CHAPTER

13

PLANE CURVES AND POLAR COORDINATES

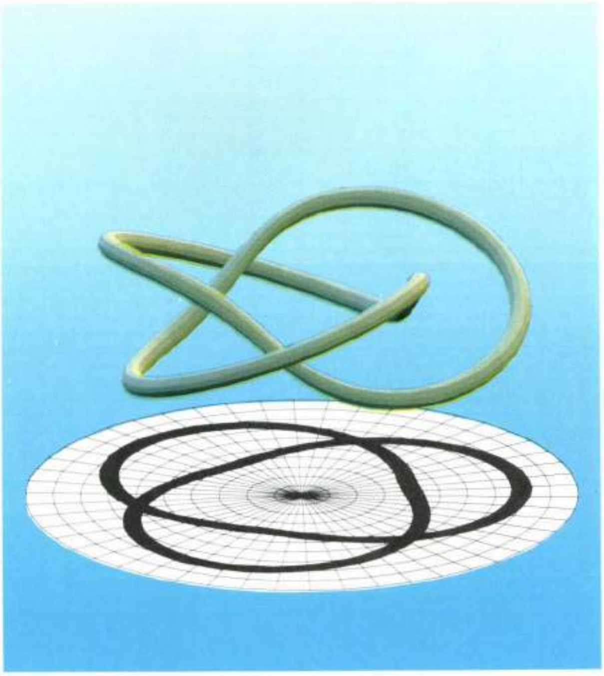
INTRODUCTION

The concept of *curve* is more general than that of the graph of a function, since a curve may cross itself in figure-eight style, be closed (as are circles and ellipses), or spiral around a fixed point. In fact, some curves studied in advanced mathematics pass through every point in a coordinate plane!

The curves discussed in this chapter lie in an xy-plane, and each has the property that the coordinates x and y of an arbitrary point P on the curve can be expressed as functions of a variable t, called a *parameter*. The reason for choosing the letter t is that in many applications this variable denotes time and P represents a moving object that has position (x, y) at time t. In later chapters we use such representations to define velocity, acceleration, and other concepts associated with motion.

In Sections 13.3 and 13.4 we discuss polar coordinates and use definite integrals to find areas enclosed by graphs of polar equations. Our methods are analogous to those developed in Chapter 6. The principal difference is that we consider limits of sums of circular sectors instead of vertical or horizontal rectangles.

The chapter closes with a unified description of conics in terms of polar equations. Such equations are indispensable in analyzing orbits of planets, satellites, and atomic particles.



13.1 PLANE CURVES

If f is a continuous function, the graph of the equation $y = f(x)$ is often called a *plane curve*. However, this definition is restrictive, because it excludes many useful graphs. The following definition is more general.

Definition **(13.1)**

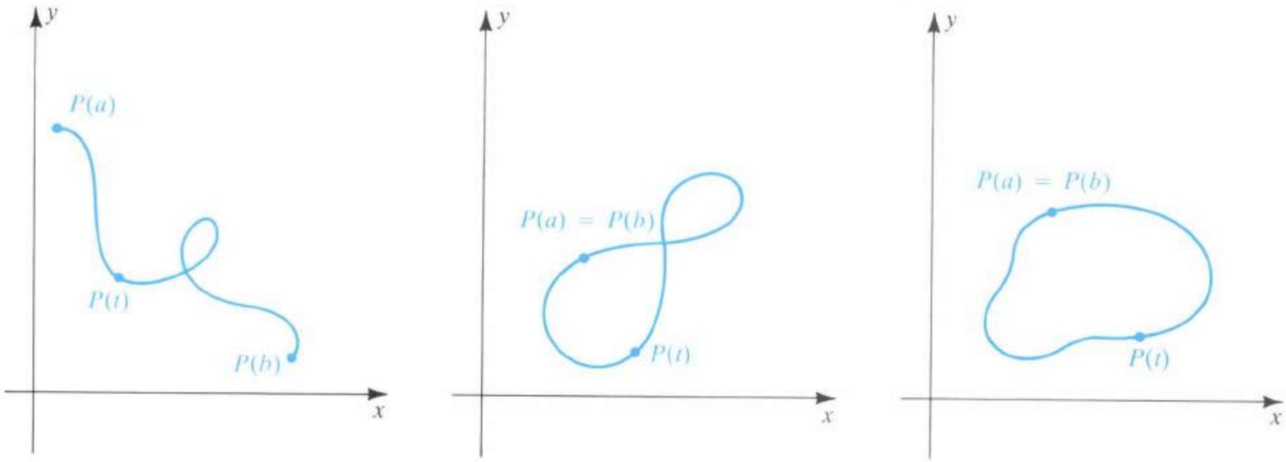
> A **plane curve** is a set C of ordered pairs $(f(t), g(t))$, where f and g are continuous functions on an interval I.

For simplicity, we often refer to a plane curve as a **curve**. The **graph** of C in Definition (13.1) consists of all points $P(t) = (f(t), g(t))$ in an xy-plane, for t in I. We shall use the term *curve* interchangeably with *graph of a curve*. We sometimes regard the point $P(t)$ as tracing the curve C as t varies through the interval I.

The graphs of several curves are sketched in Figure 13.1, where I is a closed interval $[a, b]$. In (i) of the figure, $P(a) \neq P(b)$, and $P(a)$ and $P(b)$ are called the **endpoints** of C. The curve in (i) intersects itself; that is, two different values of t produce the same point. If $P(a) = P(b)$, as in Figure 13.1(ii), then C is a **closed curve**. If $P(a) = P(b)$ and C does not intersect itself at any other point, as in (iii), then C is a **simple closed curve**.

FIGURE 13.1

(i) Curve (ii) Closed curve (iii) Simple closed curve



A convenient way to represent curves is given in the next definition.

Definition **(13.2)**

> Let C be the curve consisting of all ordered pairs $(f(t), g(t))$, where f and g are continuous on an interval I. The equations
>
> $$x = f(t), \quad y = g(t),$$
>
> for t in I, are **parametric equations** for C with **parameter** t.

The curve C in this definition is referred to as a **parametrized curve**, and the parametric equations are a **parametrization** for C. We often use the notation

$$x = f(t), \quad y = g(t); \quad t \text{ in } I$$

to indicate the domain I of f and g. Sometimes it may be possible to eliminate the parameter and obtain a familiar equation in x and y for C. In simple cases we may sketch a graph of a parametrized curve by plotting points and connecting them in the order of increasing t, as illustrated in the next example.

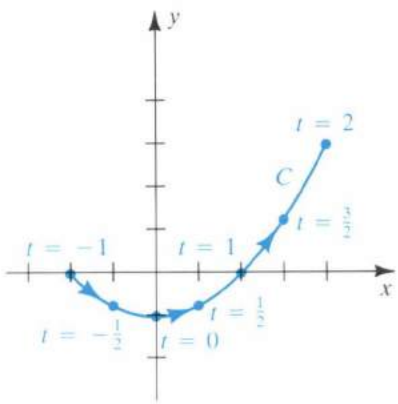
EXAMPLE 1 Sketch the graph of the curve C that has the parametrization

$$x = 2t, \quad y = t^2 - 1; \quad -1 \le t \le 2.$$

SOLUTION We use the parametric equations to tabulate coordinates of points $P(x, y)$ on C as follows.

t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
x	-2	-1	0	1	2	3	4
y	0	$-\frac{3}{4}$	-1	$-\frac{3}{4}$	0	$\frac{5}{4}$	3

FIGURE 13.2
$x = 2t, y = t^2 - 1; -1 \le t \le 2$



Plotting points leads to the sketch in Figure 13.2. The arrowheads on the graph indicate the direction in which $P(x, y)$ traces the curve as t *increases* from -1 to 2.

We may obtain a clearer description of the graph by eliminating the parameter. Solving the first parametric equation for t, we obtain $t = \frac{1}{2}x$. Substituting this expression for t in the second equation gives us

$$y = (\tfrac{1}{2}x)^2 - 1.$$

The graph of this equation in x and y is a parabola symmetric with respect to the y-axis with vertex $(0, -1)$. However, since $x = 2t$ and $-1 \le t \le 2$, we see that $-2 \le x \le 4$ for points (x, y) on C, and hence C is that part of the parabola between the points $(-2, 0)$ and $(4, 3)$ shown in Figure 13.2.

As indicated by the arrowheads in Figure 13.2, the point $P(x, y)$ traces the curve C from *left to right* as t increases. The parametric equations

$$x = -2t, \quad y = t^2 - 1; \quad -2 \le t \le 1$$

give us the same graph; however, as t increases, $P(x, y)$ traces the curve from *right to left*. For other parametrizations, the point $P(x, y)$ may oscillate back and forth as t increases.

The **orientation** of a parametrized curve C is the direction determined by *increasing* values of the parameter. We often indicate an orientation by placing arrowheads on C as in Figure 13.2. If $P(x, y)$ moves back and forth as t increases, we may place arrows *alongside* of C. As we have observed, a curve may have different orientations, depending on the parametrization.

The next example demonstrates that it is sometimes useful to eliminate the parameter *before* plotting points.

EXAMPLE 2 A point moves in a plane such that its position $P(x, y)$ at time t is given by

$$x = a\cos t, \quad y = a\sin t; \quad t \text{ in } \mathbb{R},$$

where $a > 0$. Describe the motion of the point.

SOLUTION We may eliminate the parameter by rewriting the parametric equations as

$$\frac{x}{a} = \cos t, \quad \frac{y}{a} = \sin t$$

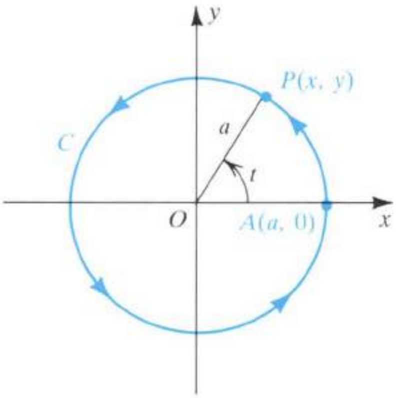
and using the identity $\cos^2 t + \sin^2 t = 1$ to obtain

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1,$$

or

$$x^2 + y^2 = a^2.$$

FIGURE 13.3
$x = a\cos t$, $y = a\sin t$; t in $\mathbb{R}$



This shows that the point $P(x, y)$ moves on the circle C of radius a with center at the origin (see Figure 13.3). The point is at $A(a, 0)$ when $t = 0$, at $(0, a)$ when $t = \pi/2$, at $(-a, 0)$ when $t = \pi$, at $(0, -a)$ when $t = 3\pi/2$, and back at $A(a, 0)$ when $t = 2\pi$. Thus, P moves around C in a counterclockwise direction, making one revolution every 2π units of time. The orientation of C is indicated by the arrowheads in the figure.

Note that in this example we may interpret t geometrically as the radian measure of the angle generated by the line segment OP.

EXAMPLE 3 Sketch the graph of the curve C that has the parametrization

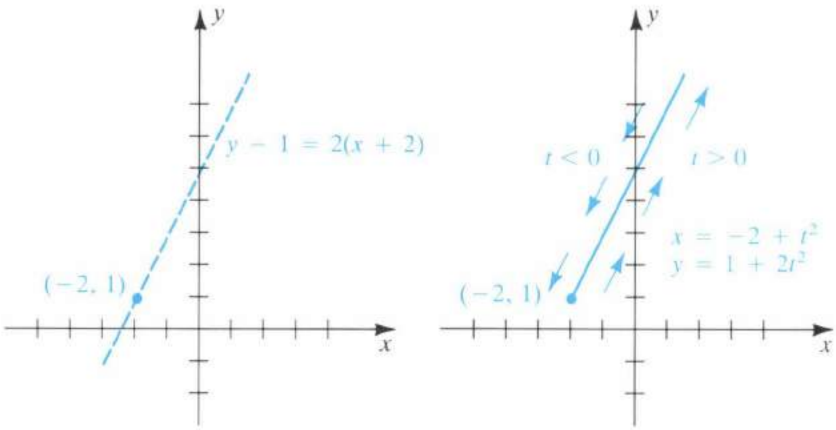
$$x = -2 + t^2, \quad y = 1 + 2t^2; \quad t \text{ in } \mathbb{R}$$

and indicate the orientation.

SOLUTION To eliminate the parameter, we use the first equation to obtain $t^2 = x + 2$ and then substitute for t^2 in the second equation. Thus,

$$y = 1 + 2(x + 2).$$

FIGURE 13.4
(i) (ii)



This is an equation of the line of slope 2 through the point $(-2, 1)$, as indicated by the dashes in Figure 13.4(i). However, since $t^2 \geq 0$, we see from the parametric equations for C that

$$x = -2 + t^2 \geq -2 \quad \text{and} \quad y = 1 + 2t^2 \geq 1.$$

Thus, the graph of C is that part of the line to the right of $(-2, 1)$ (the point corresponding to $t = 0$), as shown in Figure 13.4(ii). The orientation is indicated by the arrows alongside of C. As t increases in the interval $(-\infty, 0]$, $P(x, y)$ moves down the curve toward the point $(-2, 1)$. As t increases in $[0, \infty)$, $P(x, y)$ moves up the curve away from $(-2, 1)$.

If a curve C is described by an equation $y = f(x)$ for a continuous function f, then an easy way to obtain parametric equations for C is to let

$$x = t, \quad y = f(t),$$

where t is in the domain of f. For example, if $y = x^3$, then parametric equations are

$$x = t, \quad y = t^3; \quad t \text{ in } \mathbb{R}.$$

We can use many different substitutions for x, provided that as t varies through some interval, x takes on every value in the domain of f. Thus, the graph of $y = x^3$ is also given by

$$x = t^{1/3}, \quad y = t; \quad t \text{ in } \mathbb{R}.$$

Note, however, that the parametric equations

$$x = \sin t, \quad y = \sin^3 t; \quad t \text{ in } \mathbb{R}$$

give only that part of the graph of $y = x^3$ between the points $(-1, -1)$ and $(1, 1)$.

EXAMPLE 4 Find three parametrizations for the line of slope m through the point (x_1, y_1).

SOLUTION By the point-slope form, an equation for the line is

$$y - y_1 = m(x - x_1).$$

If we let $x = t$, then $y - y_1 = m(t - x_1)$ and we obtain the parametrization

$$x = t, \quad y = y_1 + m(t - x_1); \quad t \text{ in } \mathbb{R}.$$

We obtain another parametrization for the line if we let $x - x_1 = t$. In this case $y - y_1 = mt$, and we have

$$x = x_1 + t, \quad y = y_1 + mt; \quad t \text{ in } \mathbb{R}.$$

As a third illustration, if we let $x - x_1 = \tan t$, then

$$x = x_1 + \tan t, \quad y = y_1 + m \tan t; \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

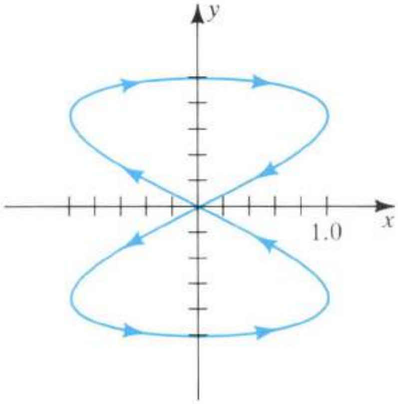
There are many other parametrizations for the line.

Parametric equations of the form

$$x = a \sin \omega_1 t, \quad y = b \cos \omega_2 t; \quad t \geq 0,$$

where a, b, ω_1, and ω_2 are constants, occur in electrical theory. The variables x and y usually represent voltages or currents at time t. The resulting curve is often difficult to sketch; however, using an oscilloscope and imposing voltages or currents on the input terminals, we can represent the graph, a **Lissajous figure**, on the screen of the oscilloscope. Computers are also useful in obtaining these complicated graphs.

FIGURE 13.5
$x = \sin 2t$, $y = \cos t$; $0 \leq t \leq 2\pi$



EXAMPLE 5 A computer-generated graph of the Lissajous figure

$$x = \sin 2t, \quad y = \cos t; \quad 0 \leq t \leq 2\pi$$

is shown in Figure 13.5, with the arrowheads indicating the orientation. Verify the orientation and find an equation in x and y for the curve.

SOLUTION Referring to the parametric equations, we see that as t increases from 0 to $\pi/2$, the point $P(x, y)$ starts at $(0, 1)$ and traces the part of the curve in quadrant I (in a generally clockwise direction). As t increases from $\pi/2$ to π, $P(x, y)$ traces the part in quadrant III (in a counterclockwise direction). For $\pi \leq t \leq 3\pi/2$, we obtain the part in quadrant IV; and $3\pi/2 \leq t \leq 2\pi$ gives us the part in quadrant II.

We may find an equation in x and y for the curve by employing trigonometric identities and algebraic manipulations. Writing $x = 2 \sin t \cos t$ and squaring, we have

$$x^2 = 4 \sin^2 t \cos^2 t,$$

or

$$x^2 = 4(1 - \cos^2 t) \cos^2 t.$$

Using $y = \cos t$ gives us

$$x^2 = 4(1 - y^2)y^2.$$

To express y in terms of x, let us rewrite the last equation as

$$4y^4 - 4y^2 + x^2 = 0$$

and use the quadratic formula to solve for y^2 as follows:

$$y^2 = \frac{4 \pm \sqrt{16 - 16x^2}}{8} = \frac{1 \pm \sqrt{1 - x^2}}{2}$$

Taking square roots, we obtain

$$y = \pm \sqrt{\frac{1 \pm \sqrt{1 - x^2}}{2}}.$$

These complicated equations should indicate the advantage of expressing the curve in parametric form.

