Mathematical Proofs
A Transition to Advanced Mathematics

Chapter 11
Cardinalities of Sets

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Cardinalities of Sets

Definition

A set S is **finite** if either $S = \emptyset$ or $|S| = n$ for some $n \in \mathbf{N}$; while a set is **infinite** if it is not finite.

It may seem that we should write $|S| = \infty$ if S is infinite but we will soon see that this is not particularly informative. Indeed, it is considerably more difficult to give a meaning to $|S|$ if S is an infinite set; however, it is precisely this topic that we are about to explore.

Numerically Equivalent Sets

Definition

Two sets A and B (finite or infinite) are said to have the **same cardinality**, written $|A| = |B|$, if either A and B are both empty or there is a bijective function from A to B.

Two sets having the same cardinality are **numerically equivalent sets**.

Two finite sets are therefore numerically equivalent if they are both empty or if both have n elements for some positive integer n.

Consequently, two nonempty sets A and B are not numerically equivalent, written $|A| \neq |B|$, if there is no bijective function from one set to the other.

Numerically Equivalent Sets

Theorem

Let $\mathcal{S}$ be a nonempty collection of nonempty sets. A relation R is defined on $\mathcal{S}$ by $A\ R\ B$ if there exists a bijective function from A to B. Then R is an equivalence relation.

Proof Let $A \in S$. Since the identity function $i_A : A \to A$ is bijective, it follows that $A\ R\ A$. Thus, R is reflexive. Next, assume that $A\ R\ B$, where $A, B \in S$. Then there is a bijective function $f : A \to B$. Therefore, f has an inverse function $f^{-1} : B \to A$ and, furthermore, f^{-1} is bijective. Therefore, $B\ R\ A$ and R is symmetric.

Finally, assume that $A\ R\ B$ and $B\ R\ C$, where $A, B, C \in S$. Then there are bijective functions $f : A \to B$ and $g : B \to C$. It follows that the composition $g \circ f : A \to C$ is bijective as well and so $A\ R\ C$. Therefore, R is transitive. Consequently, R is an equivalence relation. □

Numerically Equivalent Sets

Example 1

Let $S = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ where

$$A_1 = \{1, 2, 3\},\ A_2 = \{a, b, c, d\},\ A_3 = \{x, y, z\},$$
$$A_4 = \{r, s, t\},\ A_5 = \{m, n\},\ A_6 = \{7, 8, 9, 10\}.$$

Then every two of the sets A_1, A_3 and A_4 are numerically equivalent, while A_2 and A_6 are numerically equivalent. This says that $|A_1| = |A_3| = |A_4|$ and $|A_2| = |A_6|$. The only set in S that is numerically equivalent to A_5 is A_5 itself. Thus,

$$[A_1] = \{A_1, A_3, A_4\},\ [A_2] = \{A_2, A_6\} \text{ and } [A_5] = \{A_5\}$$

are the distinct equivalence classes of S. ♦

Numerically Equivalent Sets

Definition

A set A is called **denumerable** if $|A| = |\mathbf{N}|$, that is, if A has the same cardinality as the set of natural numbers.

Certainly, if A is denumerable, then A is infinite.

By definition, if A is a denumerable set, then there is a bijective function $f : \mathbf{N} \to A$ and so $f = \{(1, f(1)), (2, f(2)), (3, f(3)), \ldots\}$.

Consequently, $A = \{f(1), f(2), f(3), \ldots\}$ and so we can list the elements of A as $f(1), f(2), f(3), \ldots$. Equivalently, we can list the elements of A as $a_1, a_2, a_3, \ldots$, where then $a_i = f(i)$ for $i \in \mathbf{N}$.

Hence, if the elements of A can be listed as $a_1, a_2, a_3, \ldots$, where $a_i \neq a_j$ for $i \neq j$, then A is denumerable since the function $g : \mathbf{N} \to A$ defined by $g(n) = a_n$ for each $n \in \mathbf{N}$ is certainly bijective.

