# Mathematical Proofs
# A Transition to Advanced Mathematics

## Chapter 10
## Functions

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# The Definition of Function

**Definition**

Let $A$ and $B$ be nonempty sets. A **function** $f$ from $A$ to $B$, written $f : A \to B$, is a relation from $A$ to $B$ with the property that every element $a$ in $A$ is the first coordinate of exactly one ordered pair in $f$.

Since $f$ is a relation, the set $A$ in this case is the **domain** of $f$, denoted by $\mathrm{dom}(f)$.

The set $B$ is called the **codomain** of $f$.

If $(a, b) \in f$, then we write $b = f(a)$ and refer to $b$ as the **image** of $a$ under the function $f$.

Sometimes $f$ is said to **map** $a$ into $b$. Indeed, $f$ itself is sometimes called a **mapping**.

# The Definition of Function

**Definition**

The **range** of $f$ is the set

$$\begin{aligned} \mathrm{range}(f) &= \{b \in B : b \text{ is an image under } f \text{ of some element of } A\} \\ &= \{f(x) : x \in A\}. \end{aligned}$$

If $A$ is a finite set, then the function $f$ is a finite set and the number of elements in $f$ is $|A|$ since there is exactly one ordered pair in $f$ corresponding to each element of $A$.

Two functions $f : A \to B$ and $g : A \to B$ are **equal**, written $f = g$, if $f(a) = g(a)$ for all $a \in A$.

**Example 1**

Let $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$.

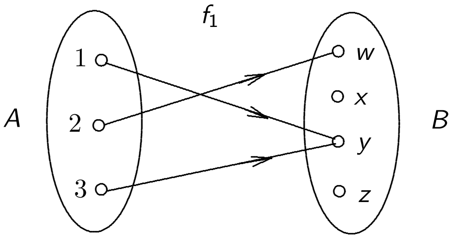
$f_1 = \{(1, y), (2, w), (3, y)\}$ is a function from $A$ to $B$ and so we may write $f_1 : A \to B$.

$f_2 = \{(1, x), (2, z), (3, y), (2, x)\}$ is not a function since there are two ordered pairs whose first coordinate is 2.

$f_3 = \{(1, z), (3, x)\}$ is not a function from $A$ to $B$ either because $\mathrm{dom}(f_3) \neq A$.
On the other hand, $f_3$ *is* a function from $A - \{2\}$ to $B$. ♦

# The Definition of Function

A function $f : A \to B$ can be represented by a diagram and drawing an arrow (a directed line segment) from an element $x \in A$ to its image $f(x) \in B$. This is illustrated for the function $f_1 = \{(1, y), (2, w), (3, y)\}$ in the figure below.



**Example 2**

In calculus, "functions" such as $f(x) = x^2$ are considered. This function $f$ is from $\mathbf{R}$ to $\mathbf{R}$, that is, $A = \mathbf{R}$ and $B = \mathbf{R}$. Although $f(x) = x^2$ is commonly referred to as a "function" in calculus and elsewhere, strictly speaking $f(x)$ is the image of a real number $x$ under $f$. The function $f$ itself is actually the set

$$f = \{(x, x^2) \ : \ x \in \mathbf{R}\}.$$

So $(2, 4)$ and $(-3, 9)$, for example, belong to $f$.
The set $\{(x, x^2) \ : \ x \in \mathbf{R}\}$ of points in the plane is the **graph** of $f$. In this case, the graph is a parabola. Here, the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$ can also be thought of as defined by a rule, namely the rule that associates the number $x^2$ with each real number $x$. ♦

# The Definition of Function

**Definition**

For a function $f : A \to B$ and a subset $C$ of $A$, the **image** $f(C)$ of $C$ is defined as

$$f(C) = \{f(x) : \ x \in C\}.$$

Therefore, $f(C) \subseteq B$ for each subset $C$ of $A$. If $C = A$, then $f(A)$ is the range of $f$.