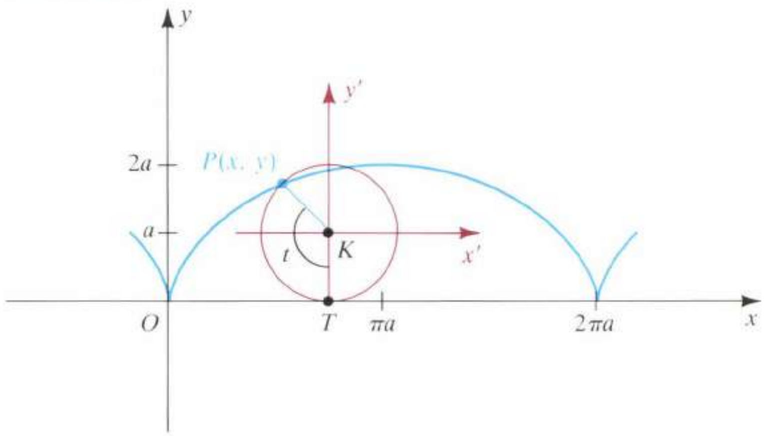
A curve C is **smooth** if it has a parametrization $x = f(t)$, $y = g(t)$ on an interval I such that the derivatives f' and g' are continuous and not simultaneously zero, except possibly at endpoints of I. A curve C is **piecewise smooth** if the interval I can be partitioned into closed subintervals with

C smooth on each subinterval. The graph of a smooth curve has no corners or cusps. The curves given in Examples 1–5 are smooth. The curve in the next example is piecewise smooth.

EXAMPLE 6 The curve traced by a fixed point P on the circumference of a circle as the circle rolls along a line in a plane is called a **cycloid**. Find parametric equations for a cycloid and determine the intervals on which it is smooth.

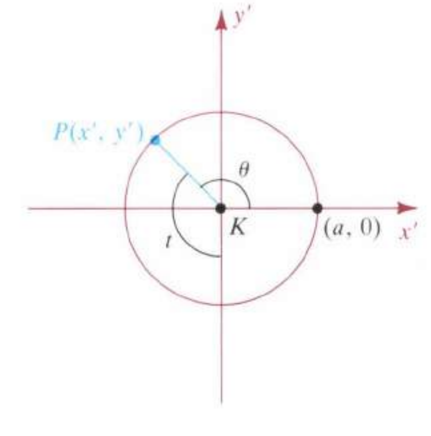
SOLUTION Suppose the circle has radius a and that it rolls along (and above) the x-axis in the positive direction. If one position of P is the origin, then Figure 13.6 displays part of the curve and a possible position of the circle.

FIGURE 13.6



Let K denote the center of the circle and T the point of tangency with the x-axis. We introduce, as a parameter t, the radian measure of angle TKP. The distance the circle has rolled is $d(O, T) = at$. Consequently the coordinates of K are (at, a). If we consider an $x'y'$-coordinate system with origin at $K(at, a)$ and if $P(x', y')$ denotes the point P relative to this system, then, by the translation of axes formulas with $h = at$ and $k = a$,

FIGURE 13.7



$$x = at + x', \quad y = a + y'.$$

If, as in Figure 13.7, θ denotes an angle in standard position on the $x'y'$-plane, then $\theta = (3\pi/2) - t$. Hence

$$x' = a \cos \theta = a \cos\left(\frac{3\pi}{2} - t\right) = -a \sin t$$

$$y' = a \sin \theta = a \sin\left(\frac{3\pi}{2} - t\right) = -a \cos t,$$

and substitution in $x = at + x'$, $y = a + y'$ gives us parametric equations for the cycloid:

$$x = a(t - \sin t), \quad y = a(1 - \cos t); \quad t \text{ in } \mathbb{R}.$$

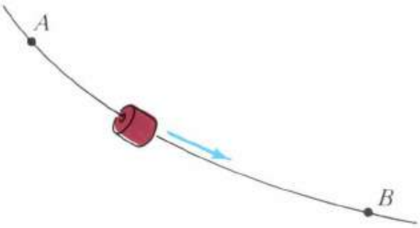
Differentiating the parametric equations of the cycloid yields

$$\frac{dx}{dt} = a(1 - \cos t), \quad \frac{dy}{dt} = a \sin t.$$

These derivatives are continuous for every t, but are simultaneously 0 at $t = 2\pi n$ for every integer n. The points corresponding to $t = 2\pi n$ are the x-intercepts of the graph, and the cycloid has a cusp at each such point (see Figure 13.6). The graph is piecewise smooth, since it is smooth on the t-interval $[2\pi n, 2\pi(n+1)]$ for every integer n.

If $a < 0$, then the graph of $x = a(t - \sin t)$, $y = a(1 - \cos t)$ is the inverted cycloid that results if the circle of Example 6 rolls *below* the x-axis. This curve has a number of important physical properties. To illustrate, suppose a thin wire passes through two fixed points A and B, as shown in Figure 13.8, and that the shape of the wire can be changed by bending it in any manner. Suppose further that a bead is allowed to slide along the wire and the only force acting on the bead is gravity. We now ask which of all the possible paths will allow the bead to slide from A to B in the least amount of time. It is natural to believe that the desired path is the straight line segment from A to B; however, this is not the correct answer. The path that requires the least time coincides with the graph of an inverted cycloid with A at the origin. Because the velocity of the bead increases more rapidly along the cycloid than along the line through A and B, the bead reaches B more rapidly, even though the distance is greater.

FIGURE 13.8



There is another interesting property of this **curve of least descent**. Suppose that A is the origin and B is the point with x-coordinate $\pi|a|$—that is, the lowest point on the cycloid in the first arc to the right of A. If the bead is released at *any* point between A and B, it can be shown that the time required for it to reach B is always the same.

Variations of the cycloid occur in applications. For example, if a motorcycle wheel rolls along a straight road, then the curve traced by a fixed point on one of the spokes is a cycloidlike curve. In this case the curve does not have corners or cusps, nor does it intersect the road (the x-axis) as does the graph of a cycloid. If the wheel of a train rolls along a railroad track, then the curve traced by a fixed point on the circumference of the wheel (which extends below the track) contains loops at regular intervals. Other cycloids are defined in Exercises 33 and 34.

EXERCISES 13.1

Exer. 1–24: (a) Find an equation in x and y whose graph contains the points on the curve C. (b) Sketch the graph of C and indicate the orientation.

1 $x = t - 2$, $y = 2t + 3$; $0 \le t \le 5$

2 $x = 1 - 2t$, $y = 1 + t$; $-1 \le t \le 4$

3 $x = t^2 + 1$, $y = t^2 - 1$; $-2 \le t \le 2$

4 $x = t^3 + 1$, $y = t^3 - 1$; $-2 \le t \le 2$

5 $x = 4t^2 - 5$, $y = 2t + 3$; t in $\mathbb{R}$

6 $x = t^3$, $y = t^2$; t in $\mathbb{R}$

7 $x = e^t$, $y = e^{-2t}$; t in $\mathbb{R}$

8 $x = \sqrt{t}$, $y = 3t + 4$; $t \ge 0$

9 $x = 2 \sin t$, $y = 3 \cos t$; $0 \le t \le 2\pi$

10 $x = \cos t - 2$, $y = \sin t + 3$; $0 \le t \le 2\pi$

11 $x = \sec t, \quad y = \tan t; \quad -\pi/2 < t < \pi/2$

12 $x = \cos 2t, \quad y = \sin t; \quad -\pi \le t \le \pi$

13 $x = t^2, \quad y = 2 \ln t; \quad t > 0$

14 $x = \cos^3 t, \quad y = \sin^3 t; \quad 0 \le t \le 2\pi$

15 $x = \sin t, \quad y = \csc t; \quad 0 < t \le \pi/2$

16 $x = e^t, \quad y = e^{-t}; \quad t$ in $\mathbb{R}$

17 $x = \cosh t, \quad y = \sinh t; \quad t$ in $\mathbb{R}$

18 $x = 3 \cosh t, \quad y = 2 \sinh t; \quad t$ in $\mathbb{R}$

19 $x = t, \quad y = \sqrt{t^2 - 1}; \quad |t| \ge 1$

20 $x = -2\sqrt{1 - t^2}, \quad y = t; \quad |t| \le 1$

21 $x = t, \quad y = \sqrt{t^2 - 2t + 1}; \quad 0 \le t \le 4$

22 $x = 2t, \quad y = 8t^3; \quad -1 \le t \le 1$

23 $x = (t + 1)^3, \quad y = (t + 2)^2; \quad 0 \le t \le 2$

24 $x = \tan t, \quad y = 1; \quad -\pi/2 < t < \pi/2$

Exer. 25–26: Curves C_1, C_2, C_3, and C_4 are given parametrically, for t in $\mathbb{R}$. Sketch their graphs and indicate orientations.

25 C_1: $x = t^2, \quad y = t$
C_2: $x = t^4, \quad y = t^2$
C_3: $x = \sin^2 t, \quad y = \sin t$
C_4: $x = e^{2t}, \quad y = -e^t$

26 C_1: $x = t, \quad y = 1 - t$
C_2: $x = 1 - t^2, \quad y = t^2$
C_3: $x = \cos^2 t, \quad y = \sin^2 t$
C_4: $x = \ln t - t, \quad y = 1 + t - \ln t; \quad t > 0$

Exer. 27–28: The parametric equations specify the position of a moving point $P(x, y)$ at time t. Sketch the graph and indicate the motion of P as t increases.

27 (a) $x = \cos t, \quad y = \sin t; \quad 0 \le t \le \pi$
(b) $x = \sin t, \quad y = \cos t; \quad 0 \le t \le \pi$
(c) $x = t, \quad y = \sqrt{1 - t^2}; \quad -1 \le t \le 1$

28 (a) $x = t^2, \quad y = 1 - t^2; \quad 0 \le t \le 1$
(b) $x = 1 - \ln t, \quad y = \ln t; \quad 1 \le t \le e$
(c) $x = \cos^2 t, \quad y = \sin^2 t; \quad 0 \le t \le 2\pi$

29 Show that

$$x = a \cos t + h, \quad y = b \sin t + k; \quad 0 \le t \le 2\pi$$

are parametric equations of an ellipse with center (h, k) and axes of lengths $2a$ and $2b$.

30 Show that

$$x = a \sec t + h, \quad y = b \tan t + k;$$
$$-\frac{\pi}{2} < t < \frac{3\pi}{2} \quad \text{and} \quad t \ne \frac{\pi}{2}$$

are parametric equations of a hyperbola with center (h, k), transverse axis of length $2a$, and conjugate axis of length $2b$. Determine the values of t for each branch.

31 If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are distinct points, show that

$$x = (x_2 - x_1)t + x_1, \quad y = (y_2 - y_1)t + y_1; \quad t \text{ in } \mathbb{R}$$

are parametric equations for the line l through P_1 and P_2.

32 Describe the difference between the graph of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ and the graph of

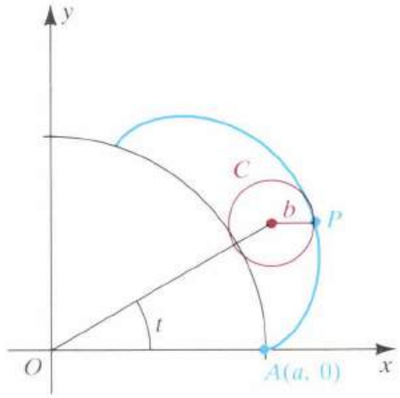
$$x = a \cosh t, \quad y = b \sinh t; \quad t \text{ in } \mathbb{R}.$$

(*Hint:* Use Theorem (8.11).)

33 A circle C of radius b rolls on the outside of the circle $x^2 + y^2 = a^2$, and $b < a$. Let P be a fixed point on C, and let the initial position of P be $A(a, 0)$, as shown in the figure. If the parameter t is the angle from the positive x-axis to the line segment from O to the center of C, show that parametric equations for the curve traced by P (an *epicycloid*) are

$$x = (a + b) \cos t - b \cos\left(\frac{a + b}{b} t\right),$$
$$y = (a + b) \sin t - b \sin\left(\frac{a + b}{b} t\right); \quad 0 \le t \le 2\pi.$$

EXERCISE 33



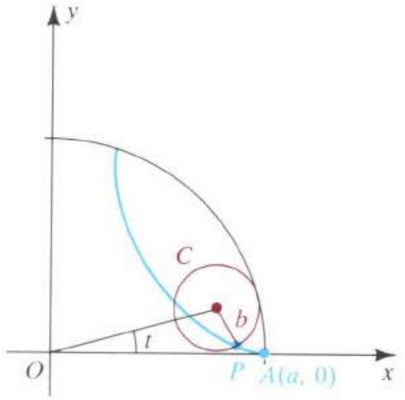
34 If the circle C of Exercise 33 rolls on the inside of the second circle (see the figure on the following page), then the curve traced by P is a *hypocycloid*.

(a) Show that parametric equations for this curve are

$$x = (a - b) \cos t + b \cos\left(\frac{a - b}{b} t\right),$$
$$y = (a - b) \sin t - b \sin\left(\frac{a - b}{b} t\right); \quad 0 \le t \le 2\pi.$$

(b) If $b = \frac{1}{4}a$, show that $x = a \cos^3 t$, $y = a \sin^3 t$ and sketch the graph.

EXERCISE 34



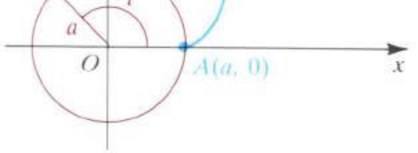
35 If $b = \frac{1}{3}a$ in Exercise 33, find parametric equations for the epicycloid and sketch the graph.

36 The radius of circle B is one-third that of circle A. How many revolutions will circle B make as it rolls around circle A until it reaches its starting point? (*Hint:* Use Exercise 35.)

37 If a string is unwound from around a circle of radius a and is kept tight in the plane of the circle, then a fixed point P on the string will trace a curve called the *involute of the circle*. Let the circle be chosen as in the figure. If the parameter t is the measure of the indicated angle and the initial position of P is $A(a, 0)$, show that parametric equations for the involute are

$$x = a(\cos t + t \sin t), \quad y = a(\sin t - t \cos t).$$

EXERCISE 37



38 Generalize the cycloid of Example 6 to the case where P is any point on a fixed line through the center C of the circle. If $b = d(C, P)$, show that

$$x = at - b \sin t, \quad y = a - b \cos t.$$

Sketch a typical graph if $b < a$ (a *curtate cycloid*) and if $b > a$ (a *prolate cycloid*). The term *trochoid* is sometimes used for either of these curves.

39 Refer to Example 5.

(a) Describe the Lissajous figure given by $f(t) = a \sin \omega t$ and $g(t) = b \cos \omega t$ for $t \geq 0$ and $a \neq b$.

(b) Suppose $f(t) = a \sin \omega_1 t$ and $g(t) = b \sin \omega_2 t$, where ω_1 and ω_2 are positive rational numbers, and write ω_2/ω_1 as m/n for positive integers m and n. Show that if $p = 2\pi n/\omega_1$, then $f(t + p) = f(t)$ and $g(t + p) = g(t)$. Conclude that the curve retraces itself every p units of time.

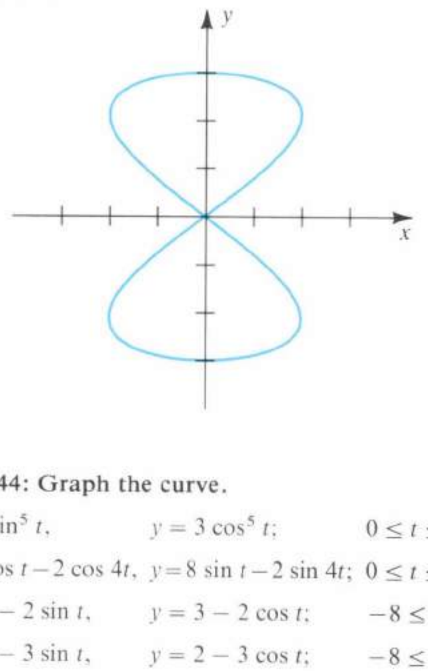
40 Shown in the figure is the Lissajous figure given by

$$x = 2 \sin 3t, \quad y = 3 \sin 1.5t; \quad t \geq 0.$$

(a) Find the period of the figure—that is, the length of the smallest t-interval that traces the curve.

(b) Find the maximum distance from the origin to a point on the graph.

EXERCISE 40



c **Exer. 41–44: Graph the curve.**

41 $x = 3 \sin^5 t, \quad y = 3 \cos^5 t; \quad 0 \leq t \leq 2\pi$

42 $x = 8 \cos t - 2 \cos 4t, \quad y = 8 \sin t - 2 \sin 4t; \quad 0 \leq t \leq 2\pi$

43 $x = 3t - 2 \sin t, \quad y = 3 - 2 \cos t; \quad -8 \leq t \leq 8$

44 $x = 2t - 3 \sin t, \quad y = 2 - 3 \cos t; \quad -8 \leq t \leq 8$

Exer. 45–48: Graph the given curves on the same coordinate axes and describe the shape of the resulting figure.

c 45
C_1: $x = 2 \sin 3t, \quad y = 3 \cos 2t; \quad -\pi/2 \leq t \leq \pi/2$
C_2: $x = \frac{1}{4} \cos t + \frac{3}{4}, \quad y = \frac{1}{4} \sin t + \frac{3}{2}; \quad 0 \leq t \leq 2\pi$
C_3: $x = \frac{1}{4} \cos t - \frac{3}{4}, \quad y = \frac{1}{4} \sin t + \frac{3}{2}; \quad 0 \leq t \leq 2\pi$
C_4: $x = \frac{3}{4} \cos t, \quad y = \frac{1}{4} \sin t; \quad 0 \leq t \leq 2\pi$
C_5: $x = \frac{1}{4} \cos t, \quad y = \frac{1}{8} \sin t + \frac{3}{4}; \quad \pi \leq t \leq 2\pi$

46
C_1: $x = \frac{3}{2} \cos t + 1, \quad y = \sin t - 1; \quad -\pi/2 \leq t \leq \pi/2$
C_2: $x = \frac{3}{2} \cos t + 1, \quad y = \sin t + 1; \quad -\pi/2 \leq t \leq \pi/2$
C_3: $x = 1, \quad y = 2 \tan t; \quad -\pi/4 \leq t \leq \pi/4$

47 C_1: $x = \tan t$, $y = 3\tan t$; $0 \le t \le \pi/4$
C_2: $x = 1 + \tan t$, $y = 3 - 3\tan t$; $0 \le t \le \pi/4$
C_3: $x = \frac{1}{2} + \tan t$, $y = \frac{3}{2}$; $0 \le t \le \pi/4$

48 C_1: $x = 1 + \cos t$, $y = 1 + \sin t$; $\pi/3 \le t \le 2\pi$
C_2: $x = 1 + \tan t$, $y = 1$; $0 \le t \le \pi/4$

13.2 TANGENT LINES AND ARC LENGTH

The curve C given parametrically by

$$x = 2t, \quad y = t^2 - 1; \quad -1 \le t \le 2$$

can also be represented by an equation of the form $y = k(x)$, where k is a function defined on a suitable interval. In Example 1 of the preceding section, we eliminated the parameter t, obtaining

$$y = k(x) = \tfrac{1}{4}x^2 - 1 \quad \text{for} \quad -2 \le x \le 4.$$

The slope of the tangent line at any point $P(x, y)$ on C is

$$k'(x) = \tfrac{1}{2}x, \quad \text{or} \quad k'(x) = \tfrac{1}{2}(2t) = t.$$

Since it is often difficult to eliminate a parameter, we shall next derive a formula that can be used to find the slope directly from the parametric equations.

