9.1 PLANE CURVES AND PARAMETRIC EQUATIONS

We often find it convenient to describe the location of a point (x, y) in the plane in terms of a parameter. For instance, in tracking the movement of a satellite, we would naturally want to give its location in terms of time. In this way, we not only know the path it follows, but we also know *when* it passes through each point.

Given any pair of functions $x(t)$ and $y(t)$ defined on the same domain D, the equations

$$x = x(t), \quad y = y(t)$$

are called **parametric equations.** Notice that for each choice of t, the parametric equations specify a point $(x, y) = (x(t), y(t))$ in the xy-plane. The collection of all such points is called the **graph** of the parametric equations. In the case where $x(t)$ and $y(t)$ are continuous functions and D is an interval of the real line, the graph is a curve in the xy-plane, referred to as a **plane curve.**

The choice of the letter t to denote the independent variable (called the **parameter**) should make you think of *time,* which is often what the parameter represents. For instance, we might represent the position $(x(t), y(t))$ of a moving object as a function of the time t. In fact, you might recall that in section 5.5, we used a pair of equations of this type to describe two-dimensional projectile motion. In many applications, the parameter has an interpretation other than time; in others, it has no physical meaning at all. In general, the parameter can be *any* quantity that is convenient for describing the relationship between x and y. In example 1.1, we can simplify our discussion by eliminating the parameter.

EXAMPLE 1.1 Graphing a Plane Curve

Sketch the plane curve defined by the parametric equations $x = 6 - t^2$, $y = t/2$, for $-2 \le t \le 4$.

Solution In the accompanying table, we list a number of values of the parameter t and the corresponding values of x and y.

t	x	y
-2	2	-1
-1	5	$-\frac{1}{2}$
0	6	0
1	5	$\frac{1}{2}$
2	2	1
3	-3	$\frac{3}{2}$
4	-10	2

We have plotted these points and connected them with a smooth curve in Figure 9.1. You might also notice that we can easily eliminate the parameter here, by solving for t in terms of y. We have $t = 2y$, so that $x = 6 - 4y^2$. The graph of this last equation is a parabola opening to the left. However, the plane curve we're looking for is the portion of this parabola corresponding to $-2 \le t \le 4$. From the table, notice that this

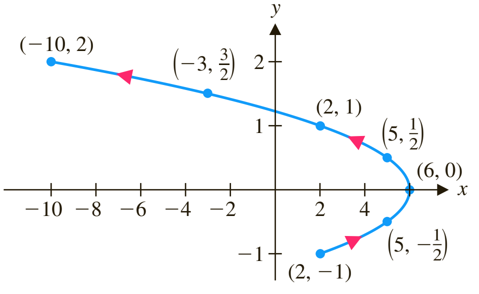


FIGURE 9.1
$x = 6 - t^2,\ y = \dfrac{t}{2},\ -2 \le t \le 4$

corresponds to $-1 \le y \le 2$, so that the plane curve is the portion of the parabola indicated in Figure 9.1, where we have also indicated a number of points on the curve. ■

You probably noticed the small arrows drawn on top of the plane curve in Figure 9.1. These indicate the **orientation** of the curve (i.e., the direction of increasing t). If t represents time and the curve represents the path of an object, the orientation indicates the direction followed by the object as it traverses the path, as in example 1.2.

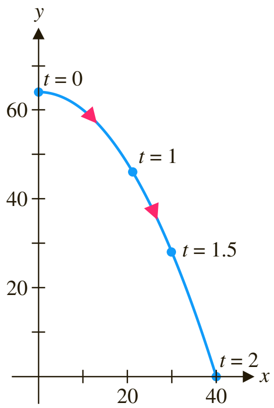


FIGURE 9.2
Path of projectile

EXAMPLE 1.2 The Path of a Projectile

Find the path of a projectile thrown horizontally with initial speed of 20 ft/s from a height of 64 feet.

Solution Following our discussion in section 5.5, the path is defined by the parametric equations

$$x = 20t, \quad y = 64 - 16t^2, \quad \text{for } 0 \le t \le 2,$$

where t represents time (in seconds). This describes the plane curve shown in Figure 9.2. Note that in this case, the orientation indicated in the graph gives the direction of motion. Although we could eliminate the parameter, as in example 1.1, the parametric equations provide us with more information. It is important to recognize that while the corresponding x-y equation $y = 64 - 16\left(\frac{x^2}{20^2}\right)$ describes the path followed by the projectile, the parametric equations provide us with additional information, as they also tell us *when* the object is located at a given point and indicate the *direction* of motion. We indicate the location of the projectile at several times in Figure 9.2. ■

Graphing calculators and computer algebra systems sketch a plane curve by plotting points corresponding to a large number of values of the parameter t and then connecting the plotted points with a curve. The appearance of the resulting graph depends greatly on the graphing window used *and also* on the particular choice of t-values. This can be seen in example 1.3.

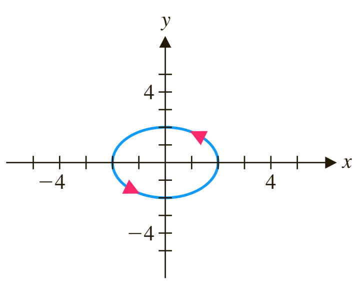


FIGURE 9.3a
$x = 2\cos t,\ y = 2\sin t$

EXAMPLE 1.3 Parametric Equations Involving Sines and Cosines

Sketch the plane curve defined by the parametric equations

$$x = 2\cos t, \quad y = 2\sin t, \quad \text{for (a) } 0 \le t \le 2\pi \text{ and (b) } 0 \le t \le \pi. \qquad (1.1)$$

Solution (a) The default graph produced by most graphing calculators looks something like the curve shown in Figure 9.3a (where we have added arrows indicating the orientation). With some thought, we can improve this sketch. First, notice that since $x = 2\cos t$, x ranges between -2 and 2. Similarly, y ranges between -2 and 2. Changing the graphing window to $-2.1 \le x \le 2.1$ and $-2.1 \le y \le 2.1$ produces the curve shown in Figure 9.3b, which is an improvement over Figure 9.3a. The curve still looks like an ellipse, but with some more thought we can identify it as a circle. Rather than eliminate the parameter by solving for t in terms of either x or y, instead notice from (1.1) that

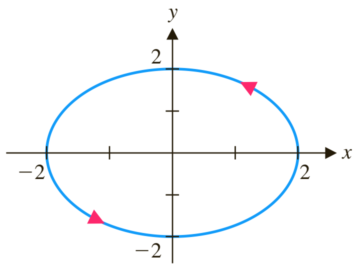


FIGURE 9.3b
$x = 2\cos t,\ y = 2\sin t$

$$x^2 + y^2 = 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4.$$

So, the plane curve lies on the circle of radius 2 centered at the origin. In fact, it's the whole circle, as we can see by recognizing what the parameter represents in this case. Recall from the definition of sine and cosine that if (x, y) is a point on the unit circle and θ is the angle from the positive x-axis to the line segment joining (x, y) and the origin, then we define $\cos\theta = x$ and $\sin\theta = y$. Since we have $x = 2\cos t$ and $y = 2\sin t$, the parameter t corresponds to the angle θ. Further, the curve is the entire circle of radius 2, traced out as the angle t ranges from 0 to 2π. A "square" graphing window is one with the same scale on the x- and y-axes (though not necessarily the same x and y ranges). Such a square window gives us the circle seen in Figure 9.3c.

(b) Finally, what would change if the domain were limited to $0 \le t \le \pi$? Since we've identified t as the angle as measured from the positive x-axis, it should be clear that you will now get the top half of the circle, as shown in Figure 9.3d.

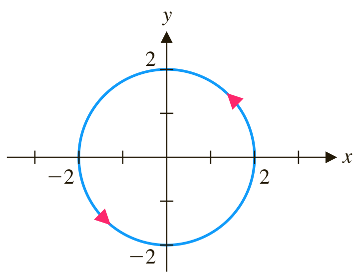


FIGURE 9.3c
A circle

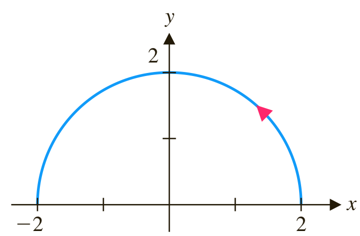


FIGURE 9.3d
Top semicircle

REMARK 1.1

To sketch a parametric graph on a CAS, you may need to write the equations in *vector format*. For instance, in the case of example 1.3, instead of entering $x = 2\cos t$ and $y = 2\sin t$, you would enter the ordered pair of functions $(2\cos t, 2\sin t)$.

Simple modifications to the parametric equations in example 1.3 will produce a variety of circles and ellipses. We explore this in example 1.4 and the exercises.

EXAMPLE 1.4 More Circles and Ellipses Defined by Parametric Equations

Identify the plane curves (a) $x = 2\cos t$, $y = 3\sin t$, (b) $x = 2 + 4\cos t$, $y = 3 + 4\sin t$ and (c) $x = 3\cos 2t$, $y = 3\sin 2t$, all for $0 \le t \le 2\pi$.

Solution A computer-generated sketch of (a) is shown in Figure 9.4a. It's difficult to determine from the sketch whether the curve is an ellipse or simply a distorted graph of a circle. You can rule out a circle, since the parametric equations produce x-values between -2 and 2 and y-values between -3 and 3. To verify that this is an ellipse, observe that

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{4\cos^2 t}{4} + \frac{9\sin^2 t}{9} = \cos^2 t + \sin^2 t = 1.$$

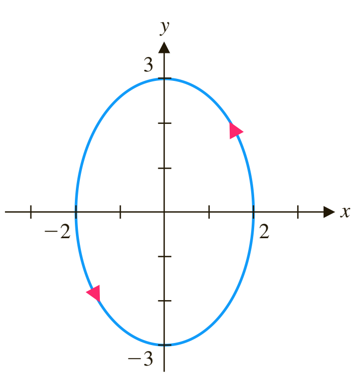


FIGURE 9.4a
$x = 2\cos t$, $y = 3\sin t$

A computer-generated sketch of (b) is shown in Figure 9.4b. You should verify that this is the circle $(x-2)^2 + (y-3)^2 = 16$. Finally, a computer sketch of (c) is shown in Figure 9.4c. You should verify that this is the circle $x^2 + y^2 = 9$. So, what is the role of the 2 in the argument of cosine and sine? If you sketched this on a calculator, you may have noticed that the circle was completed long before the calculator finished graphing. Because of the 2, a complete circle corresponds to $0 \le 2t \le 2\pi$ or $0 \le t \le \pi$. With the

REMARK 1.2

Look carefully at the plane curves in examples 1.3 and 1.4 until you can identify the roles of each of the constants in the equations $x = a + b\cos ct$, $y(t) = d + e\sin ct$. These interpretations are important in applications.