# The Definition of Function

**Example 3**

For $A = \{a, b, c, d, e\}$ and $B = \{1, 2, \ldots, 6\}$,

$$f = \{(a,3), (b,5), (c,2), (d,3), (e,6)\}$$

is a function from $A$ to $B$.
For $C_1 = \{a, b, c\}$, $C_2 = \{a, d\}$, $C_3 = \{e\}$ and $C_4 = A$,

$$f(C_1) = \{2,3,5\},\ f(C_2) = \{3\},\ f(C_3) = \{6\},$$
$$f(C_4) = \mathrm{range}(f) = \{2,3,5,6\}.$$ ♦

# The Definition of Function

**Definition**

For a function $f : A \to B$ and an element $b \in B$, the **inverse image** $f^{-1}(b)$ of $b$ is defined as

$$f^{-1}(b) = \{a \in A : \ f(a) = b\}.$$

Therefore, $f^{-1}(b) \subseteq A$ for each element $b \in B$. For a subset $D$ of $B$,

$$f^{-1}(D) = \{a \in A : \ f(a) \in D\}.$$

Necessarily then, $f^{-1}(B) = A$.

# The Definition of Function

**Example 4**

For the function $f : A = \{a, b, c, d, e\} \to B = \{1, 2, \ldots, 6\}$ defined by

$$f = \{(a, 3), (b, 5), (c, 2), (d, 3), (e, 6)\},$$

it follows that

$$f^{-1}(3) = \{a, d\},\ f^{-1}(\{1, 3\}) = \{a, d\},$$
$$f^{-1}(4) = \emptyset \text{ and } f^{-1}(B) = A.$$

◆

# One-to-one Functions

**Definition**

A function $f$ from a set $A$ to a set $B$ is called **one-to-one** or **injective** if every two distinct elements of $A$ have distinct images in $B$.

In symbols, a function $f : A \to B$ is one-to-one if whenever $x, y \in A$ and $x \neq y$, then $f(x) \neq f(y)$.

If a function $f : A \to B$ is not one-to-one, then there exist distinct elements $w$ and $z$ in $A$ such that $f(w) = f(z)$.

**Example 5**

Let $A = \{a, b, c, d\}$, $B = \{r, s, t, u, v\}$ and $C = \{x, y, z\}$. Then

$$f_1 = \{(a, s), (b, u), (c, v), (d, r)\}$$

is a one-to-one function from $A$ to $B$ since distinct elements of $A$ have distinct images in $B$; while the function

$$f_2 = \{(a, s), (b, t), (c, s), (d, u)\}$$

from $A$ to $B$ is not one-to-one since $a$ and $c$ have the same image, namely $s$. There is no one-to-one function from $A$ to $C$, however. ♦

# One-to-one Functions

Suppose that a function $f : A \to B$ is one-to-one, where $A$ and $B$ are finite sets. Since every two elements of $A$ have distinct images in $B$, there must be at least as many elements in $B$ as in $A$, that is, $|A| \leq |B|$.

There is a useful equivalent formulation of the definition using the contrapositive:

**Definition**

A function $f : A \to B$ is **one-to-one** if whenever $f(x) = f(y)$, where $x, y \in A$, then $x = y$.

**Example 6**

**Result** The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x - 5$ is one-to-one.

**Proof.** Assume that $f(a) = f(b)$, where $a, b \in \mathbf{R}$. Then

$$3a - 5 = 3b - 5.$$

Adding 5 to both sides, we obtain $3a = 3b$. Dividing by 3, we have $a = b$ and so $f$ is one-to-one. □

# One-to-one Functions

**Example 7**

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = x^2 - 3x - 2.$$

Determine whether $f$ is one-to-one.

**Solution.** Since $f(0) = -2$ and $f(3) = -2$, it follows that $f$ is not one-to-one. ♦

# Onto Functions

## Definition

A function $f : A \to B$ is called **onto** or **surjective** if every element of the codomain $B$ is the image of some element of $A$.

Equivalently, $f$ is onto if $f(A) = B$.

## Example 8

For $A = \{1, 2, 3\}$, $B = \{x, y, z, w\}$, the function $f_1 : A \to B$ defined by

$$f_1 = \{(1, y), (2, w), (3, y)\}$$

is *not* onto since neither $x$ nor $z$ is an image of some element of $A$. ♦

# Onto Functions

If $A$ and $B$ are finite sets and $f : A \to B$ is a surjective function, then $|B| \leq |A|$.