Therefore, A is a denumerable set if and only if it is possible to list the elements of A as $a_1, a_2, a_3, \ldots$ and so $A = \{a_1, a_2, a_3, \ldots\}$.

Numerically Equivalent Sets

Definition

A set is **countable** if it is either finite or denumerable. **Countably infinite** sets are then precisely the denumerable sets.

Hence, if A is a nonempty countable set, then we can either write $A = \{a_1, a_2, a_3, \ldots, a_n\}$ for some $n \in \mathbf{N}$ or $A = \{a_1, a_2, a_3, \ldots\}$.

Definition

A set that is not countable is called **uncountable**.

An uncountable set is necessarily infinite.

Certainly, $\mathbf{N}$ itself is denumerable since the identity function $i_{\mathbf{N}} : \mathbf{N} \to \mathbf{N}$ is bijective.

Numerically Equivalent Sets

Example 2

Result The set $\mathbf{Z}$ of integers is denumerable.

Proof. Observe that the elements of $\mathbf{Z}$ can be listed as $0, 1, -1, 2, -2, \ldots$. Thus, the function $f : \mathbf{N} \to \mathbf{Z}$ described in the figure below is bijective and so $\mathbf{Z}$ is denumerable. □

$$f: \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & -1 & 2 & -2 & \cdots \end{array}$$

Numerically Equivalent Sets

Theorem

Every infinite subset of a denumerable set is denumerable.

For $k \in \mathbf{N}$, the set $k\mathbf{Z}$ is defined by

$$k\mathbf{Z} = \{kn : \ n \in \mathbf{Z}\}.$$

Similarly,

$$k\mathbf{N} = \{kn : \ n \in \mathbf{N}\}.$$

Thus, $1\mathbf{Z} = \mathbf{Z}$ and $1\mathbf{N} = \mathbf{N}$, while $2\mathbf{Z}$ is the set of even integers.

Example 3

Result The set $2\mathbf{Z}$ of even integers is denumerable.

Proof. Since $2\mathbf{Z}$ is infinite and $2\mathbf{Z} \subseteq \mathbf{Z}$, it follows that $2\mathbf{Z}$ is denumerable. □

Example 4

Result If A and B are denumerable sets, then $A \times B$ is denumerable.

Proof. Since A and B are denumerable sets, we can write $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$. Construct a table which has an infinite (denumerable) number of rows and columns, where the elements $a_1, a_2, a_3, \ldots$ are written along the side and $b_1, b_2, b_3, \ldots$ are written across the top.

Numerically Equivalent Sets

Example 4 (continued)

In row i, column j of the table, we place the ordered pair (a_i, b_j). Certainly, every element of $A \times B$ appears exactly once in this table.

	b_1	b_2	b_3	$\cdots$
a_1	(a_1, b_1)	(a_1, b_2)	(a_1, b_3)	$\cdots$
a_2	(a_2, b_1)	(a_2, b_2)	(a_2, b_3)	$\cdots$
a_3	(a_3, b_1)	(a_3, b_2)	(a_3, b_3)	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Numerically Equivalent Sets

Example 4 (continued)

The directed lines indicate the order in which we will encounter the entries is the table.

	b_1	b_2	b_3	$\cdots$
a_1	(a_1, b_1)	(a_1, b_2)	(a_1, b_3)	$\cdots$
a_2	(a_2, b_1)	(a_2, b_2)	(a_2, b_3)	$\cdots$
a_3	(a_3, b_1)	(a_3, b_2)	(a_3, b_3)	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

That is, we encounter the elements of $A \times B$ in the order

$$(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_1, b_3), (a_2, b_2), \ldots .$$

Example 4 (continued)

Since every element of $A \times B$ occurs in this list exactly once, this describes a bijective function $f : \mathbf{N} \to A \times B$, where

$$f(1) = (a_1, b_1),\ f(2) = (a_1, b_2),\ f(3) = (a_2, b_1),\ f(4) = (a_1, b_3),\ f(5) = (a_2, b_2),\ \ldots.$$

Therefore, $A \times B$ is denumerable. □

Numerically Equivalent Sets

We can use a similar technique to show that another familiar set is denumerable.