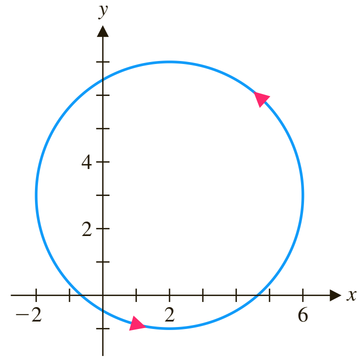


FIGURE 9.4b
$x = 2 + 4\cos t,\ y = 3 + 4\sin t$

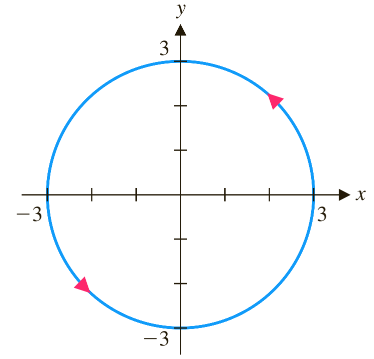


FIGURE 9.4c
$x = 3\cos 2t,\ y = 3\sin 2t$

domain $0 \le t \le 2\pi$, the circle is traced out twice. You might say that the factor of 2 in the argument doubles the speed with which the curve is traced. ■

REMARK 1.3

There are infinitely many choices of parameters that produce a given curve. For instance, you can verify that

$$x = -2 + 3t, \quad y = -3 + 5t,$$
$$\text{for } 1 \le t \le 2$$

and

$$x = t, \quad y = \frac{1 + 5t}{3},$$
$$\text{for } 1 \le t \le 4$$

both produce the line segment from example 1.5. We say that each of these pairs of parametric equations is a different **parameterization** of the curve.

In example 1.5, we see how to find parametric equations for a line segment.

EXAMPLE 1.5 Parametric Equations for a Line Segment

Find parametric equations for the line segment joining the points (1, 2) and (4, 7).

Solution For a line segment, notice that the parametric equations can be chosen to be linear functions. That is,

$$x = a + bt, \quad y = c + dt,$$

for some constants *a, b, c* and *d*. (Eliminate the parameter t to see why this generates a line.) The simplest way to choose these constants is to have $t = 0$ correspond to the starting point (1, 2). Note that if $t = 0$, the equations reduce to $x = a$ and $y = c$. To start our segment at $x = 1$ and $y = 2$, we set $a = 1$ and $c = 2$. Now note that with $t = 1$, the equations are $x = a + b$ and $y = c + d$. To produce the endpoint (4, 7), we must have $a + b = 4$ and $c + d = 7$. With $a = 1$ and $c = 2$, solve to get $b = 3$ and $d = 5$. We now have that

$$x = 1 + 3t, \quad y = 2 + 5t, \quad \text{for } 0 \le t \le 1$$

is a pair of parametric equations describing the line segment. ■

In general, for parametric equations of the form $x = a + bt$, $y = c + dt$, notice that you can always choose a and c to be the x- and y-coordinates, respectively, of the starting point (since $x = a$, $y = b$ corresponds to $t = 0$). Then b is the difference in x-coordinates (endpoint minus starting point) and d is the difference in y-coordinates. With these choices, the line segment is always sketched out for $0 \le t \le 1$.

As we illustrate in example 1.6, every equation of the form $y = f(x)$ can be simply expressed using parametric equations.

EXAMPLE 1.6 Parametric Equations from an x-y Equation

Find parametric equations for the portion of the parabola $y = x^2$ from $(-1, 1)$ to (3, 9).

Solution Any equation of the form $y = f(x)$ can be converted to parametric form simply by defining $t = x$. Here, this gives us $y = x^2 = t^2$, so that

$$x = t, \quad y = t^2, \quad \text{for } -1 \le t \le 3,$$

is a parametric representation of the curve. (Of course, you can use the letter x as the parameter instead of the letter t, if you prefer.) ■

Besides indicating an orientation, parametric representations of curves often also carry with them a built-in restriction on the portion of the curve included, as we see in example 1.7.

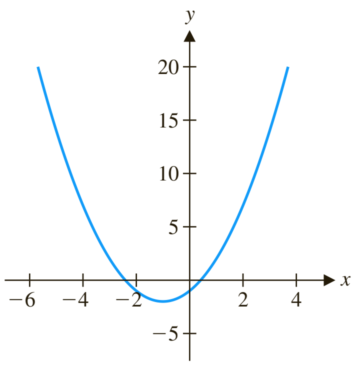


FIGURE 9.5a
$y = (x+1)^2 - 2$

EXAMPLE 1.7 Parametric Representations of a Curve with a Subtle Difference

Sketch the plane curves (a) $x = t - 1$, $y = t^2 - 2$ and (b) $x = t^2 - 1$, $y = t^4 - 2$.

Solution Since there is no restriction placed on t, we can assume that t can be any real number. Eliminating the parameter in (a), we get $t = x + 1$, so that the parametric equations in (a) correspond to the parabola $y = (x+1)^2 - 2$, shown in Figure 9.5a. Notice that the graph includes the entire parabola, since t and hence, $x = t - 1$ can be any real number. (If your calculator sketch doesn't show both sides of the parabola, adjust the range of t-values in the plot.) The importance of this check is shown by (b). When we eliminate the parameter, we get $t^2 = x + 1$ and so, $y = (x+1)^2 - 2$. This gives the same parabola as in (a). However, the initial computer sketch of the parametric equations shown in Figure 9.5b shows only the right half of the parabola. To verify that this is correct, note that since $x = t^2 - 1$, we have that $x \ge -1$ for every real number t. Therefore, the curve is only the right half of the parabola $y = (x+1)^2 - 2$, as shown. ■

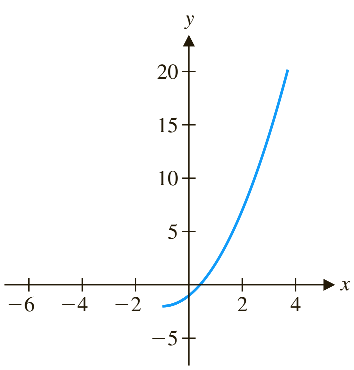


FIGURE 9.5b
$x = t^2 - 1,\ y = t^4 - 2$

Many plane curves described parametrically are unlike anything you've seen so far in your study of calculus. Many of these are difficult to draw by hand, but can be easily plotted with a graphing calculator or CAS.

EXAMPLE 1.8 Some Unusual Plane Curves

Sketch the plane curves (a) $x = t^2 - 2$, $y = t^3 - t$ and (b) $x = t^3 - t$, $y = t^4 - 5t^2 + 4$.

Solution A sketch of (a) is shown in Figure 9.6a. From the vertical line test, this is not the graph of any function. Further, converting to an x-y equation here is messy and not particularly helpful. (Try this to see why.) However, examine the parametric equations to see if important portions of the graph have been left out (e.g., is there supposed to be anything to the left of $x = -2$?). Here, $x = t^2 - 2 \ge -2$ for all t and $y = t^3 - t$ has no maximum or minimum (think about why). It seems that most of the graph is indeed shown in Figure 9.6a.

A computer sketch of (b) is shown in Figure 9.6b. Again, this is not a familiar graph. To get an idea of the scope of the graph, note that $x = t^3 - t$ has no maximum or minimum. To find the minimum of $y = t^4 - 5t^2 + 4$, note that critical numbers are at

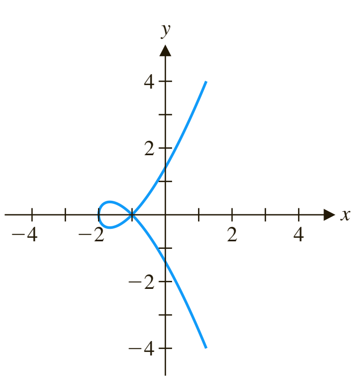


FIGURE 9.6a
$x = t^2 - 2,\ y = t^3 - t$

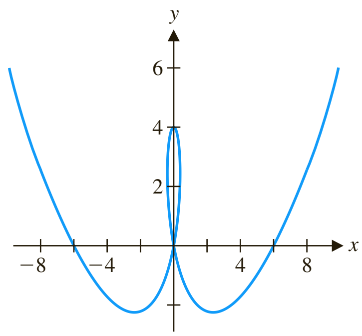


FIGURE 9.6b
$x = t^3 - t,\ y = t^4 - 5t^2 + 4$

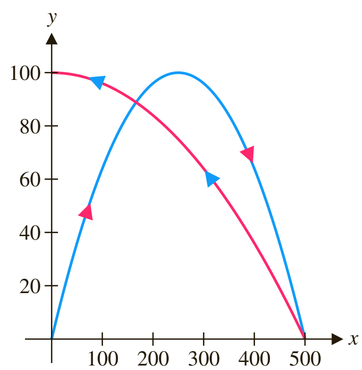


FIGURE 9.7a
Missile flight paths

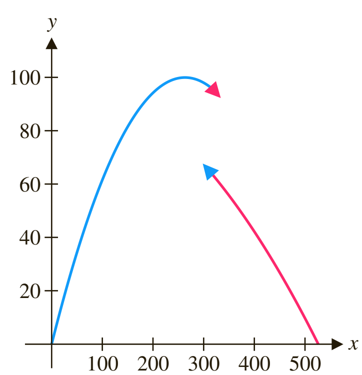


FIGURE 9.7b
Missile flight paths

$t = 0$ and $t = \pm\sqrt{\frac{5}{2}}$ with corresponding function values 4 and $-\frac{9}{4}$, respectively. You should conclude that $y \geq -\frac{9}{4}$, as indicated in Figure 9.6b. ■

You should now have some idea of the flexibility of parametric equations. Quite significantly, a large number of applications translate simply into parametric equations.

Bear in mind that parametric equations communicate more information than do the corresponding x-y equations. We illustrate this with example 1.9.