**Example 9**

For $A = \{1, 2, 3\}$ and $B = \{x, y, z, w\}$, the function $g : B \to A$ defined by

$$g = \{(x, 3), (y, 1), (z, 3), (w, 2)\}$$

*is* a surjective function. ♦

**Example 10**

**Result** The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x - 5$ is onto. [For $r \in \mathbf{R}$, we need to find a real number $x$ such that $f(x) = r$. However then, $3x - 5 = r$ and $x = (r + 5)/3$.]

**Proof.** Let $r \in \mathbf{R}$. We show that there exists $x \in \mathbf{R}$ such that $f(x) = r$. Choose $x = (r + 5)/3$. Then $x \in \mathbf{R}$ and

$$f(x) = f\left(\frac{r+5}{3}\right) = 3\left(\frac{r+5}{3}\right) - 5 = r.$$

□

# Bijective Functions

**Definition**

A function $f : A \to B$ is called **bijective** or a **one-to-one correspondence** if it is both one-to-one and onto.

If a function $f : A \to B$ is bijective and $A$ and $B$ are finite sets, then $|A| = |B|$.

If $A$ and $B$ are finite sets with $|A| = |B|$, then there exists a bijective function $f : A \to B$.

A bijective function from a set $A$ to a set $B$ creates a pairing of the elements of $A$ with the elements of $B$.

# Bijective Functions

## Example 11

For $A = \{a, b, c\}$ and $B = \{x, y, z\}$, the bijective functions from $A$ to $B$ are

$$\begin{aligned}
f_1 &= \{(a,x),(b,y),(c,z)\}\\
f_2 &= \{(a,y),(b,z),(c,x)\}\\
f_3 &= \{(a,z),(b,x),(c,y)\}\\
f_4 &= \{(a,y),(b,x),(c,z)\}\\
f_5 &= \{(a,z),(b,y),(c,x)\}\\
f_6 &= \{(a,x),(b,z),(c,y)\}.
\end{aligned}$$

That is, there are six bijective functions from $A$ to $B$; indeed there are six bijective functions from any 3-element set to any 3-element set. ♦

# Bijective Functions

**Theorem**

If $A$ and $B$ are finite sets with $|A| = |B| = n$, then there are $n!$ bijective functions from $A$ to $B$.

**Proof** Suppose that $A = \{a_1, a_2, \ldots, a_n\}$. Then any bijective function $f : A \to B$ can be expressed as

$$f = \{(a_1, -), (a_2, -), \ldots, (a_n, -)\},$$

where the second coordinate of each ordered pair of $f$ belongs to $B$.

## Proof (continued)

There are $n$ possible images for $a_1$ in $f$. Once an image for $a_1$ has been determined, then there are $n-1$ possible images for $a_2$. Since $f$ is one-to-one, no element of $B$ can be the image of two elements of $A$. Because neither of the images of $a_1$ and $a_2$ can be images of $a_3$, there are $n-2$ possibilities for $a_3$. Continuing in this manner, we see that there is only one possibility for the image of $a_n$. It turns out that the total number of possible bijective functions $f$ is obtained by multiplying these numbers and so there are

$$n(n-1)(n-2)\cdots 1 = n!$$

bijective functions from $A$ to $B$. □

# Bijective Functions

There is another interesting fact concerning the existence of bijective functions $f : A \to B$ for finite sets $A$ and $B$ with $|A| = |B|$.

**Theorem**

Let $A$ and $B$ be finite nonempty sets such that $|A| = |B|$ and let $f$ be a function from $A$ to $B$. Then $f$ is one-to-one if and only if $f$ is onto.

# Bijective Functions

**Definition**

For a nonempty set $A$, the function $i_A : A \to A$ defined by $i_A(a) = a$ for each $a \in A$ is called the **identity function** on $A$. If the set $A$ under discussion is clear, we write the identity function $i_A$ by $i$. For $S = \{1, 2, 3\}$, the identity function is

$$i_S = i = \{(1, 1), (2, 2), (3, 3)\}.$$

Not only is *this* identity function bijective, the identity function $i_A$ is bijective for *every* nonempty set $A$.