Theorem (13.3)

If a smooth curve C is given parametrically by $x = f(t)$, $y = g(t)$, then the slope dy/dx of the tangent line to C at $P(x, y)$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \text{provided} \quad \frac{dx}{dt} \ne 0.$$

PROOF If $dx/dt \ne 0$ at $x = c$, then, since f is continuous at c, $dx/dt > 0$ or $dx/dt < 0$ throughout an interval $[a, b]$, with $a < c < b$ (see Theorem (2.27)). Applying Theorem (7.6) or the analogous result for decreasing functions, we know that f has an inverse function f^{-1}, and we may consider $t = f^{-1}(x)$ for x in $[f(a), f(b)]$. Applying the chain rule to $y = g(t)$ and $t = f^{-1}(x)$, we obtain

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy/dt}{dx/dt},$$

where the last equality follows from Corollary (7.8). ■

EXAMPLE 1 Let C be the curve with parametrization

$$x = 2t, \quad y = t^2 - 1; \quad -1 \le t \le 2.$$

Find the slopes of the tangent line and normal line to C at $P(x, y)$.

SOLUTION The curve C was considered in Example 1 of the preceding section (see Figure 13.2). Using Theorem (13.3) with $x = 2t$ and $y = t^2 - 1$, we find that the slope of the tangent line at $P(x, y)$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t.$$

This result agrees with that of the discussion at the beginning of this section, where we used the form $y = k(x)$ to show that $m = \frac{1}{2}x = t$.

The slope of the normal line is the negative reciprocal $-1/t$, provided $t \neq 0$.

EXAMPLE 2 Let C be the curve with parametrization

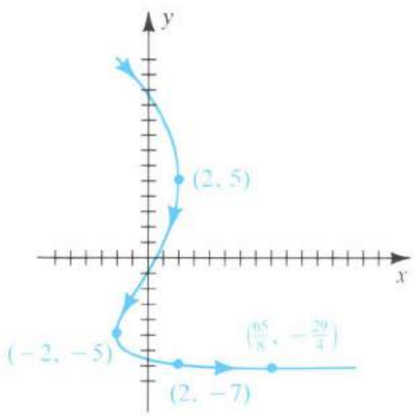
$$x = t^3 - 3t, \quad y = t^2 - 5t - 1; \quad t \text{ in } \mathbb{R}.$$

(a) Find an equation of the tangent line to C at the point corresponding to $t = 2$.

(b) For what values of t is the tangent line horizontal or vertical?

SOLUTION

FIGURE 13.9
$x = t^3 - 3t,\ y = t^2 - 5t - 1;\ t$ in $\mathbb{R}$



(a) A portion of the graph of C is sketched in Figure 13.9, where we have also plotted several points and indicated the orientation. Using the parametric equations for C, we find that the point corresponding to $t = 2$ is $(2, -7)$. By Theorem (13.3),

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t - 5}{3t^2 - 3}.$$

The slope m of the tangent line at $(2, -7)$ is

$$m = \frac{dy}{dx}\bigg]_{t=2} = \frac{2(2) - 5}{3(2^2) - 3} = -\frac{1}{9}.$$

Applying the point-slope form, we obtain an equation of the tangent line:

$$y + 7 = -\tfrac{1}{9}(x - 2), \quad \text{or} \quad x + 9y = -61$$

(b) The tangent line is horizontal if $dy/dx = 0$—that is, if $2t - 5 = 0$, or $t = \frac{5}{2}$. The corresponding point on C is $(\frac{65}{8}, -\frac{29}{4})$, as shown in Figure 13.9.

The tangent line is vertical if $3t^2 - 3 = 0$. Thus, there are vertical tangent lines at the points corresponding to $t = 1$ and $t = -1$—that is, at $(-2, 5)$ and $(2, 5)$.

If a curve C is parametrized by $x = f(t)$, $y = g(t)$ and if y' is a differentiable function of t, we can find d^2y/dx^2 by applying Theorem (13.3) to y' as follows.

Second derivative in parametric form **(13.4)**

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}}$$

It is important to observe that

$$\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2}.$$

EXAMPLE 3 Let C be the curve with parametrization

$$x = e^{-t}, \quad y = e^{2t}; \quad t \text{ in } \mathbb{R}.$$

(a) Sketch the graph of C and indicate the orientation.

(b) Use (13.3) and (13.4) to find dy/dx and d^2y/dx^2.

(c) Find a function k that has the same graph as C, and use $k'(x)$ and $k''(x)$ to check the answers to (b).

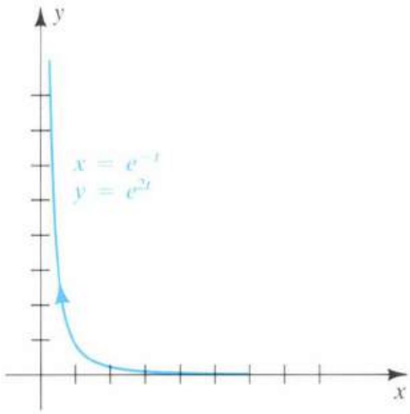
(d) Discuss the concavity of C.

SOLUTION

(a) To help us sketch the graph, let us first eliminate the parameter. Using $x = e^{-t} = 1/e^t$, we see that $e^t = 1/x$. Substituting in $y = e^{2t} = (e^t)^2$ gives us

$$y = \left(\frac{1}{x}\right)^2 = \frac{1}{x^2}.$$

FIGURE 13.10
$x = e^{-t}, y = e^{2t}; t$ in $\mathbb{R}$



Remembering that $x = e^{-t} > 0$ leads to the graph in Figure 13.10. Note that the point $(1, 1)$ corresponds to $t = 0$. If t increases in $(-\infty, 0]$, the point $P(x, y)$ approaches $(1, 1)$ from the right as indicated by the arrowhead. If t increases in $[0, \infty)$, $P(x, y)$ moves up the curve, approaching the y-axis.

(b) By (13.3) and (13.4),

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{2t}}{-e^{-t}} = -2e^{3t}$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{-6e^{3t}}{-e^{-t}} = 6e^{4t}.$$

(c) From part (a), a function k that has the same graph as C is given by

$$k(x) = \frac{1}{x^2} = x^{-2} \quad \text{for} \quad x > 0.$$

Differentiating twice yields

$$k'(x) = -2x^{-3} = -2(e^{-t})^{-3} = -2e^{3t}$$

$$k''(x) = 6x^{-4} = 6(e^{-t})^{-4} = 6e^{4t},$$

which is in agreement with part (b).

(d) Since $d^2y/dx^2 = 6e^{4t} > 0$ for every t, the curve C is concave upward at every point.

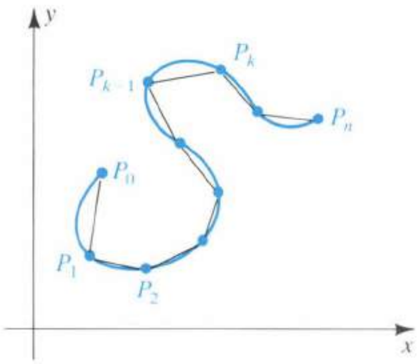
If a curve C is the graph of $y = f(x)$ and the function f is smooth on $[a, b]$, then the length of C is given by $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$ (see Definition (6.14)). We shall next obtain a formula for finding lengths of parametrized curves.

Suppose a smooth curve C is given parametrically by

$$x = f(t), \quad y = g(t); \quad a \le t \le b.$$

Furthermore, suppose C does not intersect itself—that is, different values of t between a and b determine different points on C. Consider a partition

FIGURE 13.11



P of $[a, b]$ given by $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. Let $\Delta t_k = t_k - t_{k-1}$ and let $P_k = (f(t_k), g(t_k))$ be the point on C that corresponds to t_k. If $d(P_{k-1}, P_k)$ is the length of the line segment $P_{k-1}P_k$, then the length L_P of the broken line in Figure 13.11 is

$$L_P = \sum_{k=1}^{n} d(P_{k-1}, P_k).$$

As in Section 6.5, we define

$$L = \lim_{\|P\| \to 0} L_P$$

and call L the **length of C** from P_0 to P_n if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|L_P - L| < \epsilon$ for every partition P with $\|P\| < \delta$.

By the distance formula,

$$d(P_{k-1}, P_k) = \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}.$$

By the mean value theorem (4.12), there exist numbers w_k and z_k in the open interval (t_{k-1}, t_k) such that

$$f(t_k) - f(t_{k-1}) = f'(w_k)\,\Delta t_k$$
$$g(t_k) - g(t_{k-1}) = g'(z_k)\,\Delta t_k.$$

Substituting these in the formula for $d(P_{k-1}, P_k)$ and removing the common factor $(\Delta t_k)^2$ from the radicand gives us

$$d(P_{k-1}, P_k) = \sqrt{[f'(w_k)]^2 + [g'(z_k)]^2}\,\Delta t_k.$$

Consequently

$$L = \lim_{\|P\| \to 0} L_P = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \sqrt{[f'(w_k)]^2 + [g'(z_k)]^2}\,\Delta t_k,$$

provided the limit exists. If $w_k = z_k$ for every k, then the sums are Riemann sums for the function defined by $\sqrt{[f'(t)]^2 + [g'(t)]^2}$. The limit of these sums is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt.$$

The limit exists even if $w_k \neq z_k$; however, the proof requires advanced methods and is omitted. The next theorem summarizes this discussion.

Theorem (13.5)

If a smooth curve C is given parametrically by $x = f(t)$, $y = g(t)$; $a \le t \le b$, and if C does not intersect itself, except possibly for $t = a$ and $t = b$, then the length L of C is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

The integral formula in Theorem (13.5) is not necessarily true if C intersects itself. For example, if C has the parametrization $x = \cos t$, $y = \sin t$; $0 \le t \le 4\pi$, then the graph is a unit circle with center at the origin. If t varies from 0 to 4π, the circle is traced twice and hence intersects itself infinitely many times. If we use Theorem (13.5) with $a = 0$ and $b = 4\pi$, we obtain the incorrect value 4π for the length of C. The correct value 2π can be obtained by using the t-interval $[0, 2\pi]$. Note that in this case the curve intersects itself only at the points corresponding to $t = 0$ and $t = 2\pi$, which is allowable by the theorem.

If a curve C is given by $y = k(x)$, with k' continuous on $[a, b]$, then parametric equations for C are

$$x = t, \quad y = k(t); \quad a \le t \le b.$$

In this case

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = k'(t) = k'(x), \quad dt = dx,$$

and from Theorem (13.5)

$$L = \int_a^b \sqrt{1 + [k'(x)]^2}\, dx.$$

This is in agreement with the arc length formula given in Definition (6.14).

EXAMPLE 4 Find the length of one arch of the cycloid that has the parametrization

$$x = t - \sin t, \quad y = 1 - \cos t; \quad t \text{ in } \mathbb{R}.$$

SOLUTION The graph has the shape illustrated in Figure 13.6. The radius a of the circle is 1. One arch is obtained if t varies from 0 to 2π. Applying Theorem (13.5) yields

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{1 - 2\cos^2 t + \cos^2 t + \sin^2 t}\, dt. \end{aligned}$$

Since $\cos^2 t + \sin^2 t = 1$, the integrand reduces to

$$\sqrt{2 - 2\cos t} = \sqrt{2}\sqrt{1 - \cos t}.$$

Thus,
$$L = \int_0^{2\pi} \sqrt{2}\sqrt{1 - \cos t}\, dt.$$

By a half-angle formula, $\sin^2 \frac{1}{2}t = \frac{1}{2}(1 - \cos t)$, or, equivalently,

$$1 - \cos t = 2\sin^2 \tfrac{1}{2}t.$$

Hence
$$\sqrt{1 - \cos t} = \sqrt{2\sin^2 \tfrac{1}{2}t} = \sqrt{2}\,|\sin \tfrac{1}{2}t|.$$

The absolute value sign may be deleted, since if $0 \le t \le 2\pi$, then $0 \le \frac{1}{2}t \le \pi$ and hence $\sin \frac{1}{2}t \ge 0$. Consequently

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{2}\sqrt{2}\sin \tfrac{1}{2}t\, dt = 2\int_0^{2\pi} \sin \tfrac{1}{2}t\, dt \\ &= -4\Big[\cos \tfrac{1}{2}t\Big]_0^{2\pi} = -4(-1 - 1) = 8. \end{aligned}$$

To remember Theorem (13.5), recall that if ds is the differential of arc length, then, by Theorem (6.17),

$$(ds)^2 = (dx)^2 + (dy)^2.$$

Assuming that ds and dt are positive, we have the following.

Parametric differential of arc length **(13.6)**

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Using (13.6), we can rewrite the formula for arc length in Theorem (13.5) as

$$L = \int_{t=a}^{t=b} ds.$$

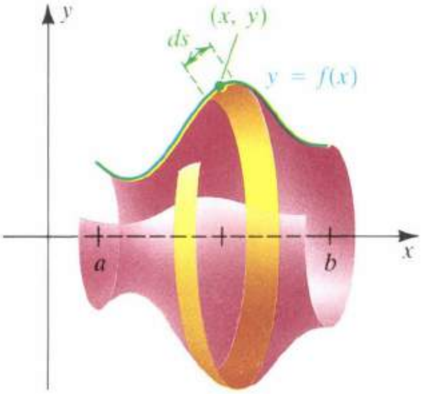
The limits of integration specify that the independent variable is t, not s.

If a function f is smooth and nonnegative for $a \le x \le b$, then, by Definition (6.19), the area S of the surface that is generated by revolving the graph of $y = f(x)$ about the x-axis (see Figure 13.12) is given by

$$S = \int_{x=a}^{x=b} 2\pi y\, ds,$$

where $ds = \sqrt{1 + [f'(x)]^2}\, dx$. We can regard $2\pi y\, ds$ as the surface area of a frustum of a cone of slant height ds and average radius y (see (6.18)).

FIGURE 13.12



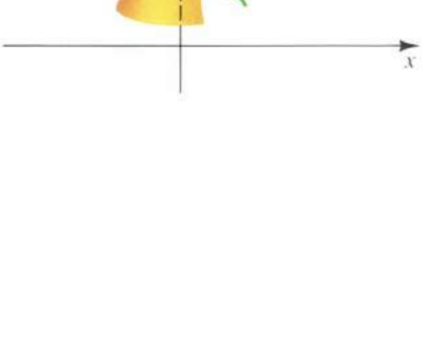
If a curve C is given parametrically by $x = f(t)$, $y = g(t)$; $a \le t \le b$ and if $g(t) \ge 0$ throughout $[a, b]$, we can use an argument similar to that given in Section 6.5 to show that the area of the surface generated by revolving C about the y-axis is $S = \int_{t=a}^{t=b} 2\pi y\, ds$, where ds is the parametric differential of arc length (13.6). Let us state this for reference as follows.

Theorem **(13.7)**

Let a smooth curve C be given by $x = f(t)$, $y = g(t)$; $a \le t \le b$, and suppose C does not intersect itself, except possibly at the point corresponding to $t = a$ and $t = b$. If $g(t) \ge 0$ throughout $[a, b]$, then the area S of the surface of revolution obtained by revolving C about the x-axis is

$$S = \int_{t=a}^{t=b} 2\pi y\, ds = \int_a^b 2\pi g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

FIGURE 13.13



The formula for S in Theorem (13.7) can be extended to the case in which $y = g(t)$ is negative for some t in $[a, b]$ by replacing the variable y that precedes ds by $|y|$.

If the curve C in Theorem (13.7) is revolved about the y-axis and if $x = f(t) \ge 0$ for $a \le t \le b$ (see Figure 13.13), then

$$S = \int_{t=a}^{t=b} 2\pi x\, ds = \int_a^b 2\pi f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In this case we may regard $2\pi x\, ds$ as the surface area of a frustum of a cone of slant height ds and average radius x.

EXAMPLE 5 Verify that the surface area of a sphere of radius a is $4\pi a^2$.

SOLUTION If C is the upper half of the circle $x^2 + y^2 = a^2$, then the spherical surface may be obtained by revolving C about the x-axis. Parametric equations for C are

$$x = a\cos t, \quad y = a\sin t; \quad 0 \le t \le \pi.$$

Applying Theorem (13.7) and using the identity $\sin^2 t + \cos^2 t = 1$, we have

$$\begin{aligned} S &= \int_0^{\pi} 2\pi a \sin t\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = 2\pi a^2\int_0^{\pi}\sin t\,dt \\ &= -2\pi a^2\Big[\cos t\Big]_0^{\pi} = -2\pi a^2[-1-1] = 4\pi a^2. \end{aligned}$$

EXERCISES 13.2

Exer. 1–8: Find the slopes of the tangent line and the normal line at the point on the curve that corresponds to $t = 1$.