EXAMPLE 1.9 Intercepting a Missile in Flight

Suppose that a missile is fired toward your location from 500 miles away and follows a flight path given by the parametric equations

$$x = 100t, \quad y = 80t - 16t^2, \quad \text{for } 0 \leq t \leq 5.$$

Two minutes later, you fire an interceptor missile from your location following the flight path

$$x = 500 - 200(t-2), \quad y = 80(t-2) - 16(t-2)^2, \quad \text{for } 2 \leq t \leq 7.$$

Determine whether the interceptor missile hits its target.

Solution In Figure 9.7a, we have plotted the flight paths for both missiles simultaneously. The two paths clearly intersect, but this does *not* necessarily mean that the two missiles collide. For that to happen, they need to be at the same point *at the same time*. To determine whether there are any values of t for which both paths are simultaneously passing through the same point, we set the two x-values equal:

$$100t = 500 - 200(t-2)$$

and obtain one solution: $t = 3$. Note that this simply says that the two missiles have the same x-coordinate when $t = 3$. Unfortunately, the y-coordinates are not the same here, since when $t = 3$, we have

$$80t - 16t^2 = 96 \quad \text{but} \quad 80(t-2) - 16(t-2)^2 = 64.$$

You can see this graphically by plotting the two paths simultaneously for $0 \leq t \leq 3$ only, as we have done in Figure 9.7b. From the graph, you can clearly see that the two missiles pass one another without colliding. So, by the time the interceptor missile intersects the flight path of the incoming missile, it is long gone! Another very nice way to observe this behavior is to plot the two sets of parametric equations on your graphing calculator in "simultaneous plot" mode. With this, you can animate the flight paths and watch the missiles pass by one another. ■

BEYOND FORMULAS

When thinking of parametric equations, it is often helpful to think of t as representing time and the graph as representing the position of a moving particle. It is important to realize that the parameter can be anything. For example, in equations of circles and ellipses, the parameter may represent the angle as you rotate around the oval. Allowing the parameter to change from problem to problem gives us an incredible flexibility to describe the relationship between x and y in the most convenient way possible.

EXERCISES 9.1

WRITING EXERCISES

1. Interpret in words the roles of each of the constants in the parametric equations $\begin{cases} x = a_1 + b_1 \cos(ct) \\ y = a_2 + b_2 \sin(ct) \end{cases}$.
2. An algorithm was given in example 1.5 for finding parametric equations of a line segment. Discuss the advantages that this method has over the other methods presented in remark 1.3.
3. As indicated in remark 1.3, a given curve can be described by numerous sets of parametric equations. Explain why several different equations can all be correct. (Hint: Emphasize the fact that t is a dummy variable.)
4. In example 1.9, you saw that missiles don't collide even though their paths intersect. If you wanted to determine the intersection point of the graphs, explain why you would need to solve for values s and t (possibly different) such that $100t = 500 - 200(s-2)$ and $80t - 16t^2 = 80(s-2) - 16(s-2)^2$.

In exercises 1–10, sketch the plane curve defined by the given parametric equations and find a corresponding x-y equation for the curve.

1. $\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}$
2. $\begin{cases} x = 1 + 2\cos t \\ y = -2 + 2\sin t \end{cases}$
3. $\begin{cases} x = -1 + 2t \\ y = 3t \end{cases}$
4. $\begin{cases} x = 4 + 3t \\ y = 2 - 4t \end{cases}$
5. $\begin{cases} x = 1 + t \\ y = t^2 + 2 \end{cases}$
6. $\begin{cases} x = 2 - t \\ y = t^2 + 1 \end{cases}$
7. $\begin{cases} x = t^2 - 1 \\ y = 2t \end{cases}$
8. $\begin{cases} x = t^2 - 1 \\ y = t^2 + 1 \end{cases}$
9. $\begin{cases} x = \cos t \\ y = 3\cos t - 1 \end{cases}$
10. $\begin{cases} x = 2\sin t \\ y = 3\cos t \end{cases}$

In exercises 11–20, use your CAS or graphing calculator to sketch the plane curves defined by the given parametric equations.

11. $\begin{cases} x = t^3 - 2t \\ y = t^2 - 3 \end{cases}$
12. $\begin{cases} x = t^3 - 2t \\ y = t^2 - 3t \end{cases}$
13. $\begin{cases} x = t^2 - 1 \\ y = t^4 - 4t \end{cases}$
14. $\begin{cases} x = t^2 - 1 \\ y = t^4 - 4t^2 \end{cases}$
15. $\begin{cases} x = \cos 2t \\ y = \sin 7t \end{cases}$
16. $\begin{cases} x = \cos 2t \\ y = \sin \pi t \end{cases}$
17. $\begin{cases} x = 3\cos 2t + \sin 5t \\ y = 3\sin 2t + \cos 5t \end{cases}$
18. $\begin{cases} x = 3\cos 2t + \sin 6t \\ y = 3\sin 2t + \cos 6t \end{cases}$
19. $\begin{cases} x = e^t \\ y = e^{-2t} \end{cases}$
20. $\begin{cases} x = e^t \\ y = e^{2t} \end{cases}$

21. Conjecture the difference between the graphs of $\begin{cases} x = \cos 2t \\ y = \sin kt \end{cases}$, where k is an integer compared to when k is an irrational number. (Hint: Use exercises 15 and 16 and try $k = 3$, $k = \sqrt{3}$ and other values.)
22. Compare the graphs of $\begin{cases} x = \cos 3t \\ y = \sin kt \end{cases}$ for $k = 1, k = 2, k = 3, k = 4$ and $k = 5$, and describe the role that k plays in the graph.
23. Compare the graphs of $\begin{cases} x = \cos t - \frac{1}{2}\cos kt \\ y = \sin t - \frac{1}{2}\sin kt \end{cases}$ for $k = 2$, $k = 3, k = 4$ and $k = 5$, and describe the role that k plays in the graph.
24. Describe the role that r plays in the graph of $\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$ and then describe how to sketch the graph of $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}$.

In exercises 25–30, match the parametric equations with the corresponding plane curve displayed in Figures A–F. Give reasons for your choices.

25. $\begin{cases} x = t^2 - 1 \\ y = t^4 \end{cases}$
26. $\begin{cases} x = t - 1 \\ y = t^3 \end{cases}$
27. $\begin{cases} x = t^2 - 1 \\ y = \sin t \end{cases}$
28. $\begin{cases} x = t^2 - 1 \\ y = \sin 2t \end{cases}$
29. $\begin{cases} x = \cos 3t \\ y = \sin 2t \end{cases}$
30. $\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}$

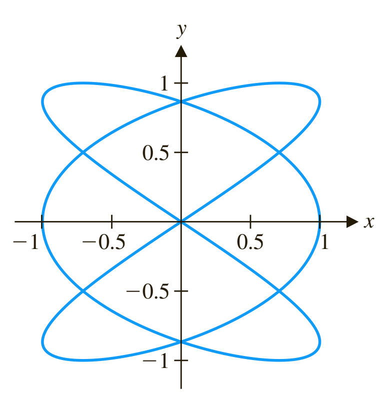


FIGURE A

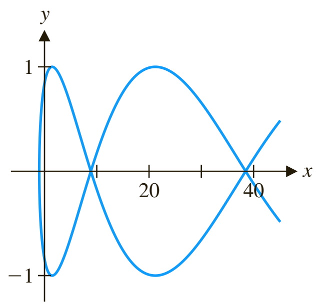


FIGURE B

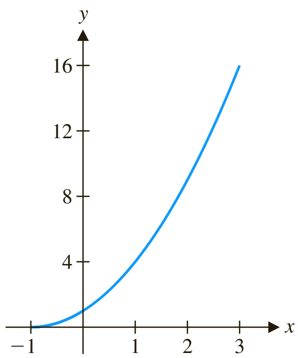


FIGURE C

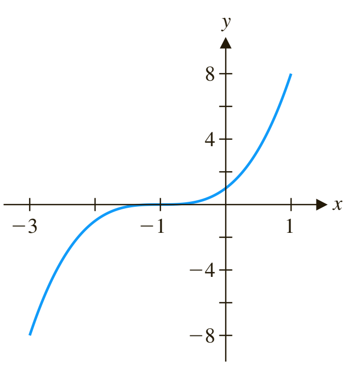


FIGURE D

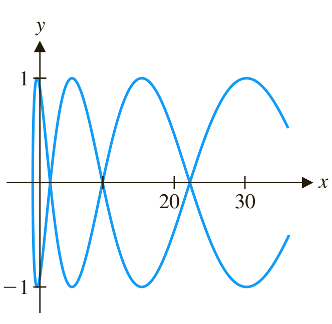


FIGURE E

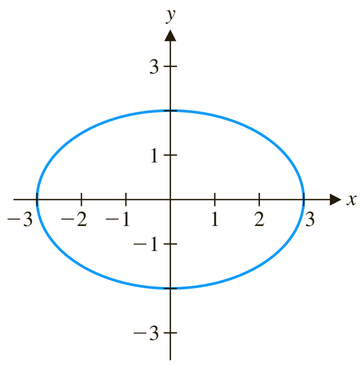


FIGURE F

In exercises 31–40, find parametric equations describing the given curve.

31. The line segment from (0, 1) to (3, 4)

32. The line segment from (3, 1) to (1, 3)

33. The line segment from (−2, 4) to (6, 1)

34. The line segment from (4, −2) to (2, −1)

35. The portion of the parabola $y = x^2 + 1$ from (1, 2) to (2, 5)

36. The portion of the parabola $y = 2x^2 - 1$ from (0, −1) to (2, 7)

37. The portion of the parabola $y = 2 - x^2$ from (2, −2) to (0, 2)

38. The portion of the parabola $y = x^2 + 1$ from (1, 2) to (−1, 2)

39. The circle of radius 3 centered at (2, 1), drawn counterclockwise

40. The circle of radius 5 centered at (−1, 3), drawn counterclockwise

In exercises 41–44, find all points of intersection of the two curves.

41. $\begin{cases} x = t \\ y = t^2 - 1 \end{cases}$ and $\begin{cases} x = 1 + s \\ y = 4 - s \end{cases}$

42. $\begin{cases} x = t^2 \\ y = t + 1 \end{cases}$ and $\begin{cases} x = 2 + s \\ y = 1 - s \end{cases}$

43. $\begin{cases} x = t + 3 \\ y = t^2 \end{cases}$ and $\begin{cases} x = 1 + s \\ y = 2 - s \end{cases}$

44. $\begin{cases} x = t^2 + 3 \\ y = t^3 + t \end{cases}$ and $\begin{cases} x = 2 + s \\ y = 1 - s \end{cases}$

45. Rework example 1.9 with the interceptor missile following the flight path $x = 500 - 500(t - 2)$ and $y = 208(t - 2) - 16(t - 2)^2$.