Example 5

Result The set $\mathbf{Q}^+$ of positive rational numbers is denumerable.
Result The set $\mathbf{Q}$ of all rational numbers is denumerable.

Proof. Since $\mathbf{Q}^+$ is denumerable, we can write $\mathbf{Q}^+ = \{q_1, q_2, q_3, \ldots\}$. Thus, $\mathbf{Q} = \{0\} \cup \{q_1, q_2, q_3, \ldots\} \cup \{-q_1, -q_2, -q_3, \ldots\}$. Therefore, $\mathbf{Q} = \{0, q_1, -q_1, q_2, -q_2, \ldots\}$ and the function $f : \mathbf{N} \to \mathbf{Q}$ shown in the figure below is bijective and so $\mathbf{Q}$ is denumerable. $\square$

$$f: \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & q_1 & -q_1 & q_2 & -q_2 & \cdots \end{array}$$

Theorem

The open interval $(0, 1)$ of real numbers is uncountable.

Proof Assume, to the contrary, that $(0, 1)$ is countable. Since $(0, 1)$ is infinite, it is denumerable. Therefore, there exists a bijective function $f : \mathbf{N} \to (0, 1)$. For $n \in \mathbf{N}$, let $f(n) = a_n$. Since $a_n \in (0, 1)$, the number a_n has a decimal expansion, say $0.a_{n1}a_{n2}a_{n3}\cdots$, where $a_{ni} \in \{0, 1, 2, \ldots, 9\}$ for all $i \in \mathbf{N}$. If a_n is irrational, then its decimal expansion is unique. If $a_n \in \mathbf{Q}$, then the expansion *may* be unique. If it is not unique, then, without loss of generality, we assume that the digits of the decimal expansion $0.a_{n1}a_{n2}a_{n3}\cdots$ are 0 from some position on. For example, since f is bijective, $2/5$ is the image of exactly one positive integer and this image is written as $0.4000\cdots$ (rather than as $0.3999\cdots$).

Proof (continued)

To summarize, we have

$$\begin{array}{rcl} f(1) & = & a_1 = 0.a_{11}a_{12}a_{13}\cdots \\ f(2) & = & a_2 = 0.a_{21}a_{22}a_{23}\cdots \\ f(3) & = & a_3 = 0.a_{31}a_{32}a_{33}\cdots \\ \vdots & & \vdots \qquad \vdots \end{array}$$

We show that the function f is not onto, however.

Proof (continued)

Define the number $b = 0.b_1 b_2 b_3 \cdots$, where $b_i \in \{0, 1, 2, \ldots, 9\}$ for all $i \in \mathbf{N}$, by

$$b_i = \begin{cases} 4 & \text{if } a_{ii} = 5 \\ 5 & \text{if } a_{ii} \neq 5. \end{cases}$$

(For example, let's suppose that $a_1 = 0.31717\cdots$, $a_2 = 0.151515\cdots$ and $a_3 = 0.04000\cdots$. Then the first three digits in the decimal expansion of b are 5, 4 and 5, that is, $b = 0.545\cdots$.)

For each $i \in \mathbf{N}$, the digit $b_i \neq a_{ii}$, implying that $b \neq a_n$ for all $n \in \mathbf{N}$ since b is not the alternate expansion of any rational number, as no digit in the expansion of b is 9. Thus, b is not an image of any element of $\mathbf{N}$. Therefore, f is not onto and, consequently, not bijective, producing a contradiction. □

Theorem

Let A and B be sets such that $A \subseteq B$. If A is uncountable, then B is uncountable.