1 $x = t^2 + 1, \quad y = t^2 - 1; \quad -2 \le t \le 2$

2 $x = t^3 + 1, \quad y = t^3 - 1; \quad -2 \le t \le 2$

3 $x = 4t^2 - 5, \quad y = 2t + 3; \quad t$ in $\mathbb{R}$

4 $x = t^3, \quad y = t^2; \quad t$ in $\mathbb{R}$

5 $x = e^t, \quad y = e^{-2t}; \quad t$ in $\mathbb{R}$

6 $x = \sqrt{t}, \quad y = 3t + 4; \quad t \ge 0$

7 $x = 2\sin t, \quad y = 3\cos t; \quad 0 \le t \le 2\pi$

8 $x = \cos t - 2, \quad y = \sin t + 3; \quad 0 \le t \le 2\pi$

Exer. 9–10: Let C be the curve with the given parametrization, for t in $\mathbb{R}$. Find the points on C at which the slope of the tangent line is m.

9 $x = -t^3, \quad y = -6t^2 - 18t; \quad m = 2$

10 $x = t^2 + t, \quad y = 5t^2 - 3; \quad m = 4$

Exer. 11–18: (a) Find the points on the curve C at which the tangent line is either horizontal or vertical. (b) Find d^2y/dx^2. (c) Sketch the graph of C.

11 $x = 4t^2, \quad y = t^3 - 12t; \quad t$ in $\mathbb{R}$

12 $x = t^3 - 4t, \quad y = t^2 - 4; \quad t$ in $\mathbb{R}$

13 $x = t^3 + 1, \quad y = t^2 - 2t; \quad t$ in $\mathbb{R}$

14 $x = 12t - t^3, \quad y = t^2 - 5t; \quad t$ in $\mathbb{R}$

15 $x = 3t^2 - 6t, \quad y = \sqrt{t}; \quad t \ge 0$

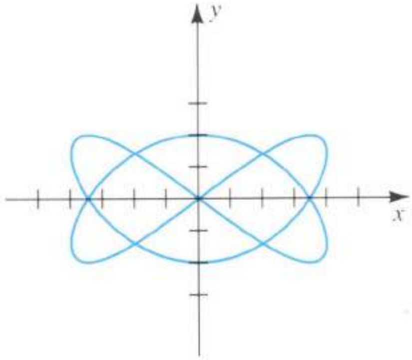
16 $x = \sqrt[3]{t}, \quad y = \sqrt[3]{t} - t; \quad t$ in $\mathbb{R}$

17 $x = \cos^3 t, \quad y = \sin^3 t; \quad 0 \le t \le 2\pi$

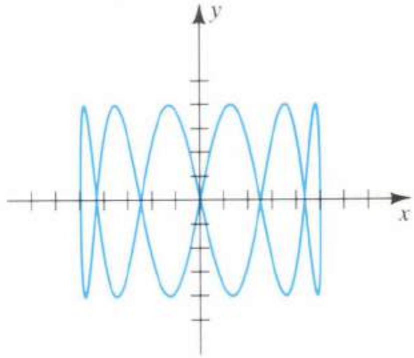
18 $x = \cosh t, \quad y = \sinh t; \quad t$ in $\mathbb{R}$

Exer. 19–20: Shown is a Lissajous figure (see Example 5, Section 13.1). Determine where the tangent line is horizontal or vertical.

19 $x = 4\sin 2t, \quad y = 2\cos 3t$



20 $x = 5\sin\frac{1}{3}t, \quad y = 4\sin 2t$



Exer. 21–26: Find the length of the curve.

21 $x = 5t^2, \quad y = 2t^3; \quad 0 \le t \le 1$

22 $x = 3t, \quad y = 2t^{3/2}; \quad 0 \le t \le 4$

23 $x = e^t\cos t, \quad y = e^t\sin t; \quad 0 \le t \le \pi/2$

24 $x = \cos 2t, \quad y = \sin^2 t; \quad 0 \le t \le \pi$

25 $x = t\cos t - \sin t, \quad y = t\sin t + \cos t; \quad 0 \le t \le \pi/2$

26 $x = \cos^3 t, \quad y = \sin^3 t; \quad 0 \le t \le \pi/2$

c **Exer. 27–28: Use Simpson's rule, with $n = 6$, to approximate the length of the curve.**

27 $x = 2\cos t,\quad y = 3\sin t;\quad 0 \le t \le 2\pi$

28 $x = 4t^3 - t,\quad y = 2t^2;\quad 0 \le t \le 1$

Exer. 29–34: Find the area of the surface generated by revolving the curve about the x-axis.

29 $x = t^2,\quad y = 2t;\quad 0 \le t \le 4$

30 $x = 4t,\quad y = t^3;\quad 1 \le t \le 2$

31 $x = t^2,\quad y = t - \frac{1}{3}t^3;\quad 0 \le t \le 1$

32 $x = 4t^2 + 1,\quad y = 3 - 2t;\quad -2 \le t \le 0$

33 $x = t - \sin t,\quad y = 1 - \cos t;\quad 0 \le t \le 2\pi$

34 $x = t,\quad y = \frac{1}{3}t^3 + \frac{1}{4}t^{-1};\quad 1 \le t \le 2$

Exer. 35–38: Find the area of the surface generated by revolving the curve about the y-axis.

35 $x = 4t^{1/2},\quad y = \frac{1}{2}t^2 + t^{-1};\quad 1 \le t \le 4$

36 $x = 3t,\quad y = t + 1;\quad 0 \le t \le 5$

37 $x = e^t \sin t,\quad y = e^t \cos t;\quad 0 \le t \le \pi/2$

38 $x = 3t^2,\quad y = 2t^3;\quad 0 \le t \le 1$

c **Exer. 39–40: Use Simpson's rule, with $n = 4$, to approximate the area of the surface generated by revolving the curve about the given axis.**

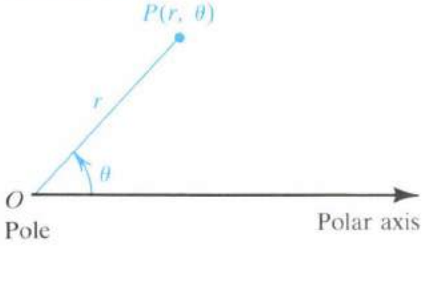
39 $x = \cos(t^2),\quad y = \sin^2 t;\quad 0 \le t \le 1;$ the x-axis

40 $x = t^2 + 2t,\quad y = t^4;\quad 0 \le t \le 1;$ the y-axis

13.3 POLAR COORDINATES

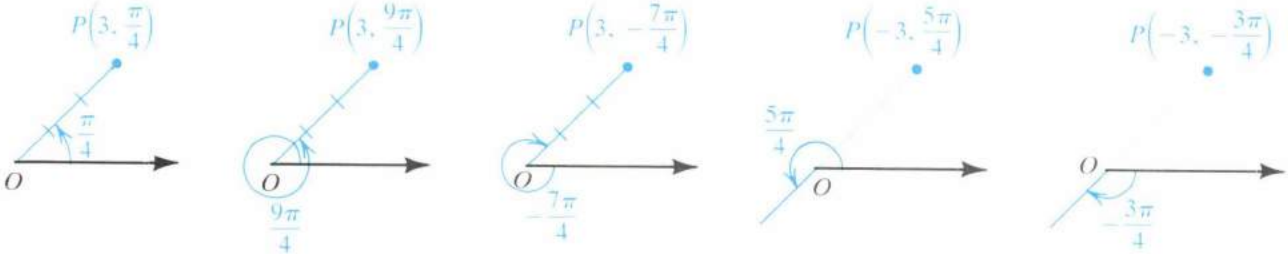
In a rectangular coordinate system, the ordered pair (a, b) denotes the point whose directed distances from the x- and y-axes are b and a, respectively. Another method for representing points is to use *polar coordinates*. We begin with a fixed point O (the **origin**, or **pole**) and a directed half-line (the **polar axis**) with endpoint O. Next we consider any point P in the plane different from O. If, as illustrated in Figure 13.14, $r = d(O, P)$ and θ denotes the measure of any angle determined by the polar axis and OP, then r and θ are **polar coordinates** of P, and the symbols (r, θ) or $P(r, \theta)$ are used to denote P. As usual, θ is considered positive if the angle is generated by a counterclockwise rotation of the polar axis and negative if the rotation is clockwise. Either radian or degree measure may be used for θ.

FIGURE 13.14



The polar coordinates of a point are not unique. For example $(3, \pi/4)$, $(3, 9\pi/4)$, and $(3, -7\pi/4)$ all represent the same point (see Figure 13.15). We shall also allow r to be negative. In this case, instead of measuring $|r|$ units along the terminal side of the angle θ, we measure along the half-line with endpoint O that has direction *opposite* that of the terminal side. The points corresponding to the pairs $(-3, 5\pi/4)$ and $(-3, -3\pi/4)$ are also plotted in Figure 13.15.

FIGURE 13.15



We agree that the pole O has polar coordinates $(0, \theta)$ for *any* θ. An assignment of ordered pairs of the form (r, θ) to points in a plane is a **polar coordinate system**, and the plane is an **$r\theta$-plane**.

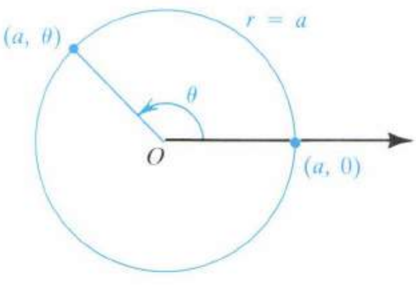


FIGURE 13.16

A **polar equation** is an equation in r and θ. A **solution** of a polar equation is an ordered pair (a, b) that leads to equality if a is substituted for r and b for θ. The **graph** of a polar equation is the set of all points (in an $r\theta$-plane) that correspond to the solutions.

The simplest polar equations are $r = a$ and $\theta = a$, where a is a nonzero real number. Since the solutions of the polar equation $r = a$ are of the form (a, θ) for *any* angle θ, it follows that the graph is a circle of radius $|a|$ with center at the pole. A graph for $a > 0$ is sketched in Figure 13.16. The same graph is obtained for $r = -a$.

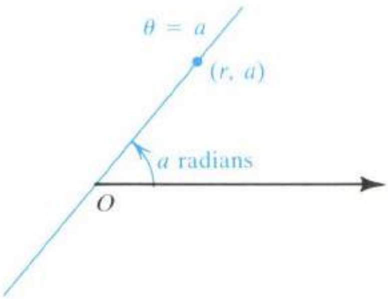


FIGURE 13.17

The solutions of the polar equation $\theta = a$ are of the form (r, a) for *any* real number r. Since the (angle) coordinate a is constant, the graph is a line through the origin, as illustrated in Figure 13.17 for the case $0 < a < \pi/2$.

In the following examples we obtain the graphs of polar equations by plotting points. As you proceed through this section, you should try to recognize forms of polar equations so that you will be able to sketch their graphs by plotting few, if any, points.

EXAMPLE 1 Sketch the graph of the polar equation $r = 4 \sin \theta$.

SOLUTION The following table displays some solutions of the equation. We have included a third row in the table that contains one-decimal-place approximations to r.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	2	$2\sqrt{2}$	$2\sqrt{3}$	4	$2\sqrt{3}$	$2\sqrt{2}$	2	0
r **(approx.)**	0	2	2.8	3.4	4	3.4	2.8	2	0

FIGURE 13.18

The points in an $r\theta$-plane that correspond to the pairs in the table appear to lie on a circle of radius 2, and we draw the graph accordingly (see Figure 13.18). As an aid to plotting points, we have extended the polar axis in the negative direction and introduced a vertical line through the pole.

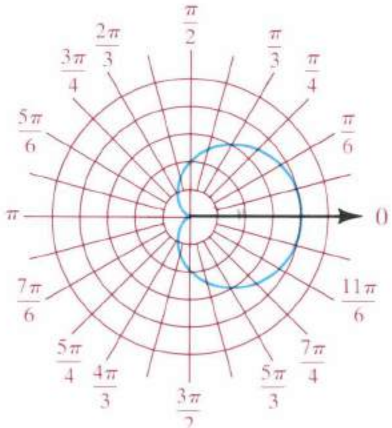
The proof that the graph of $r = 4 \sin \theta$ is a circle is given in Example 6. Additional points obtained by letting θ vary from π to 2π lie on the same circle. For example, the solution $(-2, 7\pi/6)$ gives us the same point as $(2, \pi/6)$; the point corresponding to $(-2\sqrt{2}, 5\pi/4)$ is the same as that obtained from $(2\sqrt{2}, \pi/4)$; and so on. If we let θ increase through all real numbers, we obtain the same points again and again because of the periodicity of the sine function.

EXAMPLE 2 Sketch the graph of the polar equation $r = 2 + 2 \cos \theta$.

SOLUTION Since the cosine function decreases from 1 to -1 as θ varies from 0 to π, it follows that r decreases from 4 to 0 in this θ-interval.

The following table exhibits some solutions of $r = 2 + 2\cos\theta$, together with one-decimal-place approximations to r.

FIGURE 13.19
$r = 2 + 2\cos\theta$



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	$2+\sqrt{3}$	$2+\sqrt{2}$	3	2	1	$2-\sqrt{2}$	$2-\sqrt{3}$	0
r (approx.)	4	3.7	3.4	3	2	1	0.6	0.3	0

Plotting points in an $r\theta$-plane leads to the upper half of the graph sketched in Figure 13.19. (We have used polar coordinate graph paper, which displays lines through O at various angles and concentric circles with centers at the pole.)

If θ increases from π to 2π, then $\cos\theta$ increases from -1 to 1 and, consequently, r increases from 0 to 4. Plotting points for $\pi \le \theta \le 2\pi$ gives us the lower half of the graph.

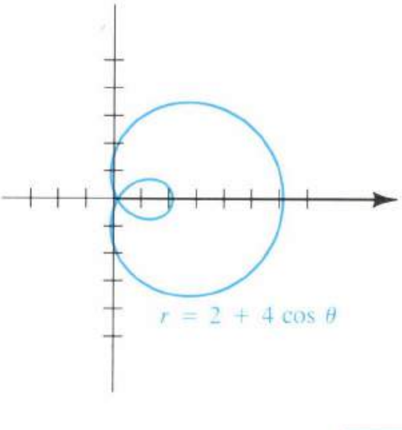
The same graph may be obtained by taking other intervals of length 2π for θ.

The heart-shaped graph in Example 2 is a **cardioid**. In general, the graph of any of the following polar equations, with $a \neq 0$, is a cardioid:

$$r = a(1 + \cos\theta) \qquad r = a(1 + \sin\theta)$$

$$r = a(1 - \cos\theta) \qquad r = a(1 - \sin\theta)$$

FIGURE 13.20



If a and b are not zero, then the graphs of the following polar equations are **limaçons**:

$$r = a + b\cos\theta \qquad r = a + b\sin\theta$$

Note that the special limaçons in which $|a| = |b|$ are cardioids. Some limaçons contain a loop, as shown in the next example.

EXAMPLE 3 Sketch the graph of the polar equation $r = 2 + 4\cos\theta$.

SOLUTION Coordinates of some points in an $r\theta$-plane that correspond to $0 \le \theta \le \pi$ are listed in the following table.

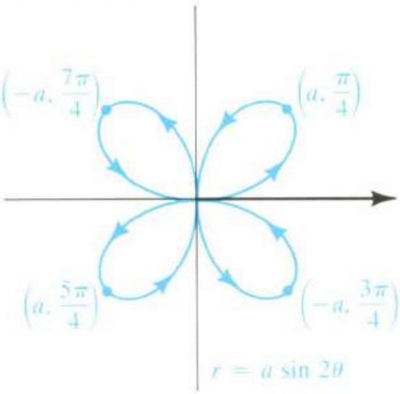
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	6	$2+2\sqrt{3}$	$2+2\sqrt{2}$	4	2	0	$2-2\sqrt{2}$	$2-2\sqrt{3}$	-2
r (approx.)	6	5.4	4.8	4	2	0	-0.8	-1.4	-2

Note that $r = 0$ at $\theta = 2\pi/3$. The values of r are negative if $2\pi/3 < \theta \le \pi$, and this leads to the lower half of the small loop in Figure 13.20. Letting θ range from π to 2π gives us the upper half of the small loop and the lower half of the large loop.

EXAMPLE 4 Sketch the graph of the polar equation $r = a \sin 2\theta$ for $a > 0$.

SOLUTION Instead of tabulating solutions, let us reason as follows. If θ increases from 0 or $\pi/4$, then 2θ varies from 0 to $\pi/2$ and hence $\sin 2\theta$ increases from 0 to 1. It follows that r increases from 0 to a in the θ-interval $[0, \pi/4]$. If we next let θ increase from $\pi/4$ to $\pi/2$, then 2θ changes from $\pi/2$ to π and hence $\sin 2\theta$ decreases from 1 to 0. Thus, r decreases from a to 0 in the θ-interval $[\pi/4, \pi/2]$. The corresponding points on the graph constitute the first-quadrant loop illustrated in Figure 13.21. Note that the point $P(r, \theta)$ traces the loop in a *counterclockwise* direction (indicated by the arrows) as θ increases from 0 to $\pi/2$.

FIGURE 13.21



If $\pi/2 \le \theta \le \pi$, then $\pi \le 2\theta \le 2\pi$ and, therefore, $r = a \sin 2\theta \le 0$. Thus, *if $\pi/2 < \theta < \pi$, then r is negative and the points $P(r, \theta)$ are in the fourth quadrant*. If θ increases from $\pi/2$ to π, then we can show, by plotting points, that $P(r, \theta)$ traces (in a counterclockwise direction) the loop shown in the fourth quadrant.

Similarly, for $\pi \le \theta \le 3\pi/2$ we get the loop in the third quadrant, and for $3\pi/2 \le \theta \le 2\pi$ we get the loop in the second quadrant. Both loops are traced in a counterclockwise direction as θ increases. You should verify these facts by plotting some points with, say, $a = 1$. In Figure 13.21 we have plotted only those points on the graph that correspond to the largest numerical values of r.