46. Rework example 1.9 with the interceptor missile following the flight path $x = 500 - 100t$ and $y = 80t - 16t^2$.

47. In example 1.9 and exercise 45, explain why the 2 in the term $t - 2$ represents the time delay between the launches of the two missiles. For the equations in example 1.9, find a value of the time delay such that the two missiles do collide.

48. Explain why the missile path in exercise 46 must produce a collision (compare the y-equations) but is unrealistic.

Exercises 49–56 explore the sound barrier problem discussed in the chapter introduction. Define 1 unit to be the distance traveled by sound in 1 second.

49. Suppose a sound wave is emitted from the origin at time 0. After t seconds ($t > 0$), explain why the position in units of the sound wave is modeled by $x = t\cos\theta$ and $y = t\sin\theta$, where the dummy parameter θ has range $0 \le \theta \le 2\pi$.

50. Find parametric equations as in exercise 49 for the position at time t seconds ($t > 0$) of a sound wave emitted at time c seconds from the point (a, b).

51. Suppose that a jet has speed 0.8 unit per second (i.e., Mach 0.8) with position function $x(t) = 0.8t$ and $y(t) = 0$. To model the position at time $t = 5$ seconds of various sound waves emitted by the jet, do the following on one set of axes. (a) Graph the position after 5 seconds of the sound wave emitted from (0, 0); (b) graph the position after 4 seconds of the sound wave emitted from (0.8, 0); (c) graph the position after 3 seconds of the sound wave emitted from (1.6, 0); (d) graph the position after 2 seconds of the sound wave emitted from (2.4, 0); (e) graph the position after 1 second of the sound wave emitted from (3.2, 0); (f) mark the position of the jet at time $t = 5$.

52. Repeat exercise 51 for a jet with speed 1.0 unit per second (Mach 1). You should notice that the sound waves all intersect at the jet's location. This is the "sound barrier" that must be broken.

53. Repeat exercise 51 for a jet with speed 1.4 units per second (Mach 1.4).

54. In exercise 53, you should find that the sound waves intersect each other. The intersections form the "shock wave" that we hear as a sonic boom. Theoretically, the angle θ between the shock wave and the x-axis satisfies the equation $\sin\theta = \frac{1}{m}$, where m is the Mach speed of the jet. Show that for $m = 1.4$, the theoretical shock wave is formed by the lines $x(t) = 7 - \sqrt{0.96}t$, $y(t) = t$ and $x(t) = 7 - \sqrt{0.96}t$, $y(t) = -t$. Superimpose these lines onto the graph of exercise 53.

55. In exercise 54, the shock wave of a jet at Mach 1.4 is modeled by two lines. Argue that in three dimensions, the shock wave has circular cross sections. Describe the three-dimensional figure formed by revolving the lines in exercise 54 about the x-axis.

56. If a pebble is dropped into water, a wave spreads out in an expanding circle. Let v be the speed of the propagation of the wave. If a boat moves through this water with speed $1.4v$, argue that the boat's wake will be described by the graphs of exercises 54 and 55. Graph the wake of a boat with speed $1.6v$.

Exercises 57–62 show that a celestial object can incorrectly appear to be moving faster than the speed of light.

57. A bright object is at position $(0, D)$ at time 0, where D is a very large positive number. The object moves toward the positive x-axis with constant speed v at an angle θ from the vertical. Find parametric equations for the position of the object at time t.

58. For the object of exercise 57, let $s(t)$ be the distance from the object to the origin at time t. Then $L(t) = \dfrac{s(t)}{c}$ gives the amount of time it takes for light emitted by the object at time t to reach the origin. Show that $L'(t) = \dfrac{1}{c}\dfrac{v^2t - Dv\cos\theta}{s(t)}$.

59. An observer stands at the origin and tracks the horizontal movement of the object in exercises 57 and 58. As computed in exercise 58, light received at time T was emitted by the object at time t, where $T = t + L(t)$. Similarly, light received at time $T + \Delta T$ was emitted at time $t + dt$, where typically $dt \ne \Delta T$. The apparent x-coordinate of the object at time T is $x_a(T) = x(t)$. The apparent horizontal speed of the object at time T as measured by the observer is $h(T) = \lim_{\Delta T \to 0} \dfrac{x_a(T + \Delta T) - x_a(T)}{\Delta T}$. Tracing back to time t, show that $h(t) = \lim_{dt \to 0} \dfrac{x(t+dt) - x(t)}{\Delta T} = \dfrac{v\sin\theta}{T'(t)} = \dfrac{v\sin\theta}{1 + L'(t)}$.

60. In exercise 59, show that $h(0) = \dfrac{cv\sin\theta}{c - v\cos\theta}$.

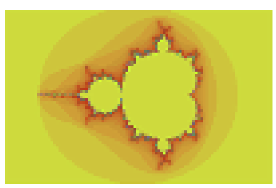
61. For the moving object of exercises 57–60, show that for a constant speed v, the maximum apparent horizontal speed $h(10)$ occurs when the object moves at an angle with $\cos\theta = \dfrac{v}{c}$. Find the maximum speed in terms of v and the contraction factor $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$.

62. For the moving object of exercises 57–61, show that as v approaches c, the apparent horizontal speed can exceed c, causing the observer to measure an object moving faster than the speed of light! As v approaches c, show that the angle producing the maximum apparent horizontal speed decreases to 0. Discuss why this is paradoxical.

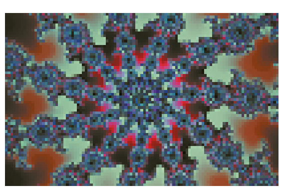
63. Compare the graphs of $\begin{cases} x = \cos 2t \\ y = \sin t \end{cases}$ and $\begin{cases} x = \cos t \\ y = \sin 2t \end{cases}$. Use the identities $\cos 2t = \cos^2 t - \sin^2 t$ and $\sin 2t = 2\cos t \sin t$ to find x-y equations for each graph.

64. Sketch the graph of $\begin{cases} x = \cosh t \\ y = \sinh t \end{cases}$. Use the identity $\cosh^2 t - \sinh^2 t = 1$ to find an x-y equation for the graph. Explain where the "hyperbolic" in hyperbolic sine and hyperbolic cosine might come from.

65. Sketch the graph of $\begin{cases} x = \frac{1}{2}\cos t - \frac{1}{4}\cos 2t \\ y = \frac{1}{2}\sin t - \frac{1}{4}\sin 2t \end{cases}$. This heart-shaped region is the largest feature of the **Mandelbrot set,** one of the most famous mathematical sets. Portions of the Mandelbrot set have been turned into colorful T-shirts and posters that you may have seen.



Mandelbrot set



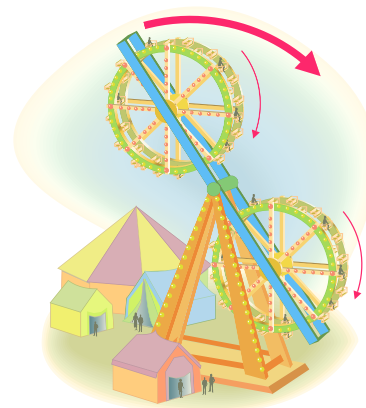
Mandelbrot zoom

To progress further on a sketch of the Mandelbrot set, add the circle $\begin{cases} x = -1 + \frac{1}{4}\cos t \\ y = \frac{1}{4}\sin t \end{cases}$ to your initial sketch.

66. Determine parametric equations for the curves defined by $x^{2n} + y^{2n} = r^{2n}$ for integers n. (Hint: Start with $n = 1$, $x^2 + y^2 = r^2$, then think of the general equation as $(x^n)^2 + (y^n)^2 = r^{2n}$.) Sketch the graphs for $n = 1$, $n = 2$ and $n = 3$, and predict what the curve will look like for large values of n.

EXPLORATORY EXERCISES

1. Many carnivals have a version of the double Ferris wheel. A large central arm rotates clockwise. At each end of the central arm is a Ferris wheel that rotates clockwise around the arm. Assume that the central arm has length 200 feet and rotates about its center. Also assume that the wheels have radius 40 feet and rotate at the same speed as the central arm. Find parametric equations for the position of a rider and graph the rider's path. Adjust the speed of rotation of the wheels to improve the ride.



2. The Flying Zucchini Circus Troupe has a human cannonball act, shooting a performer from a cannon into a specially padded seat of a turning Ferris wheel. The Ferris wheel has a radius of 40 feet and rotates counterclockwise at one revolution per minute. The special seat starts at ground level. Carefully explain why parametric equations for the seat are $\begin{cases} x = 40\cos(\frac{\pi}{30}t - \frac{\pi}{2}) \\ y = 40 + 40\sin(\frac{\pi}{30}t - \frac{\pi}{2}) \end{cases}$. The cannon is located 200 feet left of the Ferris wheel with the muzzle 10 feet above ground. The performer is launched 35 seconds after the wheel starts turning with an initial velocity of 100 ft/s at an angle of $\frac{\pi}{5}$ above the horizontal. Carefully explain why parametric equations for the human cannonball are $\begin{cases} x = (100\cos\frac{\pi}{5})(t - 35) - 200 \\ y = -16(t - 35)^2 + (100\sin\frac{\pi}{5})(t - 35) + 10 \end{cases}$ $(t \geq 35)$. Determine whether the act is safe or the Flying Zucchini comes down squash.

3. Rework exercise 2 with initial velocity 135 ft/s, launch angle 30° and a 27-second delay. How close does the Flying Zucchini get to the special seat? Given that a Ferris wheel seat actually has height, width, and depth, do you think that this is close enough? Repeat with (a) initial velocity 75 ft/s, launch angle 47° and 47.25-second delay; (b) initial velocity 118 ft/s, launch angle 35° and 28-second delay. Develop criteria for a safe and exciting human cannonball act. Consider each of the following: Should the launch velocity be large or small? Should the seat be high or low when the cannonball lands? Should the human have a positive or negative vertical velocity at landing? How close (vertically and horizontally) should the human need to get to the center of the seat? Based on your criteria, which of the launches in this exercise is the best? Find an initial velocity, launch angle and launch delay that is better.