Proof Let A and B be two sets such that $A \subseteq B$ and A is uncountable. Necessarily then A and B are infinite. Assume, to the contrary, that B is denumerable. Since A is an infinite subset of a denumerable set, it follows that A is denumerable, producing a contradiction. $\square$

Uncountable Sets

Corollary

The set $\mathbf{R}$ of real numbers is uncountable.

Theorem

Every two denumerable sets are numerically equivalent.

Theorem

If A is a denumerable set and B is an uncountable set, then A and B are not numerically equivalent

Theorem

The sets $(-1,1)$ and $\mathbf{R}$ are numerically equivalent.

Proof Consider the function $f:(-1,1)\to\mathbf{R}$ defined by

$$f(x)=\frac{x}{1-|x|}.$$

We show that f is bijective. First, we verify that f is one-to-one. Let $f(a)=f(b)$, where $a,b\in(-1,1)$. Then $\frac{a}{1-|a|}=\frac{b}{1-|b|}$. If $\frac{a}{1-|a|}=\frac{b}{1-|b|}=0$, then $a=b=0$. If $\frac{a}{1-|a|}=\frac{b}{1-|b|}>0$, then $a>0$ and $b>0$. Thus, $\frac{a}{1-a}=\frac{b}{1-b}$. Hence, $a(1-b)=b(1-a)$ and so $a=b$. If $\frac{a}{1-|a|}=\frac{b}{1-|b|}<0$, then $a<0$ and $b<0$. Thus, $\frac{a}{1+a}=\frac{b}{1+b}$. Hence, $a(1+b)=b(1+a)$ and so $a=b$. Therefore, f is one-to-one.

Proof (continued)

Next, we show that f is onto. Let $r \in \mathbf{R}$. Since $f(0) = 0$, we may assume that $r \neq 0$. If $r > 0$, then $\frac{r}{1+r} \in (0,1)$ and $f(\frac{r}{1+r}) = r$. If $r < 0$, then $\frac{r}{1-r} \in (-1,0)$ and $f(\frac{r}{1-r}) = r$. Thus, f is onto.
Since f is a bijective function, the sets $(-1,1)$ and $\mathbf{R}$ are numerically equivalent. □

Corollary

The sets $(0,1)$ and $\mathbf{R}$ are numerically equivalent.

Theorem

For every nonempty set A, the sets $\mathcal{P}(A)$ and 2^A are numerically equivalent.

Proof We show that there exists a bijective function ϕ from $\mathcal{P}(A)$ to 2^A. Define $\phi : \mathcal{P}(A) \to 2^A$ such that for $S \in \mathcal{P}(A)$, we have $\phi(S) = f_S$, where, for $x \in A$,

$$f_S(x) = \begin{cases} 1 & \text{if} \quad x \in S \\ 0 & \text{if} \quad x \notin S. \end{cases}$$

Proof (continued)

Certainly, $f_S \in 2^A$. First, we show that ϕ is one-to-one. Let $\phi(S) = \phi(T)$. Thus, $f_S = f_T$, which implies that $f_S(x) = f_T(x)$ for every $x \in A$. Therefore, $f_S(x) = 1$ if and only if $f_T(x) = 1$ for every $x \in A$; that is, $x \in S$ if and only if $x \in T$ and so $S = T$.

It remains to show that ϕ is onto. Let $f \in 2^A$. Define

$$S = \{x \in A : \ f(x) = 1\}.$$

Hence, $f_S = f$ and so $\phi(S) = f$. Thus, ϕ is onto and, consequently, ϕ is bijective. $\square$

A set A is said to have **smaller cardinality** than a set B, written as $|A| < |B|$, if there exists a one-to-one function from A to B but no bijective function from A to B.