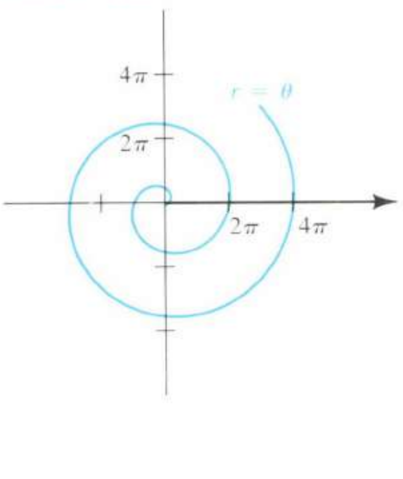
The graph in Example 4 is a **four-leafed rose**. In general, a polar equation of the form

$$r = a \sin n\theta \quad \text{or} \quad r = a \cos n\theta$$

for any positive integer n greater than 1 and any nonzero real number a has a graph that consists of a number of loops through the origin. If n is even, there are $2n$ loops, and if n is odd, there are n loops (see Exercises 15–18).

The graph of the polar equation $r = a\theta$ for any nonzero real number a is a **spiral of Archimedes**. The case $a = 1$ is considered in the next example.

FIGURE 13.22



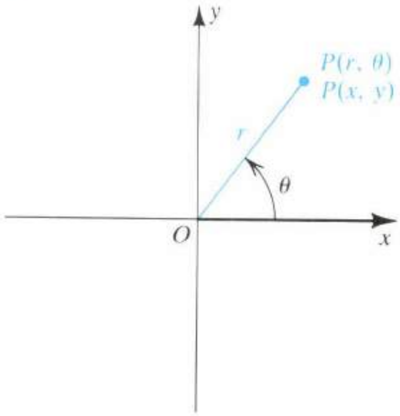
EXAMPLE 5 Sketch the graph of the polar equation $r = \theta$ for $\theta \ge 0$.

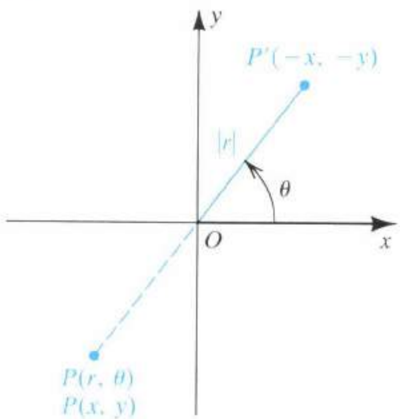
SOLUTION The graph consists of all points that have polar coordinates of the form (c, c) for every real number $c \ge 0$. Thus, the graph contains the points $(0, 0)$, $(\pi/2, \pi/2)$, (π, π), and so on. As θ increases, r increases at the same rate, and the spiral winds around the origin in a counterclockwise direction, intersecting the polar axis at $0, 2\pi, 4\pi, \ldots$, as illustrated in Figure 13.22.

If θ is allowed to be negative, then as θ decreases through negative values, the resulting spiral winds around the origin and is the symmetric image, with respect to the vertical axis, of the curve sketched in Figure 13.22.

Let us next superimpose an xy-plane on an $r\theta$-plane so that the positive x-axis coincides with the polar axis. Any point P in the plane may then be assigned rectangular coordinates (x, y) or polar coordinates (r, θ). If $r > 0$, we have a situation similar to that illustrated in Figure 13.23(i). If $r < 0$, we have that shown in (ii) of the figure, where, for later purposes, we have also plotted the point P' having polar coordinates $(|r|, \theta)$ and rectangular coordinates $(-x, -y)$.

FIGURE 13.23





The following result specifies relationships between (x, y) and (r, θ), where it is assumed that the positive x-axis coincides with the polar axis.

Relationships between rectangular and polar coordinates **(13.8)**

> The rectangular coordinates (x, y) and polar coordinates (r, θ) of a point P are related as follows:
>
> **(i)** $x = r\cos\theta, \quad y = r\sin\theta$
>
> **(ii)** $r^2 = x^2 + y^2, \quad \tan\theta = \dfrac{y}{x} \quad \text{if} \quad x \neq 0$

PROOF Although we have pictured θ as an acute angle in Figure 13.23, the discussion that follows is valid for all angles. If $r > 0$ as in Figure 13.23(i), then $\cos\theta = x/r$, $\sin\theta = y/r$, and hence

$$x = r\cos\theta, \quad y = r\sin\theta.$$

If $r < 0$, then $|r| = -r$, and from Figure 13.23(ii) we see that

$$\cos\theta = \frac{-x}{|r|} = \frac{-x}{-r} = \frac{x}{r}, \quad \sin\theta = \frac{-y}{|r|} = \frac{-y}{-r} = \frac{y}{r}.$$

Multiplication by r gives us relationship (i), and therefore these formulas hold if r is either positive or negative. If $r = 0$, then the point is the pole and we again see that the formulas in (i) are true.

The formulas in (ii) follow readily from Figure 13.23.

We may use the preceding result to change from one system of coordinates to the other. A more important use is for transforming a polar equation to an equation in x and y, and vice versa. This is illustrated in the next three examples.

EXAMPLE 6 Find an equation in x and y that has the same graph as the polar equation $r = a \sin \theta$, with $a \neq 0$. Sketch the graph.

SOLUTION From (13.8)(i), a relationship between $\sin \theta$ and y is given by $y = r \sin \theta$. To introduce this expression into the equation $r = a \sin \theta$, we multiply both sides by r, obtaining

$$r^2 = ar \sin \theta.$$

Next, using $r^2 = x^2 + y^2$ and $y = r \sin \theta$, we have

$$x^2 + y^2 = ay,$$

or

$$x^2 + y^2 - ay = 0.$$

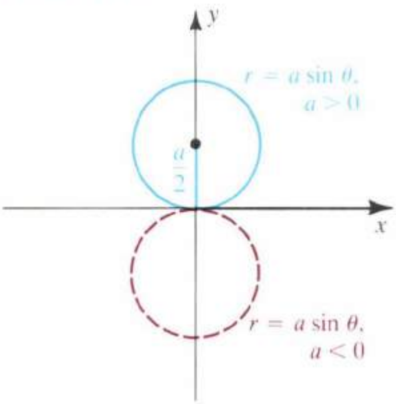
Completing the square in y gives us

$$x^2 + y^2 - ay + \left(\frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2,$$

or

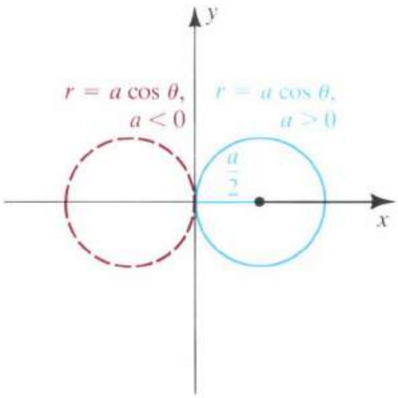
$$x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2.$$

FIGURE 13.24



In the xy-plane, the graph of the last equation is a circle with center $(0, a/2)$ and radius $|a|/2$, as illustrated in Figure 13.24 for the case $a > 0$ (the solid circle) and $a < 0$ (the dashed circle).

FIGURE 13.25



Using the same method as in the preceding example, we can show that the graph of $r = a \cos \theta$, with $a \neq 0$, is a circle of radius $a/2$ of the type illustrated in Figure 13.25.

EXAMPLE 7 Find a polar equation for the hyperbola $x^2 - y^2 = 16$.

SOLUTION Using the formulas $x = r \cos \theta$ and $y = r \sin \theta$, we obtain the following polar equations:

$$(r \cos \theta)^2 - (r \sin \theta)^2 = 16$$

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 16$$

$$r^2 (\cos^2 \theta - \sin^2 \theta) = 16$$

$$r^2 \cos 2\theta = 16$$

$$r^2 = \frac{16}{\cos 2\theta}$$

$$r^2 = 16 \sec 2\theta$$

The division by $\cos 2\theta$ is allowable because $\cos 2\theta \neq 0$. (Note that if $\cos 2\theta = 0$, then $r^2 \cos 2\theta \neq 16$.)

EXAMPLE 8 Find a polar equation of an arbitrary line.

SOLUTION Every line in an xy-coordinate plane is the graph of a linear equation $ax + by = c$. Using the formulas $x = r\cos\theta$ and $y = r\sin\theta$ gives us the following equivalent polar equations:

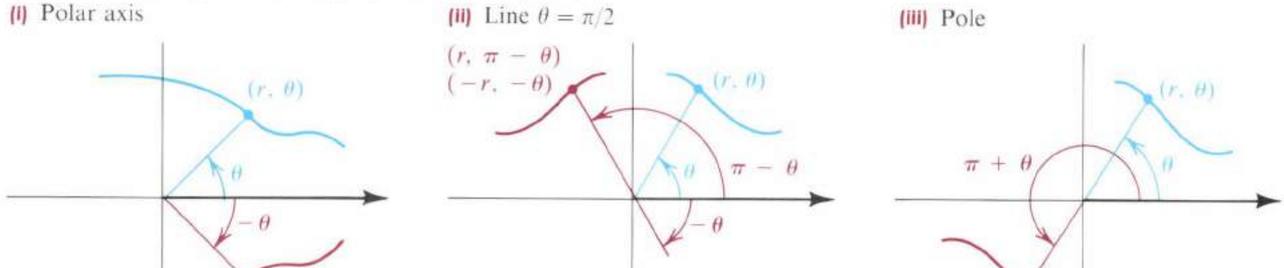
$$ar\cos\theta + br\sin\theta = c$$

$$r(a\cos\theta + b\sin\theta) = c$$

$$r = \frac{c}{a\cos\theta + b\sin\theta}$$

If we superimpose an xy-plane on an $r\theta$-plane, then the graph of a polar equation may be symmetric with respect to the x-axis (the polar axis), the y-axis (the line $\theta = \pi/2$), or the origin (the pole). Some typical symmetries are illustrated in Figure 13.26. This leads to the next result.

FIGURE 13.26 Symmetries of graphs of polar equations



Tests for symmetry (13.9)

(i) The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if substitution of $-\theta$ for θ leads to an equivalent equation.

(ii) The graph of $r = f(\theta)$ is symmetric with respect to the vertical line $\theta = \pi/2$ if substitution of either (a) $\pi - \theta$ for θ or (b) $-r$ for r and $-\theta$ for θ leads to an equivalent equation.

(iii) The graph of $r = f(\theta)$ is symmetric with respect to the pole if substitution of either (a) $-r$ for r or (b) $\pi + \theta$ for θ leads to an equivalent equation.

To illustrate, since $\cos(-\theta) = \cos\theta$, the graph of the polar equation $r = 2 + 4\cos\theta$ in Example 3 is symmetric with respect to the polar axis, by test (i). Since $\sin(\pi - \theta) = \sin\theta$, the graph in Example 1 is symmetric with respect to the line $\theta = \pi/2$, by test (ii). The graph in Example 4 is symmetric to the polar axis, the line $\theta = \pi/2$ and the pole. Other tests for symmetry may be stated; however, those we have listed are among the easiest to apply.

Unlike the graph of an equation in x and y, the graph of a polar equation $r = f(\theta)$ can be symmetric with respect to the polar axis, the line

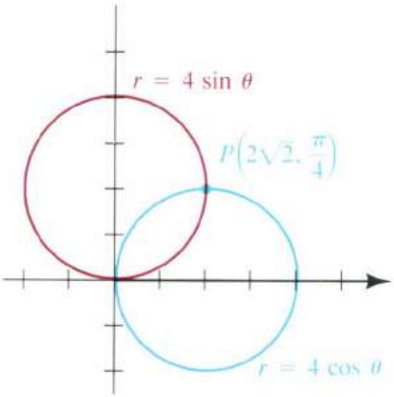


FIGURE 13.27

$\theta = \pi/2$, or the pole *without* satisfying one of the preceding tests for symmetry. This is true because of the many different ways of specifying a point in polar coordinates.

Another difference between rectangular and polar coordinate systems is that the points of intersection of two graphs cannot always be found by solving the polar equations simultaneously. To illustrate, from Example 1, the graph of $r = 4 \sin \theta$ is a circle of diameter 4 with center at $(2, \pi/2)$ (see Figure 13.27). Similarly, the graph of $r = 4 \cos \theta$ a is circle of diameter 4 with center at $(2, 0)$ on the polar axis. Referring to Figure 13.27, we see that the coordinates of the point of intersection $P(2\sqrt{2}, \pi/4)$ in quadrant I satisfy both equations; however, the origin O, which is on each circle, *cannot* be found by solving the equations simultaneously. Thus, in searching for points of intersection of polar graphs, it is sometimes necessary to refer to the graphs themselves, *in addition* to solving the two equations simultaneously. An alternative method is to use different (equivalent) equations for the graphs.

Tangent lines to graphs of polar equations may be found by means of the next theorem.

Theorem (13.10)

The slope m of the tangent line to the graph of $r = f(\theta)$ at the point $P(r, \theta)$ is

$$m = \frac{\dfrac{dr}{d\theta} \sin \theta + r \cos \theta}{\dfrac{dr}{d\theta} \cos \theta - r \sin \theta}.$$

PROOF If (x, y) are the rectangular coordinates of $P(r, \theta)$, then, by Theorem (13.8),

$$x = r \cos \theta = f(\theta) \cos \theta$$
$$y = r \sin \theta = f(\theta) \sin \theta.$$

These may be considered as parametric equations for the graph with parameter θ. Applying Theorem (13.3), we find that the slope of the tangent line at (x, y) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{f(\theta)(-\sin \theta) + f'(\theta) \cos \theta}$$
$$= \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.$$

This is equivalent to the formula in the statement of the theorem.

Horizontal tangent lines occur if the numerator in the formula for m is 0 and the denominator is not 0. Vertical tangent lines occur if the denominator is 0 and the numerator is not 0. The case 0/0 requires further investigation.

To find the slopes of the tangent lines at the pole, we must determine the values of θ for which $r = f(\theta) = 0$. For such values (and with $r = 0$

and $dr/d\theta \neq 0$), the formula in Theorem (13.10) reduces to $m = \tan\theta$. These remarks are illustrated in the next example.

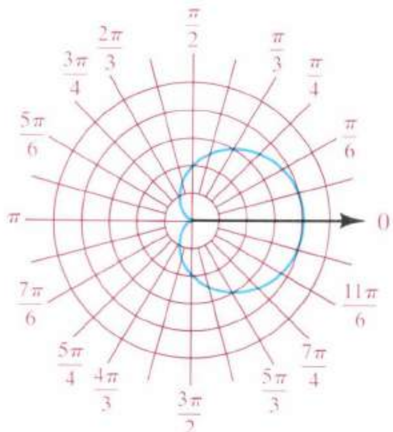
EXAMPLE 9 For the cardioid $r = 2 + 2\cos\theta$ with $0 \le \theta < 2\pi$, find

(a) the slope of the tangent line at $\theta = \pi/6$

(b) the points at which the tangent line is horizontal

(c) the points at which the tangent line is vertical

FIGURE 13.28
$r = 2 + 2\cos\theta$



SOLUTION **(a)** The graph of $r = 2 + 2\cos\theta$ was considered in Example 2 and is resketched in Figure 13.28. Applying Theorem (13.10), we find that the slope m of the tangent line is

$$\begin{aligned} m &= \frac{(-2\sin\theta)\sin\theta + (2 + 2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - (2 + 2\cos\theta)\sin\theta} \\ &= \frac{2(\cos^2\theta - \sin^2\theta) + 2\cos\theta}{-2(2\sin\theta\cos\theta) - 2\sin\theta} \\ &= -\frac{\cos 2\theta + \cos\theta}{\sin 2\theta + \sin\theta}. \end{aligned}$$

At $\theta = \pi/6$ (that is, at the point $(2 + \sqrt{3}, \pi/6)$),

$$m = -\frac{\cos(\pi/3) + \cos(\pi/6)}{\sin(\pi/3) + \sin(\pi/6)} = -\frac{(1/2) + (\sqrt{3}/2)}{(\sqrt{3}/2) + (1/2)} = -1.$$

(b) To find horizontal tangents, we let

$$\cos 2\theta + \cos\theta = 0.$$

This equation may be written as

$$2\cos^2\theta - 1 + \cos\theta = 0,$$

or

$$(2\cos\theta - 1)(\cos\theta + 1) = 0.$$

From $\cos\theta = \frac{1}{2}$ we obtain $\theta = \pi/3$ and $\theta = 5\pi/3$. The corresponding points are $(3, \pi/3)$ and $(3, 5\pi/3)$.

Using $\cos\theta = -1$ gives us $\theta = \pi$. The denominator in the formula for m is 0 at $\theta = \pi$, and hence further investigation is required. If $\theta = \pi$, then $r = 0$ and the formula for m in (13.10) reduces to $m = \tan\theta$. Thus, the slope at $(0, \pi)$ is $m = \tan\pi = 0$, and therefore the tangent line is horizontal at the pole.

(c) To find vertical tangent lines, we let

$$\sin 2\theta + \sin\theta = 0.$$

Equivalent equations are

$$2\sin\theta\cos\theta + \sin\theta = 0$$

and

$$\sin\theta\,(2\cos\theta + 1) = 0.$$

Letting $\sin\theta = 0$ and $\cos\theta = -\frac{1}{2}$ leads to the following values of θ: 0, π, $2\pi/3$, and $4\pi/3$. We found, in part (b), that π gives us a horizontal tangent. The remaining values result in the points $(4, 0)$, $(1, 2\pi/3)$, and $(1, 4\pi/3)$, at which the graph has vertical tangent lines.

EXERCISES 13.3

Exer. 1–26: Sketch the graph of the polar equation.

1 $r = 5$
2 $r = -2$
3 $\theta = -\pi/6$
4 $\theta = \pi/4$
5 $r = 3\cos\theta$
6 $r = -2\sin\theta$
7 $r = 4 - 4\sin\theta$
8 $r = -6(1 + \cos\theta)$
9 $r = 2 + 4\sin\theta$
10 $r = 1 + 2\cos\theta$
11 $r = 2 - \cos\theta$
12 $r = 5 + 3\sin\theta$
13 $r = 4\csc\theta$
14 $r = -3\sec\theta$
15 $r = 8\cos 3\theta$
16 $r = 2\sin 4\theta$
17 $r = 3\sin 2\theta$
18 $r = 8\cos 5\theta$
19 $r^2 = 4\cos 2\theta$ (lemniscate)
20 $r^2 = -16\sin 2\theta$
21 $r = e^{\theta}$, $\theta \ge 0$ (logarithmic spiral)
22 $r = 6\sin^2(\theta/2)$
23 $r = 2\theta$, $\theta \ge 0$
24 $r\theta = 1$, $\theta > 0$ (spiral)
25 $r = 2 + 2\sec\theta$ (conchoid)
26 $r = 1 - \csc\theta$

Exer. 27–36: Find a polar equation that has the same graph as the equation in x and y.