9.2 CALCULUS AND PARAMETRIC EQUATIONS

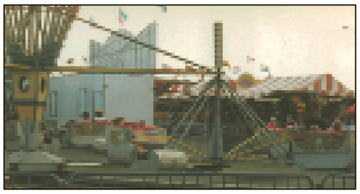


FIGURE 9.8a
The Scrambler

The Scrambler is a popular carnival ride consisting of two sets of rotating arms (see Figure 9.8a). Suppose that the inner arms have length 2 and rotate counterclockwise. In this case, we can describe the location (x_i, y_i) of the end of one of the inner arms by the parametric equations $x_i = 2\cos t$, $y_i = 2\sin t$. At the end of each inner arm, a set of outer arms rotate clockwise at roughly twice the speed. If the outer arms have length 1, parametric equations describing the outer arm rotation are $x_o = \sin 2t$, $y_o = \cos 2t$. Here, the reversal of sine and cosine terms indicates that the rotation is clockwise and the factor of 2 inside the sine and cosine terms indicates that the speed of the rotation is double that of the inner arms. The position of a person riding the Scrambler is the sum of the two component motions; that is,

$$x = 2\cos t + \sin 2t, \quad y = 2\sin t + \cos 2t.$$

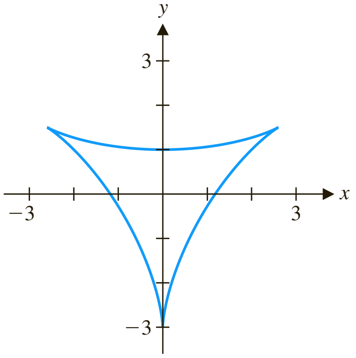


FIGURE 9.8b
Path of a Scrambler rider

The graph of these parametric equations is shown in Figure 9.8b. Passengers on the Scrambler feel like they rapidly accelerate to the outside of the ride, momentarily stop, then change direction and accelerate to a different point on the outside of the ride. Figure 9.8b suggests that this is an accurate description of the ride, but we need to develop the calculus of parametric equations to determine whether the riders actually come to a complete stop.

Our initial aim is to find a way to determine the slopes of tangent lines to curves that are defined parametrically. First, recall that for a differentiable function $y = f(x)$, the slope of the tangent line at the point $x = a$ is given by $f'(a)$. Written in Leibniz notation, the slope is $\frac{dy}{dx}(a)$. In the case of the Scrambler ride, both x and y are functions of the parameter t. Notice that if $x = x(t)$ and $y = y(t)$ both have derivatives that are continuous at $t = c$, the chain rule gives us

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

REMARK 2.1

Be careful with how you interpret equation (2.1). The primes on the right side of the equation refer to derivatives with respect to the parameter t. We recommend that you (at least initially) use the Leibniz notation, which also gives you a simple way to accurately remember the chain rule.

As long as $\frac{dx}{dt}(c) \neq 0$, we then have

$$\boxed{\frac{dy}{dx}(a) = \frac{\frac{dy}{dt}(c)}{\frac{dx}{dt}(c)} = \frac{y'(c)}{x'(c)},} \tag{2.1}$$

where $a = f(c)$. In the case where $x'(c) = y'(c) = 0$, we define

$$\boxed{\frac{dy}{dx}(a) = \lim_{t\to c}\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \lim_{t\to c}\frac{y'(t)}{x'(t)},} \tag{2.2}$$

provided the limit exists.

CAUTION

Look carefully at (2.3) and convince yourself that

$$\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}.$$

Equating these two expressions is a common error. You should be careful to avoid this trap.

We can use (2.1) to calculate second (as well as higher order) derivatives. Notice that if we replace y by $\frac{dy}{dx}$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}. \tag{2.3}$$

EXAMPLE 2.1 Slopes of Tangent Lines to the Path of the Scrambler

Find the slope of the tangent line to the path of the Scrambler $x = 2\cos t + \sin 2t$, $y = 2\sin t + \cos 2t$ at (a) $t = 0$; (b) $t = \frac{\pi}{4}$ and (c) the point $(0, -3)$.

Solution (a) First, note that

$$\frac{dx}{dt} = -2\sin t + 2\cos 2t \quad \text{and} \quad \frac{dy}{dt} = 2\cos t - 2\sin 2t.$$

From (2.1), the slope of the tangent line at $t = 0$ is then

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{\frac{dy}{dt}(0)}{\frac{dx}{dt}(0)} = \frac{2\cos 0 - 2\sin 0}{-2\sin 0 + 2\cos 0} = 1.$$

(b) The slope of the tangent line at $t = \frac{\pi}{4}$ is

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{\frac{dy}{dt}\left(\frac{\pi}{4}\right)}{\frac{dx}{dt}\left(\frac{\pi}{4}\right)} = \frac{2\cos\frac{\pi}{4} - 2\sin\frac{\pi}{2}}{-2\sin\frac{\pi}{4} + 2\cos\frac{\pi}{2}} = \frac{\sqrt{2}-2}{-\sqrt{2}}.$$

(c) To determine the slope at the point $(0, -3)$, we must first determine a value of t that corresponds to the point. In this case, notice that $t = 3\pi/2$ gives $x = 0$ and $y = -3$. Here, we have

$$\frac{dx}{dt}\left(\frac{3\pi}{2}\right) = \frac{dy}{dt}\left(\frac{3\pi}{2}\right) = 0$$

and consequently, we must use (2.2) to compute $\frac{dy}{dx}$. Since the limit has the indeterminate form $\frac{0}{0}$, we use l'Hôpital's Rule, to get

$$\frac{dy}{dx}\left(\frac{3\pi}{2}\right) = \lim_{t\to 3\pi/2} \frac{2\cos t - 2\sin 2t}{-2\sin t + 2\cos 2t} = \lim_{t\to 3\pi/2} \frac{-2\sin t - 4\cos 2t}{-2\cos t - 4\sin 2t},$$

which does not exist, since the limit in the numerator is 6 and the limit in the denominator is 0. This says that the slope of the tangent line at $t = 3\pi/2$ is undefined. In Figure 9.9, we have drawn in the tangent lines at $t = 0$, $\pi/4$ and $3\pi/2$. Notice that the tangent line at the point $(0, -3)$ is vertical. ■

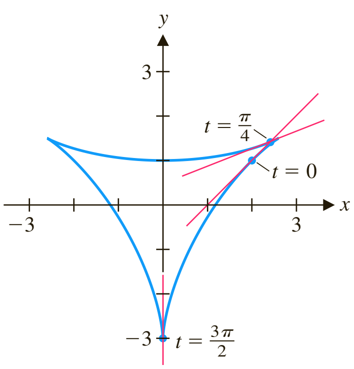


FIGURE 9.9
Tangent lines to the Scrambler path

For the passenger on the Scrambler of example 2.1, notice that the slope of the tangent line indicates the *direction* of motion and does not correspond to speed, which we discuss shortly.

Finding slopes of tangent lines can help us identify many points of interest.

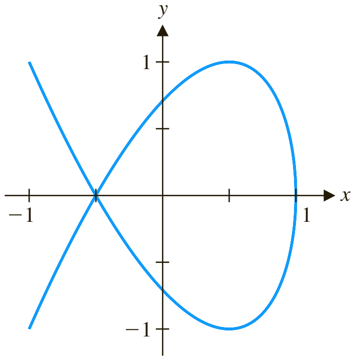


FIGURE 9.10
$x = \cos 2t,\ y = \sin 3t$

EXAMPLE 2.2 Finding Vertical and Horizontal Tangent Lines

Identify all points at which the plane curve $x = \cos 2t$, $y = \sin 3t$ has a horizontal or vertical tangent line.

Solution A sketch of the curve is shown in Figure 9.10. There appear to be two locations (the top and bottom of the bow) with horizontal tangent lines and one point (the far right edge of the bow) with a vertical tangent line. Recall that horizontal tangent lines occur where $\frac{dy}{dx} = 0$. From (2.1), we then have $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = 0$, which can occur only when

$$0 = y'(t) = 3\cos 3t,$$

provided that $x'(t) = -2\sin 2t \neq 0$ for the same value of t. Since $\cos\theta = 0$ only when θ is an odd multiple of $\frac{\pi}{2}$, we have that $y'(t) = 3\cos 3t = 0$, only when $3t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ and so, $t = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \ldots$. The corresponding points on the curve are then

$$\left(x\left(\frac{\pi}{6}\right), y\left(\frac{\pi}{6}\right)\right) = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{2}\right) = \left(\frac{1}{2}, 1\right),$$

$$\left(x\left(\frac{3\pi}{6}\right), y\left(\frac{3\pi}{6}\right)\right) = \left(\cos\pi, \sin\frac{3\pi}{2}\right) = (-1, -1),$$

$$\left(x\left(\frac{7\pi}{6}\right), y\left(\frac{7\pi}{6}\right)\right) = \left(\cos\frac{7\pi}{3}, \sin\frac{7\pi}{2}\right) = \left(\frac{1}{2}, -1\right)$$

and
$$\left(x\left(\frac{9\pi}{6}\right), y\left(\frac{9\pi}{6}\right)\right) = \left(\cos 3\pi, \sin\frac{9\pi}{2}\right) = (-1, 1).$$

Note that $t = \frac{5\pi}{6}$ and $t = \frac{11\pi}{6}$ reproduce the first and third points, respectively, and so on. The points $(\frac{1}{2}, 1)$ and $(\frac{1}{2}, -1)$ are on the top and bottom of the bow, respectively, where there clearly are horizontal tangents. The points $(-1, -1)$ and $(-1, 1)$ should not seem quite right, though. These points are on the extreme ends of the bow and certainly don't look like they have vertical or horizontal tangents. In fact, they don't. Notice that at both $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$, we have $x'(t) = y'(t) = 0$ and so, the slope must be computed as a limit using (2.2). We leave it as an exercise to show that the slopes at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ are $\frac{9}{4}$ and $-\frac{9}{4}$, respectively.

To find points where there is a vertical tangent, we need to see where $x'(t) = 0$ but $y'(t) \neq 0$. Setting $0 = x'(t) = -2\sin 2t$, we get $\sin 2t = 0$, which occurs if $2t = 0, \pi, 2\pi, \ldots$ or $t = 0, \frac{\pi}{2}, \pi, \ldots$. The corresponding points are

$$(x(0), y(0)) = (\cos 0, \sin 0) = (1, 0),$$

$$(x(\pi), y(\pi)) = (\cos 2\pi, \sin 3\pi) = (1, 0)$$

and the points corresponding to $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$, which we have already discussed (where $y'(t) = 0$, also). Since $y'(t) = 3\cos 3t \neq 0$, for $t = 0$ or $t = \pi$, there is a vertical tangent line only at the point $(1, 0)$. ■

Theorem 2.1 generalizes what we observed in example 2.2.