Example 6

Since $\mathbf{N}$ is denumerable and $\mathbf{R}$ is uncountable, there is no bijective function from $\mathbf{N}$ to $\mathbf{R}$. Since the function $f : \mathbf{N} \to \mathbf{R}$ defined by $f(n) = n$ for all $n \in \mathbf{N}$ is injective, it follows that $|\mathbf{N}| < |\mathbf{R}|$. ♦

To verify that $|A| \leq |B|$, we need only show the existence of a one-to-one function from A to B.

The cardinality of the set **N** of natural numbers is often denoted by $\aleph_o$; so $|\mathbf{N}| = \aleph_o$. ($\aleph_o$ is read as aleph null.)

The set **R** of real numbers is also referred to as the **continuum** and its cardinality is denoted by c. Hence, $|\mathbf{R}| = c$ and from what we have seen, $\aleph_o < c$.

The Continuum Hypothesis

It was the German mathematician Georg Cantor who helped to put the theory of sets on a firm foundation. An interesting conjecture of his became known as:

The Continuum Hypothesis

There exists no set S such that

$$\aleph_o < |S| < c.$$

Theorem

If A is a set, then $|A| < |\mathcal{P}(A)|$.

Proof. If $A = \emptyset$, then $|A| = 0$ and $|\mathcal{P}(A)| = 1$; so $|A| < |\mathcal{P}(A)|$. Hence, we may assume that $A \neq \emptyset$. First, we show that there is a one-to-one function from A to $\mathcal{P}(A)$. Define the function $f : A \to \mathcal{P}(A)$ by $f(x) = \{x\}$ for each $x \in A$. Let $f(x_1) = f(x_2)$. Then $\{x_1\} = \{x_2\}$. So, $x_1 = x_2$ and f is one-to-one.

To prove that $|A| < |\mathcal{P}(A)|$, it remains to show that there is no bijective function from A to $\mathcal{P}(A)$. Assume, to the contrary, that there exists a bijective function $g : A \to \mathcal{P}(A)$. For each $x \in A$, let $g(x) = A_x$, where $A_x \subseteq A$. We show that there is a subset of A that is distinct from A_x for each $x \in A$. Define the subset B of A by

$$B = \{x \in A : \; x \notin A_x\}.$$

Proof (continued)

By assumption, there exists an element $y \in A$ such that $B = A_y$. If $y \in A_y$, then $y \notin B$ by the definition of B. On the other hand, if $y \notin A_y$, then, according to the definition of the set B, it follows that $y \in B$. In either case, y belongs to exactly one of A_y and B. Hence, $B \neq A_y$, producing a contradiction. □

According to this theorem, there is no largest set.

In particular, there is a set S with $|S| > c$.

The Schröder – Bernstein Theorem

From what we know of inequalities (of real numbers), it might seem that if A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. This is indeed the case. This theorem is often referred to as the Schröder – Bernstein Theorem.

The Schröder – Bernstein Theorem

If A and B are sets such that $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

The Schröder – Bernstein Theorem

Consider the following two theorems:

Theorem A

For any two cardinal numbers a and b, exactly one of the following occurs:

$$(1)\ \ a = b, \quad (2)\ \ a < b, \quad (3)\ \ a > b.$$

Theorem B

If A and B are two sets for which there exist a one-to-one function from A to B and a one-to-one function from B to A, then $|A| = |B|$.

The Axiom of Choice

For every collection of pairwise disjoint nonempty sets, there exists at least one set that contains exactly one element from each of these nonempty sets.

As it turned out, not only can the Axiom of Choice be used to prove Theorem A, but Theorem A is true if and only if the Axiom of Choice is true.

Theorem

The sets $\mathcal{P}(\mathbf{N})$ and $\mathbf{R}$ are numerically equivalent.

Corollary

The sets $2^{\mathbf{N}}$ and $\mathbf{R}$ are numerically equivalent.

We have already mentioned that $|A| = \aleph_o$ for every denumerable set A and that $|\mathbf{R}| = c$. If A is denumerable, then we represent the cardinality of the set 2^A by $2^{\aleph_o}$. By this corollary, $2^{\aleph_o} = c$.