27 $x = -3$
28 $y = 2$
29 $x^2 + y^2 = 16$
30 $x^2 = 8y$
31 $2y = -x$
32 $y = 6x$
33 $y^2 - x^2 = 4$
34 $xy = 8$
35 $(x^2 + y^2)\tan^{-1}(y/x) = ay$, $a > 0$ (cochleoid, or *Oui-ja board curve*)
36 $x^3 + y^3 - 3axy = 0$ (Folium of Descartes)

Exer. 37–50: Find an equation in x and y that has the same graph as the polar equation and use it to help sketch the graph in an $r\theta$-plane.

37 $r\cos\theta = 5$
38 $r\sin\theta = -2$
39 $r = -3\csc\theta$
40 $r = 4\sec\theta$
41 $r^2\cos 2\theta = 1$
42 $r^2\sin 2\theta = 4$
43 $r(\sin\theta - 2\cos\theta) = 6$
44 $r(3\cos\theta - 4\sin\theta) = 12$
45 $r(\sin\theta + r\cos^2\theta) = 1$
46 $r(r\sin^2\theta - \cos\theta) = 3$
47 $r = 8\sin\theta - 2\cos\theta$
48 $r = 2\cos\theta - 4\sin\theta$
49 $r = \tan\theta$
50 $r = 6\cot\theta$

Exer. 51–60: Find the slope of the tangent line to the graph of the polar equation at the point corresponding to the given value of θ.

51 $r = 2\cos\theta$; $\theta = \pi/3$
52 $r = -2\sin\theta$; $\theta = \pi/6$
53 $r = 4(1 - \sin\theta)$; $\theta = 0$
54 $r = 1 + 2\cos\theta$; $\theta = \pi/2$
55 $r = 8\cos 3\theta$; $\theta = \pi/4$
56 $r = 2\sin 4\theta$; $\theta = \pi/4$
57 $r^2 = 4\cos 2\theta$; $\theta = \pi/6$
58 $r^2 = -2\sin 2\theta$; $\theta = 3\pi/4$
59 $r = 2^{\theta}$; $\theta = \pi$
60 $r\theta = 1$; $\theta = 2\pi$

61 If $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ are points in an $r\theta$-plane, use the law of cosines to prove that

$$[d(P_1, P_2)]^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1).$$

62 If a and b are nonzero real numbers, prove that the graph of $r = a\sin\theta + b\cos\theta$ is a circle, and find its center and radius.

63 If the graphs of the polar equations $r = f(\theta)$ and $r = g(\theta)$ intersect at $P(r, \theta)$, prove that the tangent lines at P are perpendicular if and only if

$$f'(\theta)g'(\theta) + f(\theta)g(\theta) = 0.$$

(The graphs are said to be *orthogonal* at P.)

64 Use Exercise 63 to prove that the graphs of each pair of equations are orthogonal at their point of intersection:

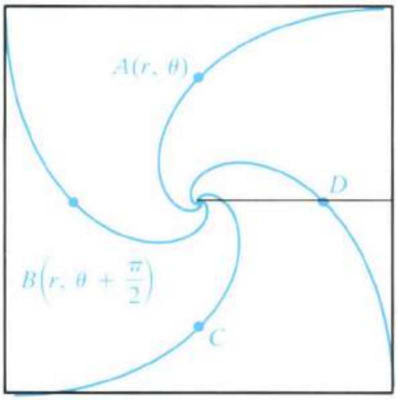
(a) $r = a\sin\theta$, $r = a\cos\theta$ (b) $r = a\theta$, $r\theta = a$

65 If $\cos\theta \ne 0$, show that the slope of the tangent line to the graph of $r = f(\theta)$ is

$$m = \frac{(dr/d\theta)\tan\theta + r}{(dr/d\theta) - r\tan\theta}.$$

66 A logarithmic spiral has a polar equation of the form $r = ae^{b\theta}$ for nonzero constants a and b (see Exercise 21). A famous *four bugs problem* illustrates such a curve. Four bugs A, B, C, and D are placed at the four corners of a square. The center of the square corresponds to the pole. The bugs begin to crawl simultaneously—bug A crawls toward B, B toward C, C toward D, and D toward A, as shown in the figure on the following page. Assume that all bugs crawl at the same rate, that they move directly toward the next bug at all times, and that they approach one another but never meet. (The bugs are infinitely small!) At any instant, the positions of the bugs are the vertices of a square, which shrinks and rotates toward the center of the original square as the bugs continue to crawl. If the position of bug A has polar coordinates (r, θ), then the position of B has coordinates $(r, \theta + \pi/2)$.

EXERCISE 66



(a) Show that the line through A and B has slope

$$\frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}.$$

(b) The line through A and B is tangent to the path of bug A. Use the formula in Exercise 65 to conclude that $dr/d\theta = -r$.

(c) Prove that the path of bug A is a logarithmic spiral. (*Hint:* Solve the differential equation in (b) by separating variables.)

c **Exer. 67–68: Graph the polar equation for the given values of θ, and use the graph to determine symmetries.**

67 $r = 2 \sin^2 \theta \tan^2 \theta; \quad -\pi/3 \le \theta \le \pi/3$

68 $r = \dfrac{4}{1 + \sin^2 \theta}; \quad 0 \le \theta \le 2\pi$

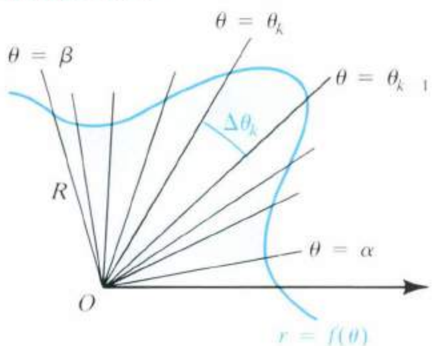
c **Exer. 69–70: Graph the polar equations on the same coordinate plane, and estimate the points of intersection of the graphs.**

69 $r = 8 \cos 3\theta, \quad r = 4 - 2.5 \cos \theta$

70 $r = 2 \sin^2 \theta, \quad r = \frac{3}{4}(\theta + \cos^2 \theta)$

13.4 INTEGRALS IN POLAR COORDINATES

FIGURE 13.29



The areas of certain regions bounded by graphs of polar equations can be found by using limits of sums of areas of circular sectors. We shall call a region R in the $r\theta$-plane an $\boldsymbol{R_\theta}$ **region** (for integration with respect to θ) if R is bounded by lines $\theta = \alpha$ and $\theta = \beta$ for $0 \le \alpha < \beta \le 2\pi$ and by the graph of a polar equation $r = f(\theta)$, where f is continuous and $f(\theta) \ge 0$ on $[\alpha, \beta]$. An R_θ region is illustrated in Figure 13.29.

Let P denote a partition of $[\alpha, \beta]$ determined by

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = \beta$$

and let $\Delta\theta_k = \theta_k - \theta_{k-1}$ for $k = 1, 2, \ldots, n$. The lines $\theta = \theta_k$ divide R into wedge-shaped subregions. If $f(u_k)$ is the minimum value and $f(v_k)$ is the maximum value of f on $[\theta_{k-1}, \theta_k]$, then, as illustrated in Figure 13.30, the area ΔA_k of the kth subregion is between the areas of the inscribed and circumscribed circular sectors having central angle $\Delta\theta_k$ and radii $f(u_k)$

FIGURE 13.30

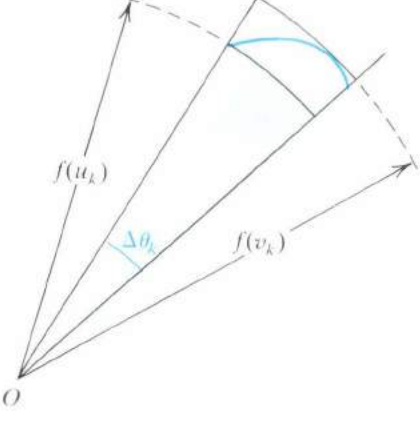
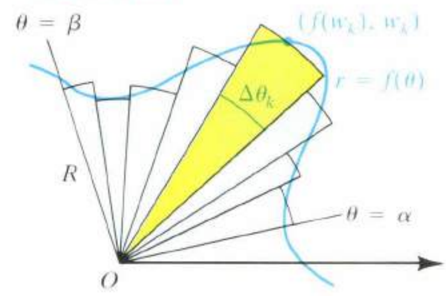


FIGURE 13.31



and $f(v_k)$, respectively. Hence, by Theorem (1.15),

$$\tfrac{1}{2}[f(u_k)]^2\,\Delta\theta_k \le \Delta A_k \le \tfrac{1}{2}[f(v_k)]^2\,\Delta\theta_k.$$

Summing from $k = 1$ to $k = n$ and using the fact that the sum of the ΔA_k is the area A of R, we obtain

$$\sum_{k=1}^{n} \tfrac{1}{2}[f(u_k)]^2\,\Delta\theta_k \le A \le \sum_{k=1}^{n} \tfrac{1}{2}[f(v_k)]^2\,\Delta\theta_k.$$

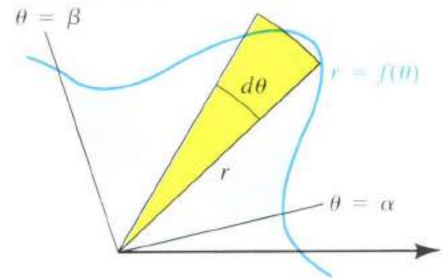
The limits of the sums, as the norm $\|P\|$ of the subdivision approaches zero, both equal the integral $\int_\alpha^\beta \frac{1}{2}[f(\theta)]^2\,d\theta$. This gives us the following result.

Theorem (13.11)

If f is continuous and $f(\theta) \ge 0$ on $[\alpha, \beta]$, where $0 \le \alpha < \beta \le 2\pi$, then the area A of the region bounded by the graphs of $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$ is

$$A = \int_\alpha^\beta \tfrac{1}{2}[f(\theta)]^2\,d\theta = \int_\alpha^\beta \tfrac{1}{2}r^2\,d\theta.$$

FIGURE 13.32



The integral in Theorem (13.11) may be interpreted as a limit of sums by writing

$$A = \int_\alpha^\beta \tfrac{1}{2}[f(\theta)]^2\,d\theta = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \tfrac{1}{2}[f(w_k)]^2\,\Delta\theta_k$$

for *any* number w_k in the subinterval $[\theta_{k-1}, \theta_k]$ of $[\alpha, \beta]$. Figure 13.31 is a geometric illustration of a typical Riemann sum.

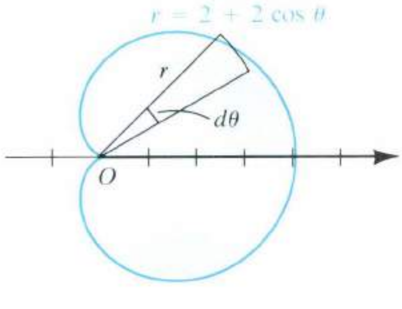
The following guidelines may be useful for remembering this limit of sums formula (see Figure 13.32).

Guidelines for finding the area of an R_θ region (13.12)

1. Sketch the region, labeling the graph of $r = f(\theta)$. Find the smallest value $\theta = \alpha$ and the largest value $\theta = \beta$ for points (r, θ) in the region.
2. Sketch a typical circular sector and label its central angle $d\theta$.
3. Express the area of the sector in guideline 2 as $\frac{1}{2}r^2\,d\theta$.
4. Apply the limit of sums operator $\int_\alpha^\beta$ to the expression in guideline 3 and evaluate the integral.

EXAMPLE 1 Find the area of the region bounded by the cardioid $r = 2 + 2\cos\theta$.

FIGURE 13.33



SOLUTION Following guideline 1, we first sketch the region as in Figure 13.33. The cardioid is obtained by letting θ vary from 0 to 2π; however, using symmetry we may find the area of the top half and multiply by 2. Thus, we use $\alpha = 0$ and $\beta = \pi$ for the smallest and largest values of θ. As in guideline 2, we sketch a typical circular sector and label its central angle $d\theta$. To apply guideline 3, we refer to the figure, obtaining the following:

radius of circular sector: $r = 2 + 2\cos\theta$

area of sector: $\frac{1}{2}r^2\,d\theta = \frac{1}{2}(2 + 2\cos\theta)^2\,d\theta$

$4 + 8\cos\theta + 4\cos^2\theta$

$4 + 8\cos\theta + 4(\frac{1}{2} + \frac{1}{2}\cos 2\theta)$

We next use guideline 4, with $\alpha = 0$ and $\beta = \pi$, remembering that applying $\int_0^\pi$ to the expression $\frac{1}{2}(2 + 2\cos\theta)^2\,d\theta$ represents taking a limit of sums of areas of circular sectors, *sweeping out* the region by letting θ vary from 0 to π. Thus,

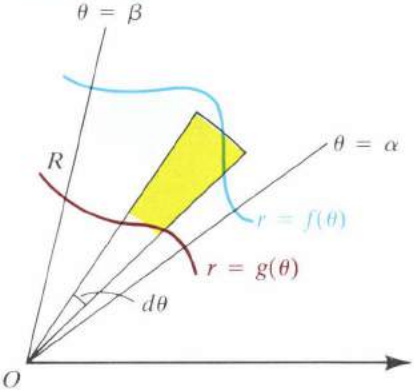
$$\begin{aligned} A &= 2\int_0^\pi \tfrac{1}{2}(2 + 2\cos\theta)^2\,d\theta \\ &= \int_0^\pi (4 + 8\cos\theta + 4\cos^2\theta)\,d\theta. \end{aligned}$$

Using the fact that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ yields

$$\begin{aligned} A &= \int_0^\pi (6 + 8\cos\theta + 2\cos 2\theta)\,d\theta \\ &= \Big[6\theta + 8\sin\theta + \sin 2\theta\Big]_0^\pi = 6\pi. \end{aligned}$$

We could also have found the area by using $\alpha = 0$ and $\beta = 2\pi$.

FIGURE 13.34



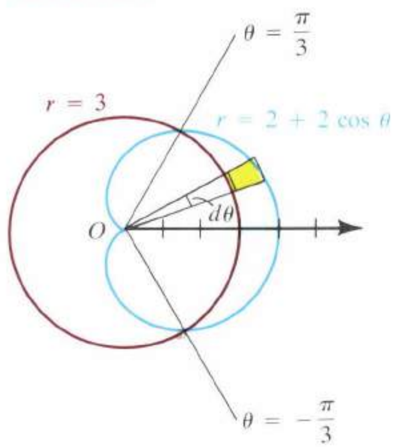
A region R between the graphs of two polar equations $r = f(\theta)$ and $r = g(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is sketched in Figure 13.34. We may find the area A of R by subtracting the area of the *inner* region bounded by $r = g(\theta)$ from the area of the *outer* region bounded by $r = f(\theta)$ as follows:

$$A = \int_\alpha^\beta \tfrac{1}{2}[f(\theta)]^2\,d\theta - \int_\alpha^\beta \tfrac{1}{2}[g(\theta)]^2\,d\theta$$

We use this technique in the next example.

EXAMPLE 2 Find the area A of the region R that is inside the cardioid $r = 2 + 2\cos\theta$ and outside the circle $r = 3$.

FIGURE 13.35



SOLUTION Figure 13.35 shows the region R and circular sectors that extend from the pole to the graphs of the two polar equations. The points of intersection $(3, -\pi/3)$ and $(3, \pi/3)$ can be found by solving the equations simultaneously. Since the angles α and β in Guidelines 13.12 are nonnegative, we shall find the area of the top half of R (using $\alpha = 0$ and $\beta = \pi/3$) and then double the result. Subtracting the area of the inner region (bounded by $r = 3$) from the area of the outer region (bounded by $r = 2 + 2\cos\theta$), we obtain

$$\begin{aligned} A &= 2\left[\int_0^{\pi/3} \tfrac{1}{2}(2 + 2\cos\theta)^2\,d\theta - \int_0^{\pi/3} \tfrac{1}{2}(3)^2\,d\theta\right] \\ &= \int_0^{\pi/3} (4\cos^2\theta + 8\cos\theta - 5)\,d\theta \end{aligned}$$

As in Example 1, the integral may be evaluated by using the substitution $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$. It can be shown that

$$A = \tfrac{9}{2}\sqrt{3} - \pi \approx 4.65.$$

If a curve C is the graph of a polar equation $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we can find its length L by using parametric equations. Thus, as in the proof of Theorem (13.10), a parametrization for C is

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta; \quad \alpha \le \theta \le \beta.$$

Differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta$$

$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta.$$

Using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, we can show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

Substitution in Theorem (13.5) with $t = \theta$, $a = \alpha$, and $b = \beta$ gives us

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

As an aid to remembering this formula, we may use the differential of arc length $ds = \sqrt{(dx)^2 + (dy)^2}$ in (13.6). The preceding manipulations give us the following.

Differential of arc length in polar coordinates **(13.13)**

$$\boxed{ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

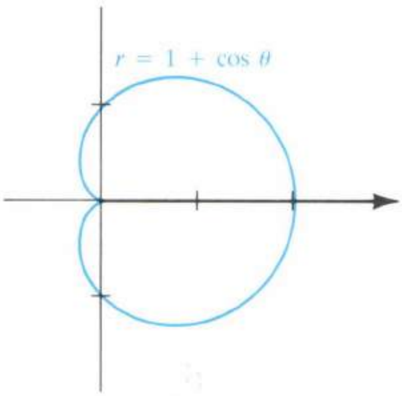
We may now write the formula for L as

$$L = \int_{\theta=\alpha}^{\theta=\beta} ds.$$

The limits of integration specify that the independent variable is θ, not s.