THEOREM 2.1

Suppose that $x'(t)$ and $y'(t)$ are continuous. Then for the curve defined by the parametric equations $x = x(t)$ and $y = y(t)$,

(i) if $y'(c) = 0$ and $x'(c) \neq 0$, there is a horizontal tangent line at the point $(x(c), y(c))$;
(ii) if $x'(c) = 0$ and $y'(c) \neq 0$, there is a vertical tangent line at the point $(x(c), y(c))$.

PROOF

The proof depends on the calculation of derivatives for parametric curves and is left as an exercise. ■

Recall that our introductory question about the Scrambler was whether or not the rider ever comes to a complete stop. To answer this question, we will need to be able to compute velocities. Recall that if the position of an object moving along a straight line is given by the differentiable function $f(t)$, the object's velocity is given by $f'(t)$. The situation with parametric equations is completely analogous. If the position is given by $(x(t), y(t))$, for differentiable functions $x(t)$ and $y(t)$, then the **horizontal component of velocity** is given by $x'(t)$ and the **vertical component of velocity** is given by $y'(t)$ (see Figure 9.11). We define the **speed** to be $\sqrt{[x'(t)]^2 + [y'(t)]^2}$. From this, note that the speed is 0 if and only if $x'(t) = y'(t) = 0$. In this event, there is no horizontal or vertical motion.

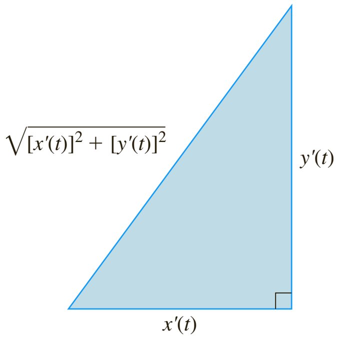


FIGURE 9.11
Horizontal and vertical components of velocity and speed

EXAMPLE 2.3 Velocity of the Scrambler

For the path of the Scrambler $x = 2\cos t + \sin 2t$, $y = 2\sin t + \cos 2t$, find the horizontal and vertical components of velocity and speed at times $t = 0$ and $t = \frac{\pi}{2}$, and indicate the direction of motion. Also determine all times at which the speed is zero.

Solution Here, the horizontal component of velocity is $\dfrac{dx}{dt} = -2\sin t + 2\cos 2t$ and the vertical component is $\dfrac{dy}{dt} = 2\cos t - 2\sin 2t$. At $t = 0$, the horizontal and vertical components of velocity both equal 2 and the speed is $\sqrt{4+4} = \sqrt{8}$. The rider is located at the point $(x(0), y(0)) = (2, 1)$ and is moving to the right [since $x'(0) > 0$] and up [since $y'(0) > 0$]. At $t = \frac{\pi}{2}$, the velocity has components -4 (horizontal) and 0 (vertical) and the speed is $\sqrt{16+0} = 4$. At this time, the rider is located at the point (0, 1) and is moving to the left [since $x'\left(\frac{\pi}{2}\right) < 0$].

In general, the speed of the rider at time t is given by

$$\begin{aligned} s(t) &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-2\sin t + 2\cos 2t)^2 + (2\cos t - 2\sin 2t)^2} \\ &= \sqrt{4\sin^2 t - 8\sin t\cos 2t + 4\cos^2 2t + 4\cos^2 t - 8\cos t\sin 2t + 4\sin^2 2t} \\ &= \sqrt{8 - 8\sin t\cos 2t - 8\cos t\sin 2t} \\ &= \sqrt{8 - 8\sin 3t}, \end{aligned}$$

using the identities $\sin^2 t + \cos^2 t = 1$, $\cos^2 2t + \sin^2 2t = 1$ and $\sin t\cos 2t + \sin 2t\cos t = \sin 3t$. So, the speed is 0 whenever $\sin 3t = 1$.

This occurs when $3t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$, or $t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6}, \ldots$. The corresponding points on the curve are $\left(x\left(\frac{\pi}{6}\right), y\left(\frac{\pi}{6}\right)\right) = \left(\frac{3}{2}\sqrt{3}, \frac{3}{2}\right)$, $\left(x\left(\frac{5\pi}{6}\right), y\left(\frac{5\pi}{6}\right)\right) = \left(-\frac{3}{2}\sqrt{3}, \frac{3}{2}\right)$ and $\left(x\left(\frac{9\pi}{6}\right), y\left(\frac{9\pi}{6}\right)\right) = (0, -3)$. You can easily verify that these points are the three tips of the path seen in Figure 9.8b. ■

We just showed that riders in the Scrambler of Figure 9.8b actually come to a brief stop at the outside of each loop. As you will explore in the exercises, for similar Scrambler paths, the riders slow down but have a positive speed at the outside of each loop. This is true of the Scrambler at most carnivals, for which a more complicated path makes up for the lack of stopping.

Notice that the Scrambler path shown in Figure 9.8b begins and ends at the same point and so, encloses an area. An interesting question is to determine the area enclosed by such a curve. Computing areas in parametric equations is a straightforward extension of our original development of integration. Recall that for a continuous function f defined on $[a, b]$, where $f(x) \geq 0$ on $[a, b]$, the area under the curve $y = f(x)$ for $a \leq x \leq b$ is given by

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx.$$

Now, suppose that this same curve is described parametrically by $x = x(t)$ and $y = y(t)$, where the curve is traversed exactly once for $c \leq t \leq d$. We can then compute the area by making the substitution $x = x(t)$. It then follows that $dx = x'(t)\,dt$ and so, the area is given by

$$A = \int_a^b \underbrace{y}_{y(t)}\ \underbrace{dx}_{x'(t)dt} = \int_c^d y(t)x'(t)\,dt,$$

where you should notice that we have also changed the limits of integration to match the new variable of integration. We generalize this result in Theorem 2.2.

THEOREM 2.2 (Area Enclosed by a Curve Defined Parametrically)

Suppose that the parametric equations $x = x(t)$ and $y = y(t)$, with $c \leq t \leq d$, describe a curve that is traced out *clockwise* exactly once, as t increases from c to d and where the curve does not intersect itself, except that the initial and terminal points are the same [i.e., $x(c) = x(d)$ and $y(c) = y(d)$]. Then the enclosed area is given by

$$A = \int_c^d y(t)x'(t)\,dt = -\int_c^d x(t)y'(t)\,dt. \tag{2.4}$$

If the curve is traced out *counterclockwise,* then the enclosed area is given by

$$A = -\int_c^d y(t)x'(t)\,dt = \int_c^d x(t)y'(t)\,dt. \tag{2.5}$$

PROOF

This result is a special case of Green's Theorem, which we will develop in section 14.4. ■

The new area formulas given in Theorem 2.2 turn out to be quite useful. As we see in example 2.4, we can use these to find the area enclosed by a parametric curve.

EXAMPLE 2.4 Finding the Area Enclosed by a Curve

Find the area enclosed by the path of the Scrambler $x = 2\cos t + \sin 2t$, $y = 2\sin t + \cos 2t$.

Solution Notice that the curve is traced out counterclockwise once for $0 \le t \le 2\pi$. From (2.5), the area is then

$$A = \int_0^{2\pi} x(t)y'(t)\,dt = \int_0^{2\pi} (2\cos t + \sin 2t)(2\cos t - 2\sin 2t)\,dt$$

$$= \int_0^{2\pi} (4\cos^2 t - 2\cos t \sin 2t - 2\sin^2 2t)\,dt = 2\pi,$$

where we evaluated the integral using a CAS. ■

In example 2.5, we use Theorem 2.2 to derive a formula for the area enclosed by an ellipse. Pay particular attention to how much easier this is to do with parametric equations than it is to do with the original x-y equation.

EXAMPLE 2.5 Finding the Area Enclosed by an Ellipse

Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (for constants $a, b > 0$).

Solution One way to compute the area is to solve the equation for y to obtain $y = \pm b\sqrt{1 - \dfrac{x^2}{a^2}}$ and then integrate:

$$A = \int_{-a}^{a} \left[b\sqrt{1 - \frac{x^2}{a^2}} - \left(-b\sqrt{1 - \frac{x^2}{a^2}} \right) \right] dx.$$

You can evaluate this integral by trigonometric substitution or by using a CAS, but a simpler, more elegant way to compute the area is to use parametric equations. Notice that the ellipse is described parametrically by $x = a\cos t$, $y = b\sin t$, for $0 \le t \le 2\pi$. The ellipse is then traced out counterclockwise exactly once for $0 \le t \le 2\pi$, so that the area is given by (2.5) to be

$$A = -\int_0^{2\pi} y(t)x'(t)\,dt = -\int_0^{2\pi} (b\sin t)(-a\sin t)\,dt = ab\int_0^{2\pi} \sin^2 t\,dt = ab\pi,$$

where this last integral can be evaluated by using the half-angle formula:

$$\sin^2 t = \frac{1}{2}(1 - \cos 2t).$$

We leave the details of this calculation as an exercise. ■

BEYOND FORMULAS

The formulas in this section are not new, but are simply modifications of the well-established rules for differentiation and integration. If you think of them this way, they are not complicated memorization exercises, but instead are old standards expressed in a slightly different way.

EXERCISES 9.2

WRITING EXERCISES

1. In the derivation of parametric equations for the Scrambler, we used the fact that reversing the sine and cosine functions to $\begin{cases} x = \sin t \\ y = \cos t \end{cases}$ causes the circle to be traced out clockwise. Explain why this is so by starting at $t = 0$ and following the graph as t increases to 2π.

2. Explain why Theorem 2.1 makes sense. (Hint: If $y'(c) = 0$, what does that say about the change in y-coordinates on the graph? Why do you also need $x'(c) \neq 0$ to guarantee a horizontal tangent?)

3. Imagine an object with position given by $x(t)$ and $y(t)$. If a right triangle has a horizontal leg of length $x'(t)$ and a vertical leg of length $y'(t)$, what would the length of the hypotenuse represent? Explain why this makes sense.

4. Explain why the sign ($\pm$) of $\int_c^d y(t)x'(t)\,dt$ in Theorem 2.2 is different for curves traced out clockwise and counterclockwise.

In exercises 1–6, find the slopes of the tangent lines to the given curves at the indicated points.