### Example 12

**Result** The function $f : \mathbf{R} - \{2\} \to \mathbf{R} - \{3\}$ defined by

$$f(x) = \frac{3x}{x-2}$$

is bijective.

**Proof.** Here it is necessary to show that $f$ is both one-to-one and onto. We begin with the first of these. Assume that $f(a) = f(b)$, where $a, b \in \mathbf{R} - \{2\}$. Then

$$\frac{3a}{a-2} = \frac{3b}{b-2}.$$

**Example 12 (continued)**

Multiplying both sides by $(a-2)(b-2)$, we obtain $3a(b-2) = 3b(a-2)$. Simplifying, we have $3ab - 6a = 3ab - 6b$. Adding $-3ab$ to both sides and dividing by $-6$, we obtain $a = b$. Thus, $f$ is one-to-one.

To show that $f$ is onto, let $r \in \mathbf{R} - \{3\}$. We show that there exists $x \in \mathbf{R} - \{2\}$ such that $f(x) = r$. Choose $x = \dfrac{2r}{r-3}$. Then

$$f(x) = f\left(\frac{2r}{r-3}\right) = \frac{3\left(\frac{2r}{r-3}\right)}{\frac{2r}{r-3} - 2} = \frac{6r}{2r - 2(r-3)} = \frac{6r}{6} = r,$$

implying that $f$ is onto. Therefore $f$ is bijective. □

# Composition of Functions

## Definition

For nonempty sets $A, B$ and $C$ and functions $f : A \to B$ and $g : B \to C$, it is possible to create a new function from $f$ and $g$, called their composition. The **composition** $g \circ f$ of $f$ and $g$ is the function from $A$ to $C$ defined by

$$(g \circ f)(a) = g(f(a)) \quad \text{for all } a \in A.$$

## Example 13

Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$ and $C = \{r, s, t, u, v\}$ and define the functions $f : A \to B$ and $g : B \to C$ by

$$\begin{aligned} f &= \{(1, b), (2, d), (3, a), (4, a)\}, \\ g &= \{(a, u), (b, r), (c, r), (d, s)\}. \end{aligned}$$

We now have the correct arrangement of sets and functions to consider the composition $g \circ f$.

Since $g \circ f$ is a function from $A$ to $C$, it follows that $g \circ f$ has the following appearance:

$$g \circ f = \{(1, \alpha), (2, \beta), (3, \gamma), (4, \delta)\},$$

where $\alpha, \beta, \gamma, \delta \in C$.

# Composition of Functions

**Example 13 (continued)**

It remains only to determine the image of each element of $A$. First, we find the image of 1. According to the definition of $g \circ f$,

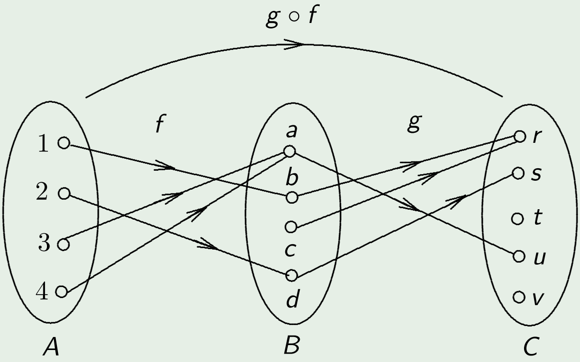
$$(g \circ f)(1) = g(f(1)) = g(b) = r,$$

so $(1, r) \in g \circ f$. Similarly, $(g \circ f)(2) = g(f(2)) = g(d) = s$ and so $(2, s) \in g \circ f$.

## Example 13 (continued)

Continuing in this manner, we obtain

$$g \circ f = \{(1, r), (2, s), (3, u), (4, u)\}. \quad \blacklozenge$$



Composition of functions was also encountered in calculus.

**Example 14**

Suppose that the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are defined by $f(x) = x^2$ and $g(x) = \sin x$. In this case, we can determine both $g \circ f$ and $f \circ g$; namely,

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) = g(x^2) = \sin\left(x^2\right) \\ (f \circ g)(x) &= f(g(x)) = f(\sin x) = (\sin x)^2 = \sin^2 x. \end{aligned}$$

Secondly, this example also serves to illustrate that even when $g \circ f$ and $f \circ g$ are both defined, they need not be equal. ♦

# Composition of Functions

**Theorem**

Let $f : A \to B$ and $g : B \to C$ be two functions.