EXAMPLE 3 Find the length of the cardioid $r = 1 + \cos\theta$.

FIGURE 13.36



SOLUTION The cardioid is sketched in Figure 13.36. Making use of symmetry, we shall find the length of the upper half and double the result. Applying (13.13), we have

$$\begin{aligned} ds &= \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta \\ &= \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \\ &= \sqrt{2 + 2\cos\theta}\, d\theta \\ &= \sqrt{2}\sqrt{1+\cos\theta}\, d\theta. \end{aligned}$$

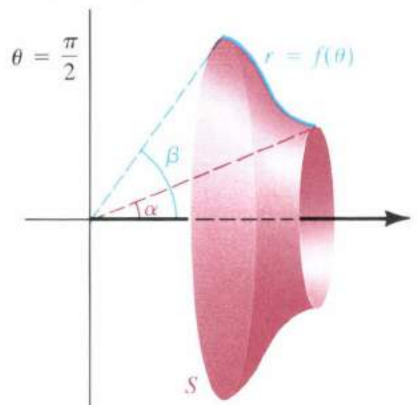
Hence

$$L = 2\int_{\theta=0}^{\theta=\pi} ds = 2\int_0^\pi \sqrt{2}\sqrt{1+\cos\theta}\, d\theta.$$

The last integral may be evaluated by employing the trigonometric identity $\cos^2 \frac{1}{2}\theta = \frac{1}{2}(1 + \cos\theta)$, or, equivalently, $1 + \cos\theta = 2\cos^2\frac{1}{2}\theta$. Thus,

$$\begin{aligned} L &= 2\sqrt{2}\int_0^\pi \sqrt{2\cos^2 \tfrac{1}{2}\theta}\, d\theta \\ &= 4\int_0^\pi \cos\tfrac{1}{2}\theta\, d\theta \\ &= 8\Big[\sin\tfrac{1}{2}\theta\Big]_0^\pi = 8. \end{aligned}$$

FIGURE 13.37



In the solution to Example 3, it was legitimate to replace $\sqrt{\cos^2 \frac{1}{2}\theta}$ by $\cos \frac{1}{2}\theta$, because if $0 \le \theta \le \pi$, then $0 \le \frac{1}{2}\theta \le \pi/2$, and hence $\cos \frac{1}{2}\theta$ is *positive* on $[0, \pi]$. If we had *not* used symmetry, but had written L as $\int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$, this simplification would not have been valid. Generally, in determining areas or arc lengths that involve polar coordinates, it is a good idea to use any symmetries that exist.

Let C be the graph of a polar equation $r = f(\theta)$ for $\alpha \le \theta \le \beta$. Let us obtain a formula for the area S of the surface generated by revolving C about the polar axis, as illustrated in Figure 13.37. Since parametric equations for C are

$$x = f(\theta) \cos \theta, \quad y = f(\theta) \sin \theta; \quad \alpha \le \theta \le \beta,$$

we may find S by using Theorem (13.7) with $\theta = t$. This gives us the following result, where the arc length differential ds is given by (13.13).

Surfaces of revolution in polar coordinates **(13.14)**

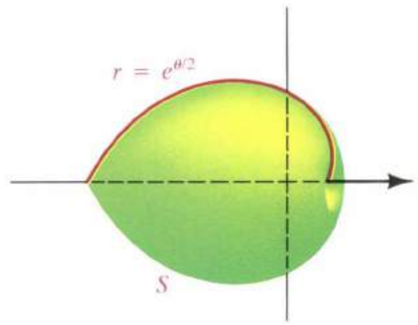
$$\textit{About the polar axis:} \quad S = \int_{\theta=\alpha}^{\theta=\beta} 2\pi y\, ds = \int_{\theta=\alpha}^{\theta=\beta} 2\pi r \sin \theta\, ds$$

$$\textit{About the line } \theta = \pi/2\textit{:} \quad S = \int_{\theta=\alpha}^{\theta=\beta} 2\pi x\, ds = \int_{\theta=\alpha}^{\theta=\beta} 2\pi r \cos \theta\, ds$$

When applying (13.14), *we must choose α and β so that the surface does not retrace itself when C is revolved*, as would be the case if the circle $r = \cos \theta$, with $0 \le \theta \le \pi$, were revolved about the polar axis.

EXAMPLE 4 The part of the spiral $r = e^{\theta/2}$ from $\theta = 0$ to $\theta = \pi$ is revolved about the polar axis. Find the area of the resulting surface.

FIGURE 13.38



SOLUTION The surface is illustrated in Figure 13.38. By (13.13), the polar differential of arc length in polar coordinates is

$$\begin{aligned} ds &= \sqrt{(e^{\theta/2})^2 + (\tfrac{1}{2}e^{\theta/2})^2}\, d\theta \\ &= \sqrt{\frac{5}{4} e^{\theta}}\, d\theta = \frac{\sqrt{5}}{2} e^{\theta/2}\, d\theta. \end{aligned}$$

Hence, by (13.14),

$$\begin{aligned} S &= \int_{\theta=0}^{\theta=\pi} 2\pi y\, ds = \int_{\theta=0}^{\theta=\pi} 2\pi r \sin \theta\, ds \\ &= \int_0^{\pi} 2\pi e^{\theta/2} \sin \theta \left(\frac{\sqrt{5}}{2} e^{\theta/2} \right) d\theta \\ &= \sqrt{5}\pi \int_0^{\pi} e^{\theta} \sin \theta\, d\theta. \end{aligned}$$

Using integration by parts or Formula 98 in the table of integrals (see Appendix IV), we have

$$S = \frac{\sqrt{5}\pi}{2} \Big[e^{\theta} (\sin \theta - \cos \theta) \Big]_0^{\pi} = \frac{\sqrt{5}\pi}{2} (e^{\pi} + 1) \approx 84.8.$$

EXERCISES 13.4

Exer. 1–6: Find the area of the region bounded by the graph of the polar equation.

1 $r = 2\cos\theta$ 2 $r = 5\sin\theta$

3 $r = 1 - \cos\theta$ 4 $r = 6 - 6\sin\theta$

5 $r = \sin 2\theta$ 6 $r^2 = 9\cos 2\theta$ $r = \sqrt{9 \cos 2\theta}$

Exer. 7–8: Find the area of region R.

7 $R = \{(r, \theta): 0 \le \theta \le \pi/2, 0 \le r \le e^{\theta}\}$

8 $R = \{(r, \theta): 0 \le \theta \le \pi, 0 \le r \le 2\theta\}$

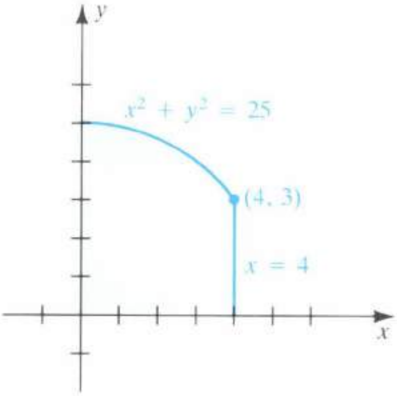
Exer. 9–12: Find the area of the region bounded by one loop of the graph of the polar equation.

9 $r^2 = 4\cos 2\theta$ 10 $r = 2\cos 3\theta$

11 $r = 3\cos 5\theta$ 12 $r = \sin 6\theta$

Exer. 13–16: Set up integrals in polar coordinates that can be used to find the area of the region shown in the figure.

13



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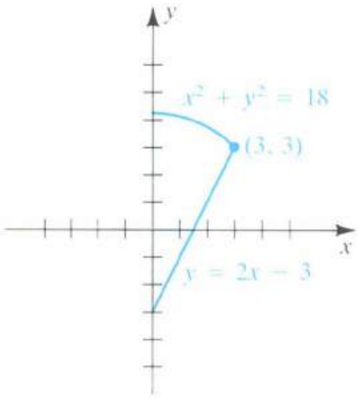

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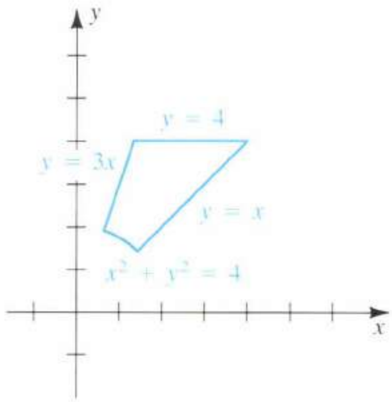

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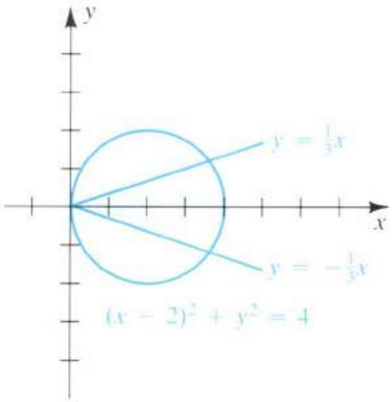
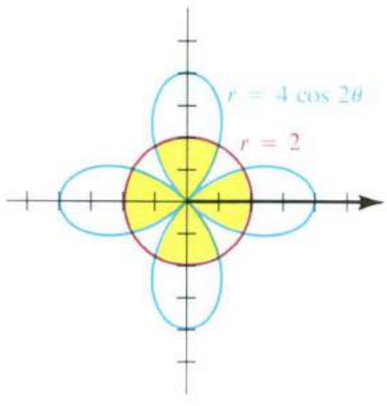


Exer. 17–18: Set up integrals in polar coordinates that can be used to find the area of (a) the blue region and (b) the green region.

17



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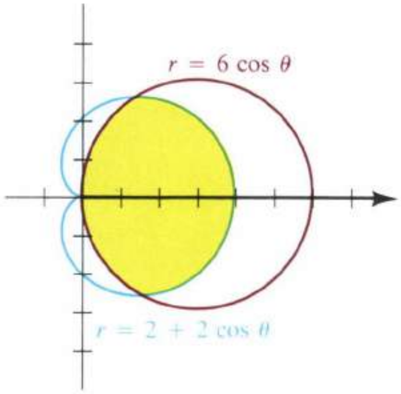


Exer. 19–22: Find the area of the region that is outside the graph of the first equation and inside the graph of the second equation.

19 $r = 2 + 2\cos\theta,\quad r = 3$

20 $r = 2,\quad r = 4\cos\theta$

21 $r = 2,\quad r^2 = 8\sin 2\theta$

22 $r = 1 - \sin\theta,\quad r = 3\sin\theta$

Exer. 23–26: Find the area of the region that is inside the graphs of *both* equations.

23 $r = \sin\theta,\quad r = \sqrt{3}\cos\theta$

24 $r = 2(1 + \sin\theta),\quad r = 1$

25 $r = 1 + \sin\theta,\quad r = 5\sin\theta$

26 $r^2 = 4\cos 2\theta,\quad r = 1$

$60° = \pi/3$

$\tan\theta$

$0, \pi/3$

Exer. 27–32: Find the length of the curve.

27 $r = e^{-\theta}$ from $\theta = 0$ to $\theta = 2\pi$

28 $r = \theta$ from $\theta = 0$ to $\theta = 4\pi$

29 $r = \cos^2 \frac{1}{2}\theta$ from $\theta = 0$ to $\theta = \pi$

30 $r = 2^\theta$ from $\theta = 0$ to $\theta = \pi$

31 $r = \sin^3 \frac{1}{3}\theta$

32 $r = 2 - 2\cos\theta$

c **Exer. 33–34: Use Simpson's rule, with $n = 4$, to approximate the length of the curve.**

33 $r = \theta + \cos\theta$ from $\theta = 0$ to $\theta = \pi/2$

34 $r = \sin\theta + \cos^2\theta$ from $\theta = 0$ to $\theta = \pi$

Exer. 35–38: Find the area of the surface generated by revolving the graph of the equation about the polar axis.

35 $r = 2 + 2\cos\theta$

36 $r^2 = 4\cos 2\theta$

37 $r = 2a\sin\theta$

38 $r = 2a\cos\theta$

c **Exer. 39–40: Use the trapezoidal rule, with $n = 4$, to approximate the area of the surface generated by revolving the graph of the polar equation about the line $\theta = \pi/2$. (Use symmetry when setting up the integral.)**

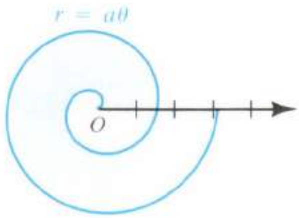
39 $r = \sin^2\theta$

40 $r = \cos^2\theta$

41 A *torus* is the surface generated by revolving a circle about a nonintersecting line in its plane. Use polar coordinates to find the surface area of the torus generated by revolving the circle $x^2 + y^2 = a^2$ about the line $x = b$, where $0 < a < b$.

42 Let OP be the ray from the pole to the point $P(r, \theta)$ on the spiral $r = a\theta$, where $a > 0$. If the ray makes two revolutions (starting from $\theta = 0$), find the area of the region swept out in the second revolution that was not swept out in the first revolution (see figure).

EXERCISE 42



43 The part of the spiral $r = e^{-\theta}$ from $\theta = 0$ to $\theta = \pi/2$ is revolved about the line $\theta = \pi/2$. Find the area of the resulting surface.

13.5 POLAR EQUATIONS OF CONICS

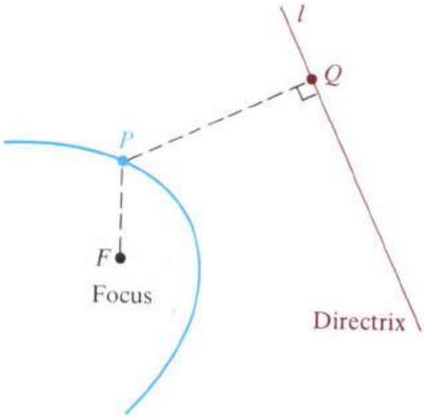
The following theorem combines the definitions of parabola, ellipse, and hyperbola into a unified description of the conic sections. The constant e in the statement of the theorem is the **eccentricity** of the conic. The point

F is a **focus** of the conic, and the line l is a **directrix**. Possible positions of F and l are illustrated in Figure 13.39.

Theorem **(13.15)**

> Let F be a fixed point and l a fixed line in a plane. The set of all points P in the plane, such that the ratio $d(P, F)/d(P, Q)$ is a positive constant e with $d(P, Q)$ the distance from P to l, is a conic section. The conic is a parabola if $e = 1$, an ellipse if $0 < e < 1$, and a hyperbola if $e > 1$.

FIGURE 13.39

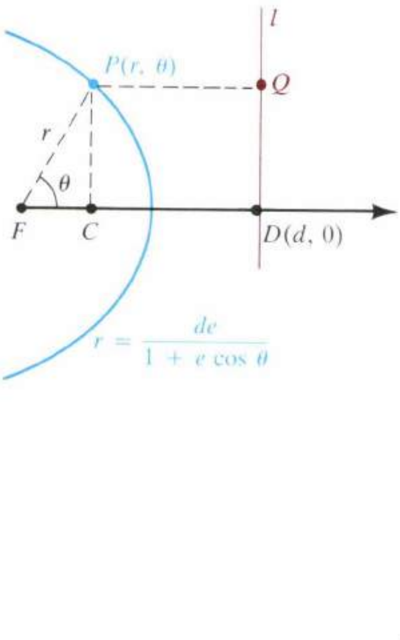


PROOF If $e = 1$, then $d(P, F) = d(P, Q)$ and, by definition, the resulting conic is a parabola with focus F and directrix l.

Suppose next that $0 < e < 1$. It is convenient to introduce a polar coordinate system in the plane with F as the pole and l perpendicular to the polar axis at the point $D(d, 0)$, with $d > 0$, as illustrated in Figure 13.40. If $P(r, \theta)$ is a point in the plane such that $d(P, F)/d(P, Q) = e < 1$, then P lies to the left of l. Let C be the projection of P on the polar axis. Since

$$d(P, F) = r \quad \text{and} \quad d(P, Q) = d - r\cos\theta,$$

FIGURE 13.40



it follows that P satisfies the condition in the theorem if and only if the following are true:

$$\frac{r}{d - r\cos\theta} = e$$

$$r = de - er\cos\theta$$

$$r(1 + e\cos\theta) = de$$

$$r = \frac{de}{1 + e\cos\theta}$$

The same equations are obtained if $e = 1$; however, there is no point (r, θ) on the graph if $1 + \cos\theta = 0$.

An equation in x and y corresponding to $r = de - er\cos\theta$ is

$$\sqrt{x^2 + y^2} = de - ex.$$

Squaring both sides and rearranging terms leads to

$$(1 - e^2)x^2 + 2de^2x + y^2 = d^2e^2.$$

Completing the square and simplifying, we obtain

$$\left(x + \frac{de^2}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{d^2e^2}{(1 - e^2)^2}.$$

Finally, dividing both sides by $d^2e^2/(1 - e^2)^2$ gives us an equation of the form

$$\frac{(x - h)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with $h = -de^2/(1 - e^2)$. Consequently the graph is an ellipse with center at the point $(h, 0)$ on the x-axis and with

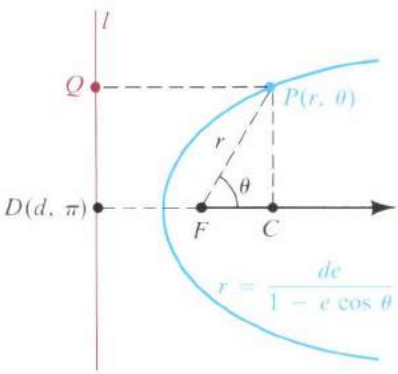
$$a^2 = \frac{d^2e^2}{(1 - e^2)^2}, \qquad b^2 = \frac{d^2e^2}{1 - e^2}.$$

Since

$$c^2 = a^2 - b^2 = \frac{d^2e^4}{(1 - e^2)^2},$$

we obtain $c = de^2/(1 - e^2)$ and hence $|h| = c$. This proves that F is a focus of the ellipse. It also follows that $e = c/a$. A similar proof may be given for the case $e > 1$.