1. $\begin{cases} x = t^2 - 2 \\ y = t^3 - t \end{cases}$ (a) $t = -1$, (b) $t = 1$, (c) $(-2, 0)$

2. $\begin{cases} x = t^3 - t \\ y = t^4 - 5t^2 + 4 \end{cases}$ (a) $t = -1$, (b) $t = 1$, (c) $(0, 4)$

3. $\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}$ (a) $t = \frac{\pi}{4}$, (b) $t = \frac{\pi}{2}$, (c) $(0, 3)$

4. $\begin{cases} x = 2\cos 2t \\ y = 3\sin 2t \end{cases}$ (a) $t = \frac{\pi}{4}$, (b) $t = \frac{\pi}{2}$, (c) $(-2, 0)$

5. $\begin{cases} x = \cos 2t \\ y = \sin 4t \end{cases}$ (a) $t = \frac{\pi}{4}$, (b) $t = \frac{\pi}{2}$, (c) $\left(\frac{\sqrt{2}}{2}, 1\right)$

6. $\begin{cases} x = \cos 2t \\ y = \sin 3t \end{cases}$ (a) $t = \frac{\pi}{2}$, (b) $t = \frac{3\pi}{2}$, (c) $(1, 0)$

In exercises 7 and 8, sketch the graph and find the slope of the curve at the given point.

7. $\begin{cases} x = t^2 - 2 \\ y = t^3 - t \end{cases}$ at $(-1, 0)$

8. $\begin{cases} x = t^3 - t \\ y = t^4 - 5t^2 + 4 \end{cases}$ at $(0, 0)$

In exercises 9–14, identify all points at which the curve has (a) a horizontal tangent and (b) a vertical tangent.

9. $\begin{cases} x = \cos 2t \\ y = \sin 4t \end{cases}$

10. $\begin{cases} x = \cos 2t \\ y = \sin 7t \end{cases}$

11. $\begin{cases} x = t^2 - 1 \\ y = t^4 - 4t \end{cases}$

12. $\begin{cases} x = t^2 - 1 \\ y = t^4 - 4t^2 \end{cases}$

13. $\begin{cases} x = 2\cos t + \sin 2t \\ y = 2\sin t + \cos 2t \end{cases}$

14. $\begin{cases} x = 2\cos 2t + \sin t \\ y = 2\sin 2t + \cos t \end{cases}$

In exercises 15–20, parametric equations for the position of an object are given. Find the object's velocity and speed at the given times and describe its motion.

15. $\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$ (a) $t = 0$, (b) $t = \frac{\pi}{2}$

16. $\begin{cases} x = 2\cos 2t \\ y = 2\sin 2t \end{cases}$ (a) $t = 0$, (b) $t = \frac{\pi}{2}$

17. $\begin{cases} x = 20t \\ y = 30 - 2t - 16t^2 \end{cases}$ (a) $t = 0$, (b) $t = 2$

18. $\begin{cases} x = 40t + 5 \\ y = 20 + 3t - 16t^2 \end{cases}$ (a) $t = 0$, (b) $t = 2$

19. $\begin{cases} x = 2\cos 2t + \sin 5t \\ y = 2\sin 2t + \cos 5t \end{cases}$ (a) $t = 0$, (b) $t = \frac{\pi}{2}$

20. $\begin{cases} x = 3\cos t + \sin 3t \\ y = 3\sin t + \cos 3t \end{cases}$ (a) $t = 0$, (b) $t = \frac{\pi}{2}$

In exercises 21–28, find the area enclosed by the given curve.

21. $\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}$

22. $\begin{cases} x = 6\cos t \\ y = 2\sin t \end{cases}$

23. $\begin{cases} x = \frac{1}{2}\cos t - \frac{1}{4}\cos 2t \\ y = \frac{1}{2}\sin t - \frac{1}{4}\sin 2t \end{cases}$

24. $\begin{cases} x = 2\cos 2t + \cos 4t \\ y = 2\sin 2t + \sin 4t \end{cases}$

25. $\begin{cases} x = \cos t \\ y = \sin 2t \end{cases}, \quad \frac{\pi}{2} \le t \le \frac{3\pi}{2}$

26. $\begin{cases} x = t\sin t \\ y = t\cos t \end{cases}, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

27. $\begin{cases} x = t^3 - 4t \\ y = t^2 - 3 \end{cases}, \quad -2 \le t \le 2$

28. $\begin{cases} x = t^3 - 4t \\ y = t^4 - 1 \end{cases}, \quad -2 \le t \le 2$

In exercises 29 and 30, find the speed of the object each time it crosses the x-axis.

29. $\begin{cases} x = 2\cos^2 t + 2\cos t - 1 \\ y = 2(1 - \cos t)\sin t \end{cases}$ **30.** $\begin{cases} x = 6\cos t + 5\cos 3t \\ y = 6\sin t - 5\sin 3t \end{cases}$

31. A modification of the Scrambler in example 2.1 is $\begin{cases} x = 2\cos 3t + \sin 5t \\ y = 2\sin 3t + \cos 5t \end{cases}$. In example 2.1, the ratio of the speed of the outer arms to the speed of the inner arms is 2-to-1. What is the ratio in this version of the Scrambler? Sketch a graph showing the motion of this new Scrambler.

32. Compute the speed of the Scrambler in exercise 31. Using trigonometric identities as in example 2.3, show that the speed is at a minimum when $\sin 8t = 1$ but that the speed is never zero. Show that the minimum speed is reached at the outer points of the path.

33. Find parametric equations for a Scrambler that is the same as in example 2.1 except that the outer arms rotate three times as fast as the inner arms. Sketch a graph of its motion and determine its minimum and maximum speeds.

34. Find parametric equations for a Scrambler that is the same as in example 2.1 except that the inner arms have length 3. Sketch a graph of its motion and determine its minimum and maximum speeds.

35. Suppose an object follows the path $\begin{cases} x = \sin 4t \\ y = -\cos 4t \end{cases}$. Show that its speed is constant. Show that, at any time t, the tangent line is perpendicular to a line connecting the origin and the object.

36. A Ferris wheel has height 100 feet and completes one revolution in 3 minutes at a constant speed. Compute the speed of a rider in the Ferris wheel.

37. Suppose you are standing at the origin watching an object that has position $(x(t), y(t))$ at time t. Show that, from your perspective, the object is moving clockwise if $\left(\dfrac{y(t)}{x(t)}\right)' < 0$ and is moving counterclockwise if $\left(\dfrac{y(t)}{x(t)}\right)' > 0$.

38. In the Ptolemaic model of planetary motion, the earth was at the center of the solar system and the sun and planets orbited the earth. Circular orbits, which were preferred for aesthetic reasons, could not account for the actual motion of the planets as viewed from the earth. Ptolemy modified the circles into epicycloids, which are circles on circles similar to the Scrambler of example 2.1. Suppose that a planet's motion is given by $\begin{cases} x = 10\cos 16\pi t + 20\cos 4\pi t \\ y = 10\sin 16\pi t + 20\sin 4\pi t \end{cases}$. Using the result of exercise 37, find the intervals in which the planet rotates clockwise and the intervals in which the planet rotates counterclockwise.

39. Find parametric equations for the path traced out by a specific point on a circle of radius r rolling from left to right at a constant speed $v > r$. Assume that the point starts at (r, r) at time $t = 0$. (Hint: First, find parametric equations for the center of the circle. Then, add on parametric equations for the point going around the center of the circle.) Find the minimum and maximum speeds of the point and the locations where each occurs. Graph the curve for $v = 3$ and $r = 2$. This curve is called a **cycloid.**

40. Find parametric equations for the path traced out by a specific point inside the circle as the circle rolls from left to right. (Hint: If r is the radius of the circle, let $d < r$ be the distance from the point to the center.) Find the minimum and maximum speeds of the point and the locations where each occurs. Graph the curve for $v = 3$, $r = 2$ and $d = 1$. This curve is called a **trochoid.**

41. A **hypocycloid** is the path traced out by a point on a smaller circle of radius b that is rolling inside a larger circle of radius $a > b$. Find parametric equations for the hypocycloid and graph it for $a = 5$ and $b = 3$. Find an equation in terms of the parameter t for the slope of the tangent line to the hypocycloid and determine one point at which the tangent line is vertical. What interesting simplification occurs if $a = 2b$?

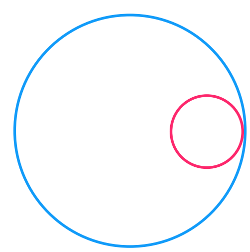


Figure for exercise 41

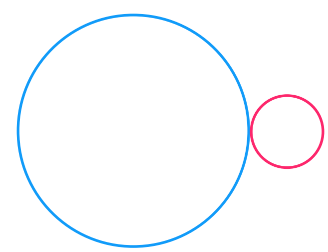


Figure for exercise 42

42. An **epicycloid** is the path traced out by a point on a smaller circle of radius b that is rolling outside a larger circle of radius $a > b$. Find parametric equations for the epicycloid and graph it for $a = 8$ and $b = 5$. Find an equation in terms of the parameter t for the slope of the tangent line to the epicycloid and determine one point at which the slope is vertical. What interesting simplification occurs if $a = 2b$?

43. Suppose that $x = 2\cos t$ and $y = 2\sin t$. At the point $(\sqrt{3}, 1)$, show that $\dfrac{d^2y}{dx^2}(\sqrt{3}) \neq \dfrac{\frac{d^2y}{dt^2}(\pi/6)}{\frac{d^2x}{dt^2}(\pi/6)}$.

44. For $x = at^2$ and $y = bx^2$ for nonzero constants a and b, determine whether there are any values of t such that

$$\frac{d^2y}{dx^2}(x(t)) = \frac{\frac{d^2y}{dt^2}(t)}{\frac{d^2x}{dt^2}(t)}.$$

EXPLORATORY EXERCISES

1. By varying the speed of the outer arms, the Scrambler of example 2.1 can be generalized to $\begin{cases} x = 2\cos t + \sin kt \\ y = 2\sin t + \cos kt \end{cases}$ for some positive constant k. Show that the minimum speed for any such Scrambler is reached at the outside of a loop. Show that the only value of k that actually produces a speed of 0 is $k = 2$. By varying the lengths of the arms, you can further generalize the Scrambler to $\begin{cases} x = r\cos t + \sin kt \\ y = r\sin t + \cos kt \end{cases}$ for positive constants $r > 1$ and k. Sketch the paths for several such Scramblers and determine the relationship between r and k needed to produce a speed of 0.