(a) If $f$ and $g$ are injective, then so is $g \circ f$.

(b) If $f$ and $g$ are surjective, then so is $g \circ f$.

**Proof** Let $f : A \to B$ and $g : B \to C$ be injective functions. Assume that $(g \circ f)(a_1) = (g \circ f)(a_2)$, where $a_1, a_2 \in A$. By definition, $g(f(a_1)) = g(f(a_2))$. Since $g$ is injective, it follows that $f(a_1) = f(a_2)$. However, since $f$ is injective, it follows that $a_1 = a_2$. This implies that $g \circ f$ is injective.

# Composition of Functions

**Proof (continued)**

Next, let $f : A \to B$ and $g : B \to C$ be surjective functions and let $c \in C$. Since $g$ is surjective, there exists $b \in B$ such that $g(b) = c$. On the other hand, since $f$ is surjective, it follows that there exists $a \in A$ such that $f(a) = b$. Hence, $(g \circ f)(a) = g(f(a)) = g(b) = c$, implying that $g \circ f$ is also surjective. □

# Composition of Functions

**Corollary**

If $f : A \to B$ and $g : B \to C$ are bijective functions, then $g \circ f$ is bijective.

**Theorem**

For nonempty sets $A, B, C$ and $D$, let $f : A \to B$, $g : B \to C$, and $h : C \to D$ be functions. Then $(h \circ g) \circ f = h \circ (g \circ f)$.

### Definition

For a relation $R$ from a set $A$ to a set $B$, the **inverse relation** $R^{-1}$ from $B$ to $A$ is defined as

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

For example, if $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$ and

$$R = \{(a, 1), (a, 3), (c, 2), (c, 3), (d, 1)\}$$

is a relation from $A$ to $B$, then

$$R^{-1} = \{(1, a), (3, a), (2, c), (3, c), (1, d)\}$$

is the inverse relation of $R$.

# Inverse Functions

Of course, every function $f : A \to B$ is also a relation from $A$ to $B$ and so there is an inverse relation $f^{-1}$ from $B$ to $A$. This brings up a natural question: Under what conditions is the inverse relation $f^{-1}$ from $B$ to $A$ also a function from $B$ to $A$? If the inverse relation $f^{-1}$ is a function from $B$ to $A$, then certainly $\mathrm{dom}(f^{-1}) = B$. This means that $f$ must be onto. If $f$ is not one-to-one, then $f(a_1) = f(a_2) = b$ for some $a_1, a_2 \in A$ and $b \in B$, where $a_1 \neq a_2$. But then $(b, a_1), (b, a_2) \in f^{-1}$, which cannot occur if $f^{-1}$ is a function. Consequently, if $f^{-1}$ is a function from $B$ to $A$, then $f$ must be one-to-one.

This leads us to the following theorem. In a proof, two basic facts are used repeatedly, namely

(1) *$f(a) = b$ if and only if $(a, b) \in f$ and*

(2) *if $f^{-1}$ is a function and $f(a) = b$, then $(b, a) \in f^{-1}$.*

**Theorem**

Let $f : A \to B$ be a function. Then the inverse relation $f^{-1}$ is a function from $B$ to $A$ if and only if $f$ is bijective. Furthermore, if $f$ is bijective, then $f^{-1}$ is also bijective.

**Theorem**

If $f : A \to B$ and $g : B \to A$ are two functions such that $g \circ f = i_A$ and $f \circ g = i_B$, then $f$ and $g$ are bijective and $g = f^{-1}$.

## Example 15

The function $f : \mathbf{R} - \{2\} \to \mathbf{R} - \{3\}$ defined by

$$f(x) = \frac{3x}{x-2}$$

is known to be bijective. Determine $f^{-1}(x)$, where $x \in \mathbf{R} - \{3\}$.

