \frac{f(x) - f(a)}{x - a}(x - a) + f(a).$$

Now, using properties of limits, we have

$$\begin{aligned} \lim_{x \to a} f(x) &= \left[\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right] \lim_{x \to a}(x - a) + \lim_{x \to a} f(a) \\ &= f'(a) \cdot 0 + f(a) = f(a). \end{aligned}$$

□

The converse of this theorem is not true. For example, the functions f and g defined by $f(x) = |x|$ and $g(x) = \sqrt[3]{x}$ are continuous at 0 but neither is differentiable at 0.