FIGURE 13.41

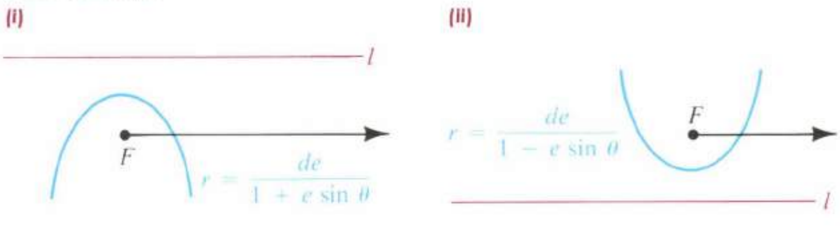


We also can show that every conic that is not degenerate may be described by means of the statement in Theorem (13.15). This gives us a formulation of conic sections that is equivalent to the approach used previously. Since the theorem includes all three types of conics, it is sometimes regarded as a definition for the conic sections.

If we had chosen the focus F to the *right* of the directrix, as illustrated in Figure 13.41 (with $d > 0$), then the equation $r = de/(1 - e\cos\theta)$ would have resulted. (Note the minus sign in place of the plus sign.) Other sign changes occur if d is allowed to be negative.

If we had taken l *parallel* to the polar axis through one of the points $(d, \pi/2)$ or $(d, 3\pi/2)$, as illustrated in Figure 13.42, then the corresponding equations would have contained $\sin\theta$ instead of $\cos\theta$.

FIGURE 13.42



The following theorem summarizes our discussion.

Theorem **(13.16)**

> A polar equation that has one of the four forms
>
> $$r = \frac{de}{1 \pm e\cos\theta}, \qquad r = \frac{de}{1 \pm e\sin\theta}$$
>
> is a conic section. The conic is a parabola if $e = 1$, an ellipse if $0 < e < 1$, or a hyperbola if $e > 1$.

EXAMPLE 1 Describe and sketch the graph of the polar equation $r = \dfrac{10}{3 + 2\cos\theta}$.

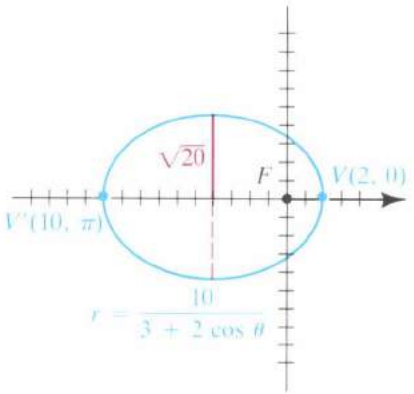
SOLUTION We first divide numerator and denominator of the fraction by 3:

$$r = \frac{\frac{10}{3}}{1 + \frac{2}{3}\cos\theta}$$

This equation has one of the forms in Theorem (13.16), with $e = \frac{2}{3}$. Thus, the graph is an ellipse with focus F at the pole and major axis along the polar axis. We find the endpoints of the major axis by letting $\theta = 0$ and $\theta = \pi$. This gives us $V(2, 0)$ and $V'(10, \pi)$. Hence

$$2a = d(V', V) = 12, \quad \text{or} \quad a = 6.$$

FIGURE 13.43



The center of the ellipse is the midpoint $(4, \pi)$ of the segment $V'V$. Using the fact that $e = c/a$, we obtain

$$c = ae = 6(\tfrac{2}{3}) = 4.$$

Hence $$b^2 = a^2 - c^2 = 36 - 16 = 20.$$

Thus, $b = \sqrt{20}$. The graph is sketched in Figure 13.43. For reference, we have superimposed an xy-coordinate system on the polar system.

EXAMPLE 2 Describe and sketch the graph of the polar equation $r = \dfrac{10}{2 + 3\sin\theta}$.

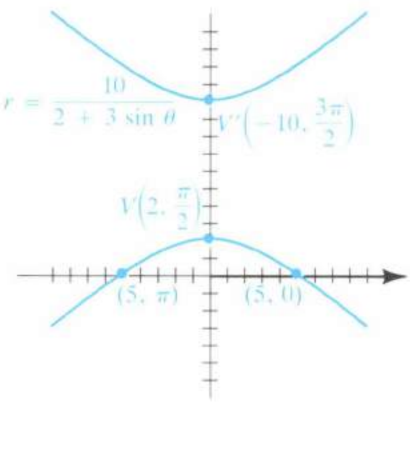
SOLUTION To express the equation in the proper form, we divide numerator and denominator of the fraction by 2:

$$r = \frac{5}{1 + \frac{3}{2}\sin\theta}$$

Thus, $e = \frac{3}{2}$, and, by Theorem (13.16), the graph is a hyperbola with a focus at the pole. The expression $\sin\theta$ tells us that the transverse axis of the hyperbola is perpendicular to the polar axis. To find the vertices, we let $\theta = \pi/2$ and $\theta = 3\pi/2$ in the given equation. This gives us the points $V(2, \pi/2)$, $V'(-10, 3\pi/2)$, and hence

$$2a = d(V, V') = 8, \quad \text{or} \quad a = 4.$$

FIGURE 13.44



The points $(5, 0)$ and $(5, \pi)$ on the graph can be used to sketch the lower branch of the hyperbola. The upper branch is obtained by symmetry, as illustrated in Figure 13.44. If we desire more accuracy or additional information, we calculate

$$c = ae = 4(\tfrac{3}{2}) = 6$$

and $$b^2 = c^2 - a^2 = 36 - 16 = 20.$$

Asymptotes may then be constructed in the usual way.

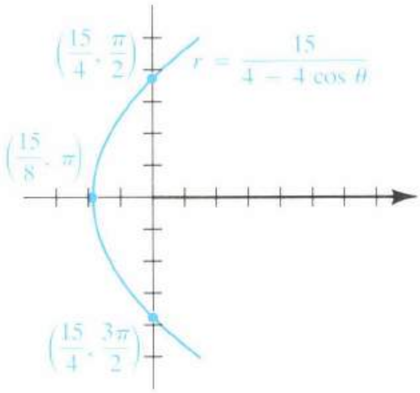
EXAMPLE 3 Sketch the graph of the polar equation $r = \dfrac{15}{4 - 4\cos\theta}$.

SOLUTION To obtain the proper form, we divide numerator and denominator by 4:

$$r = \frac{\frac{15}{4}}{1 - \cos\theta}$$

Consequently $e = 1$, and, by Theorem (13.16), the graph is a parabola with focus at the pole. We may obtain a sketch by plotting the points that correspond to the x- and y-intercepts. These are indicated in the following table.

FIGURE 13.45



$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\boldsymbol{r}$	undefined	$\frac{15}{4}$	$\frac{15}{8}$	$\frac{15}{4}$

Note that there is no point on the graph corresponding to $\theta = 0$, since the denominator $1 - \cos\theta$ is 0 for that value. Plotting the three points and using the fact that the graph is a parabola with focus at the pole gives us the sketch in Figure 13.45.

If we desire only a rough sketch of a conic, then the technique employed in Example 3 is recommended. To use this method, we plot (if possible) points corresponding to $\theta = 0$, $\pi/2$, π, and $3\pi/2$. These points, together with the type of conic (obtained from the value of e), readily lead to the sketch.

EXAMPLE 4 Find an equation in x and y that has the same graph as the polar equation

$$r = \frac{15}{4 - 4\sin\theta}.$$

SOLUTION We multiply both sides of the polar equation by the lcd, $4 - 4\sin\theta$, and then use relationships between rectangular and polar coordinates as follows:

$$r(4 - 4\sin\theta) = 15$$
$$4r = 15 + 4r\sin\theta$$
$$4(\pm\sqrt{x^2 + y^2}) = 15 + 4y$$

Squaring both sides and simplifying yields

$$16(x^2 + y^2) = 225 + 120y + 16y^2$$
$$16x^2 = 120y + 225.$$

EXAMPLE 5 Find a polar equation of the conic with a focus at the pole, eccentricity $e = \frac{1}{2}$, and directrix $r = -3 \sec \theta$.

SOLUTION The equation $r = -3 \sec \theta$ of the directrix may be written $r \cos \theta = -3$, which is equivalent to $x = -3$ in a rectangular coordinate system. This gives us the situation illustrated in Figure 13.41, with $d = 3$. Hence a polar equation has the form

$$r = \frac{de}{1 - e \cos \theta}.$$

We now substitute $d = 3$ and $e = \frac{1}{2}$:

$$r = \frac{3(\frac{1}{2})}{1 - \frac{1}{2} \cos \theta}$$

$$r = \frac{3}{2 - \cos \theta}$$

EXERCISES 13.5

Exer. 1–10: (a) Find the eccentricity and classify the conic. (b) Sketch the graph and label the vertices.

1 $r = \dfrac{12}{6 + 2 \sin \theta}$

2 $r = \dfrac{12}{6 - 2 \sin \theta}$

3 $r = \dfrac{12}{2 - 6 \cos \theta}$

4 $r = \dfrac{12}{2 + 6 \cos \theta}$

5 $r = \dfrac{3}{2 + 2 \cos \theta}$

6 $r = \dfrac{3}{2 - 2 \sin \theta}$

7 $r = \dfrac{4}{\cos \theta - 2}$

8 $r = \dfrac{4 \sec \theta}{2 \sec \theta - 1}$

9 $r = \dfrac{6 \csc \theta}{2 \csc \theta + 3}$

10 $r = \csc \theta (\csc \theta - \cot \theta)$

Exer. 11–20: Find equations in x and y for the polar equations in Exercises 1–10.

Exer. 21–26: Find a polar equation of the conic with focus at the pole and the given eccentricity and equation of the directrix.

21 $e = \frac{1}{3}$; $r = 2 \sec \theta$

22 $e = \frac{2}{5}$; $r = 4 \csc \theta$

23 $e = 4$; $r = -3 \csc \theta$

24 $e = 3$; $r = -4 \sec \theta$

25 $e = 1$; $r \cos \theta = 5$

26 $e = 1$; $r \sin \theta = -2$

27 Find a polar equation of the parabola with focus at the pole and vertex $V(4, \pi/2)$.

28 An ellipse has a focus at the pole, center $C(3, \pi/2)$, and vertex $V(1, 3\pi/2)$.

(a) Find the eccentricity.

(b) Find a polar equation for the ellipse.

Exer. 29–32: Find the slope of the tangent line to the conic at the point corresponding to the given value of θ.

29 $r = \dfrac{12}{6 + 2 \sin \theta}$; $\theta = \dfrac{\pi}{6}$

30 $r = \dfrac{12}{6 - 2 \sin \theta}$; $\theta = 0$

31 $r = \dfrac{12}{2 - 6 \cos \theta}$; $\theta = \dfrac{\pi}{2}$

32 $r = \dfrac{12}{2 + 6 \cos \theta}$; $\theta = \dfrac{\pi}{3}$

Exer. 33–36: Set up an integral in polar coordinates that can be used to find the area of the region bounded by the graphs of the equations.

33 $r = 2 \sec \theta$; $\theta = \pi/6$, $\theta = \pi/3$

34 $r = \csc \theta \cot \theta$; $\theta = \pi/6$, $\theta = \pi/4$

35 $r(1 - \cos \theta) = 4$; $\theta = \pi/4$

36 $r(1 + \sin \theta) = 2$; $\theta = \pi/3$

37 Kepler's first law states that planets travel in elliptical orbits with the sun at a focus. To find an equation of an orbit, place the pole O at the center of the sun and the polar axis along the major axis of the ellipse (see figure).

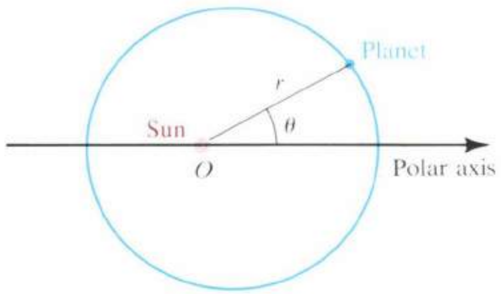
(a) Show that an equation of the orbit is

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta},$$

where e is the eccentricity and $2a$ is the length of the major axis.

(b) The *perihelion distance* r_{per} and *aphelion distance* r_{aph} are defined as the minimum and maximum distances, respectively, of a planet from the sun. Show that $r_{\text{per}} = a(1 - e)$ and $r_{\text{aph}} = a(1 + e)$.

EXERCISE 37



38 Refer to Exercise 37. The planet Pluto travels in an elliptical orbit of eccentricity 0.249. If the perihelion distance is 29.62 AU, find a polar equation for the orbit and estimate the aphelion distance.

39 (a) Use Theorem (9.6) to show that

$$\int_{-\pi}^{\pi} \frac{1}{1 - e\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{1 - e^2}}.$$

(b) Use Definition (5.29) to find the average distance of a planet from the sun.

40 Refer to Exercises 37 and 39. Eros, with an average distance from the sun of 1.46 AU, is an asteroid in the solar system. If Eros has an elliptical orbit of eccentricity 0.223, find a polar equation that approximates the orbit and estimate how close Eros comes to the sun.

13.6 REVIEW EXERCISES

Exer. 1–4: (a) Find an equation in x and y whose graph contains the points on the curve C. (b) Sketch the graph of C and indicate the orientation.

1 $x = \dfrac{1}{t} + 1, \quad y = \dfrac{2}{t} - t; \quad 0 < t \le 4$

2 $x = \cos^2 t - 2, \quad y = \sin t + 1; \quad 0 \le t \le 2\pi$

3 $x = \sqrt{t}, \quad y = 2^{-t}; \quad t \ge 0$

4 $x = 3\cos t + 2, \quad y = -3\sin t - 1; \quad 0 \le t \le 2\pi$

Exer. 5–6: Sketch the graphs of C_1, C_2, C_3, and C_4, and indicate their orientations.

5 C_1: $x = t, \quad y = \sqrt{16 - t^2}; \quad -4 \le t \le 4$
C_2: $x = -\sqrt{16 - t}, \quad y = -\sqrt{t}; \quad 0 \le t \le 16$
C_3: $x = 4\cos t, \quad y = 4\sin t; \quad 0 \le t \le 2\pi$
C_4: $x = e^t, \quad y = -\sqrt{16 - e^{2t}}; \quad t \le \ln 4$

6 C_1: $x = t^2, \quad y = t^3; \quad t$ in $\mathbb{R}$
C_2: $x = t^4, \quad y = t^6; \quad t$ in $\mathbb{R}$
C_3: $x = e^{2t}, \quad y = e^{3t}; \quad t$ in $\mathbb{R}$
C_4: $x = 1 - \sin^2 t, \quad y = \cos^3 t; \quad t$ in $\mathbb{R}$

Exer. 7–8: Let C be the given parametrized curve. (a) Express dy/dx in terms of t. (b) Find the values of t that correspond to horizontal or vertical tangent lines to the graph of C. (c) Express d^2y/dx^2 in terms of t.

7 $x = t^2, \quad y = 2t^3 + 4t - 1; \quad t$ in $\mathbb{R}$

8 $x = t - 2\sin t, \quad y = 1 - 2\cos t; \quad t$ in $\mathbb{R}$

Exer. 9–26: Sketch the graph of the polar equation.

9 $r = -4\sin\theta$

10 $r = 10\cos\theta$

11 $r = 6 - 3\cos\theta$

12 $r = 3 + 2\cos\theta$

13 $r^2 = 9\sin 2\theta$

14 $r^2 = -4\sin 2\theta$

15 $r = 3\sin 5\theta$

16 $r = 2\sin 3\theta$

17 $2r = \theta$

18 $r = e^{-\theta}, \quad \theta \ge 0$

19 $r = 8\sec\theta$

20 $r(3\cos\theta - 2\sin\theta) = 6$

21 $r = 4 - 4\cos\theta$

22 $r = 4\cos^2 \frac{1}{2}\theta$

23 $r = 6 - r\cos\theta$

24 $r = 6\cos 2\theta$

25 $r = \dfrac{8}{3 + \cos\theta}$

26 $r = \dfrac{8}{1 - 3\sin\theta}$

Exer. 27–32: Find a polar equation that has the same graph as the given equation.

27 $y^2 = 4x$

28 $x^2 + y^2 - 3x + 4y = 0$

29 $2x - 3y = 8$

30 $x^2 + y^2 = 2xy$

31 $y^2 = x^2 - 2x$

32 $x^2 = y^2 + 3y$

Exer. 33–38: Find an equation in x and y that has the same graph as the polar equation.

33 $r^2 = \tan\theta$

34 $r = 2\cos\theta + 3\sin\theta$

35 $r^2 = 4\sin 2\theta$

36 $r^2 = \sec 2\theta$

37 $\theta = \sqrt{3}$

38 $r = -6$

Exer. 39–40: Find the slope of the tangent line to the graph of the polar equation at the point corresponding to the given value of θ.

39 $r = \dfrac{3}{2 + 2\cos\theta}; \quad \theta = \pi/2$

40 $r = e^{3\theta}; \quad \theta = \pi/4$

41 Find the area of the region bounded by one loop of $r^2 = 4\sin 2\theta$.

42 Find the area of the region that is inside the graph of $r = 3 + 2\sin\theta$ and outside the graph of $r = 4$.

43 The position (x, y) of a moving point at time t is given by $x = 2\sin t$, $y = \sin^2 t$. Find the distance the point travels from $t = 0$ to $t = \pi/2$.

44 Find the length of the spiral $r = 1/\theta$ from $\theta = 1$ to $\theta = 2$.

45 The curve with parametrization $x = 2t^2 + 1$, $y = 4t - 3$; $0 \le t \le 1$ is revolved about the y-axis. Find the area of the resulting surface.

46 The arc of the spiral $r = e^\theta$ from $\theta = 0$ to $\theta = 1$ is revolved about the line $\theta = \pi/2$. Find the area of the resulting surface.

47 Find the area of the surface generated by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about the polar axis.

48 A line segment of fixed length has endpoints A and B on the y-axis and x-axis, respectively. A fixed point P on AB is selected with $d(A, P) = a$ and $d(B, P) = b$ (see figure). If A and B may slide freely along their respective axes, a curve C is traced by P. If t is the radian measure of angle ABO, find parametric equations for C with parameter t and describe C.

EXERCISE 48