**Solution.** Since $\left(f \circ f^{-1}\right)(x) = x$ for all $x \in \mathbf{R} - \{3\}$, it follows that

$$\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = \frac{3f^{-1}(x)}{f^{-1}(x) - 2} = x.$$

Thus, $3f^{-1}(x) = x(f^{-1}(x) - 2)$ and $3f^{-1}(x) = xf^{-1}(x) - 2x$.

## Example 15 (continued)

Collecting the terms involving $f^{-1}(x)$ on the same side of the equation and then factoring out the term $f^{-1}(x)$, we have

$$xf^{-1}(x) - 3f^{-1}(x) = 2x;$$

so

$$f^{-1}(x)(x-3) = 2x.$$

Solving for $f^{-1}(x)$, we obtain

$$f^{-1}(x) = \frac{2x}{x-3}. \quad \blacklozenge$$

# Permutations

**Permutation**

A **permutation** of (or on) a nonempty set $A$ is a bijective function on $A$.

The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x - 5$ is a permutation of $\mathbf{R}$.

For $A = \{1, 2, 3\}$, let $f$ be a permutation of $A$. Then $f$ is completely determined once we know the images of 1, 2 and 3 under $f$. We saw that there are three possible choices for $f(1)$, two choices for $f(2)$ once $f(1)$ has been specified and one choice for $f(3)$ once $f(1)$ and $f(2)$ have been specified. From this, it follows that there are $3 \cdot 2 \cdot 1 = 3! = 6$ different permutations $f$ of the set $A = \{1, 2, 3\}$.

# Permutations

One of these functions is the identity function defined on $\{1, 2, 3\}$, which we denote by $\alpha_1$; that is,

$$\alpha_1 = \{(1, 1), (2, 2), (3, 3)\}.$$

Another permutation of $\{1, 2, 3\}$ is

$$\alpha_2 = \{(1, 1), (2, 3), (3, 2)\}.$$

There are other common ways to represent these permutations. A permutation of $\{1, 2, 3\}$ is also written as

$$\begin{pmatrix} 1 & 2 & 3 \\ - & - & - \end{pmatrix},$$

where the numbers immediately below $1, 2$ and $3$ are their images.

# Permutations

Hence, $\alpha_1, \alpha_2$ and the other four permutations of $\{1,2,3\}$ can be expressed as:

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\alpha_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \alpha_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

# Permutations

Since each permutation $\alpha_i$ $(1 \leq i \leq 6)$ is a bijective function from $\{1, 2, 3\}$ to $\{1, 2, 3\}$, it follows that the composition of any two permutations of $\{1, 2, 3\}$ is again a permutation of $\{1, 2, 3\}$. For example, let's consider

$$\alpha_2 \circ \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ - & - & - \end{pmatrix}.$$

Since $(\alpha_2 \circ \alpha_5)(1) = \alpha_2(\alpha_5(1)) = \alpha_2(2) = 3$, $(\alpha_2 \circ \alpha_5)(2) = 2$ and $(\alpha_2 \circ \alpha_5)(3) = 1$, it follows that

$$\alpha_2 \circ \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \alpha_3.$$

# Permutations

Composition of permutations on the same nonempty set $A$ is associative. Hence, for all integers $i, j, k \in \{1, 2, \cdots, 6\}$,

$$(\alpha_i \circ \alpha_j) \circ \alpha_k = \alpha_i \circ (\alpha_j \circ \alpha_k).$$

Since a permutation is a bijective function, each permutation has an inverse, which is also a permutation. Thus, for each $i$ $(1 \le i \le 6)$, $\alpha_i^{-1} = \alpha_j$ for some $j$ $(1 \le j \le 6)$. The inverse of a permutation can be found by interchanging the two rows and then re-ordering the columns so that the top row is in the natural order $1, 2, 3, \ldots$. Thus,

$$\alpha_5^{-1} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \alpha_6.$$

**Definition**

The set of all $n!$ permutations of the set $\{1, 2, , \cdots, n\}$ is denoted by $\mathcal{S}_n$. Thus,

$$\mathcal{S}_3 = \{\alpha_1, \alpha_2, \ldots, \alpha_6\}.$$

As we have seen with $\mathcal{S}_3$, the elements of $\mathcal{S}_n$ satisfy the properties of closure, associativity and the existence of inverses for every positive integer $n$.