2. **Bézier curves** are essential in almost all areas of modern engineering design. (For instance, Bézier curves were used for sketching many of the figures for this book.) One version of a Bézier curve starts with **control points** at (a, y_a), (b, y_b), (c, y_c) and (d, y_d). The Bézier curve passes through the points (a, y_a) and (d, y_d). The tangent line at $x = a$ passes through (b, y_b) and the tangent line at $x = d$ passes through (c, y_c). Show that these criteria are met, for $0 \le t \le 1$, with

$$\begin{cases} x = (a + b - c - d)t^3 + (2d - 2b + c - a)t^2 \\ \qquad + (b - a)t + a \\ y = (y_a + y_b - y_c - y_d)t^3 + (2y_d - 2y_b + y_c - y_a)t^2 \\ \qquad + (y_b - y_a)t + y_a \end{cases}$$

Use this formula to find and graph the Bézier curve with control points (0, 0), (1, 2), (2, 3) and (3, 0). Explore the effect of moving the middle control points, for example, moving them up to (1, 3) and (2, 4), respectively.

9.3 ARC LENGTH AND SURFACE AREA IN PARAMETRIC EQUATIONS

In this section, we investigate arc length and surface area for curves defined parametrically. Along the way, we explore one of the most famous and interesting curves in mathematics.

Let C be the curve defined by the parametric equations $x = x(t)$ and $y = y(t)$, for $a \le t \le b$ (see Figure 9.12a), where x, x', y and y' are continuous on the interval $[a, b]$. We further assume that the curve does not intersect itself, except possibly at a finite number of points. Our goal is to compute the length of the curve (the **arc length**). As we have done countless times now, we begin by constructing an approximation.

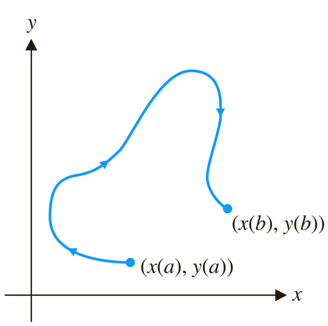


FIGURE 9.12a
The plane curve C

First, we divide the t-interval $[a, b]$ into n subintervals of equal length, Δt:

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b,$$

where $t_i - t_{i-1} = \Delta t = \dfrac{b-a}{n}$, for each $i = 1, 2, 3, \ldots, n$. For each subinterval $[t_{i-1}, t_i]$, we approximate the arc length s_i of the portion of the curve joining the point $(x(t_{i-1}), y(t_{i-1}))$ to the point $(x(t_i), y(t_i))$ with the length of the line segment joining these points. This approximation is shown in Figure 9.12b for the case where $n = 4$. We have

$$\begin{aligned} s_i &\approx d\{(x(t_{i-1}), y(t_{i-1})), (x(t_i), y(t_i))\} \\ &= \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2}. \end{aligned}$$

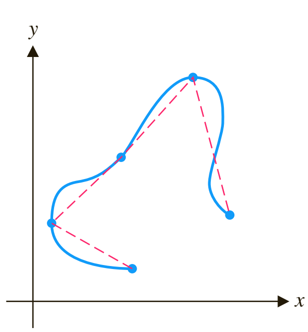


FIGURE 9.12b
Approximate arc length

Recall that from the Mean Value Theorem (see section 2.9 and make sure you know why we can apply it here), we have that

$$x(t_i) - x(t_{i-1}) = x'(c_i)(t_i - t_{i-1}) = x'(c_i)\,\Delta t$$

and

$$y(t_i) - y(t_{i-1}) = y'(d_i)(t_i - t_{i-1}) = y'(d_i)\,\Delta t,$$

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2. **Bézier curves** are essential in almost all areas of modern engineering design. (For instance, Bézier curves were used for sketching many of the figures for this book.) One version of a Bézier curve starts with **control points** at (a, y_a), (b, y_b), (c, y_c) and (d, y_d). The Bézier curve passes through the points (a, y_a) and (d, y_d). The tangent line at $x = a$ passes through (b, y_b) and the tangent line at $x = d$ passes through (c, y_c). Show that these criteria are met, for $0 \le t \le 1$, with

$$\begin{cases} x = (a + b - c - d)t^3 + (2d - 2b + c - a)t^2 \\ \qquad + (b - a)t + a \\ y = (y_a + y_b - y_c - y_d)t^3 + (2y_d - 2y_b + y_c - y_a)t^2 \\ \qquad + (y_b - y_a)t + y_a \end{cases}$$

Use this formula to find and graph the Bézier curve with control points (0, 0), (1, 2), (2, 3) and (3, 0). Explore the effect of moving the middle control points, for example, moving them up to (1, 3) and (2, 4), respectively.

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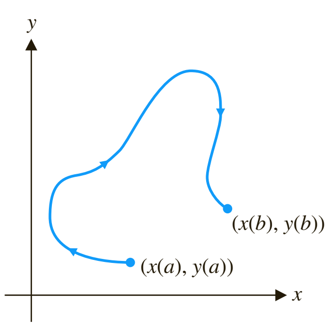

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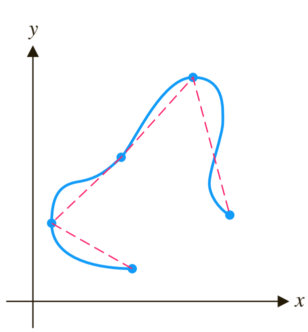

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and

$$y(t_i) - y(t_{i-1}) = y'(d_i)(t_i - t_{i-1}) = y'(d_i)\,\Delta t,$$

where c_i and d_i are some points in the interval (t_{i-1}, t_i). This gives us

$$\begin{aligned} s_i &\approx \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2} \\ &= \sqrt{[x'(c_i)\,\Delta t]^2 + [y'(d_i)\,\Delta t]^2} \\ &= \sqrt{[x'(c_i)]^2 + [y'(d_i)]^2}\,\Delta t. \end{aligned}$$

Notice that if Δt is small, then c_i and d_i are close together. So, we can make the further approximation

$$s_i \approx \sqrt{[x'(c_i)]^2 + [y'(c_i)]^2}\,\Delta t,$$

for each $i = 1, 2, \ldots, n$. The total arc length is then approximately

$$s \approx \sum_{i=1}^{n} \sqrt{[x'(c_i)]^2 + [y'(c_i)]^2}\,\Delta t.$$

Taking the limit as $n \to \infty$ then gives us the exact arc length, which you should recognize as an integral:

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[x'(c_i)]^2 + [y'(c_i)]^2}\,\Delta t = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt.$$

We summarize this discussion in Theorem 3.1.

THEOREM 3.1 (Arc Length for a Curve Defined Parametrically)

For the curve defined parametrically by $x = x(t)$, $y = y(t)$, $a \le t \le b$, if x' and y' are continuous on $[a, b]$ and the curve does not intersect itself (except possibly at a finite number of points), then the arc length s of the curve is given by

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt. \tag{3.1}$$

In example 3.1, we illustrate the use of (3.1) to find the arc length of the Scrambler curve from example 2.1.

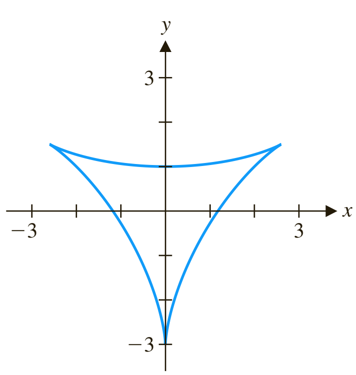


FIGURE 9.13
$x = 2\cos t + \sin 2t$,
$y = 2\sin t + \cos 2t$,
$0 \le t \le 2\pi$

EXAMPLE 3.1 Finding the Arc Length of a Plane Curve

Find the arc length of the Scrambler curve $x = 2\cos t + \sin 2t$, $y = 2\sin t + \cos 2t$, for $0 \le t \le 2\pi$. Also, find the average speed of the Scrambler over this interval.

Solution The curve is shown in Figure 9.13. First, note that x, x', y and y' are all continuous on the interval $[0, 2\pi]$. From (3.1), we then have

$$\begin{aligned} s &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi} \sqrt{(-2\sin t + 2\cos 2t)^2 + (2\cos t - 2\sin 2t)^2}\,dt \\ &= \int_0^{2\pi} \sqrt{4\sin^2 t - 8\sin t\cos 2t + 4\cos^2 2t + 4\cos^2 t - 8\cos t\sin 2t + 4\sin^2 2t}\,dt \\ &= \int_0^{2\pi} \sqrt{8 - 8\sin t\cos 2t - 8\cos t\sin 2t}\,dt = \int_0^{2\pi} \sqrt{8 - 8\sin 3t}\,dt \approx 16, \end{aligned}$$

since $\sin^2 t + \cos^2 t = 1$, $\cos^2 2t + \sin^2 2t = 1$ and $\sin t \cos 2t + \sin 2t \cos t = \sin 3t$ and where we have approximated the last integral numerically. To find the average speed over the given interval, we simply divide the arc length (i.e., the distance traveled), by the total time, 2π, to obtain

$$s_{\text{ave}} \approx \frac{16}{2\pi} \approx 2.546.$$

We want emphasize that Theorem 3.1 allows the curve to intersect itself at a *finite* number of points, but a curve cannot intersect itself over an entire interval of values of the parameter t. To see why this requirement is needed, notice that the parametric equations $x = \cos t$, $y = \sin t$, for $0 \le t \le 4\pi$, describe the circle of radius 1 centered at the origin. However, the circle is traversed *twice* as t ranges from 0 to 4π. If you were to apply (3.1) to this curve, you'd obtain

$$\int_0^{4\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{4\pi} \sqrt{(-\sin t)^2 + \cos^2 t}\, dt = 4\pi,$$

which corresponds to twice the arc length (circumference) of the circle. As you can see, if a curve intersects itself over an entire interval of values of t, the arc length of such a portion of the curve is counted twice by (3.1).

EXAMPLE 3.2 Finding the Arc Length of a Complicated Plane Curve

Find the arc length of the plane curve $x = \cos 5t$, $y = \sin 7t$, for $0 \le t \le 2\pi$.

Solution This unusual curve (an example of a **Lissajous curve**) is sketched in Figure 9.14. We leave it as an exercise to verify that the hypotheses of Theorem 3.1 are met. From (3.1), we then have that

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(-5\sin 5t)^2 + (7\cos 7t)^2}\, dt \approx 36.5,$$

where we have approximated the integral numerically. This is a long curve to be confined within the rectangle $-1 \le x \le 1$, $-1 \le y \le 1$!

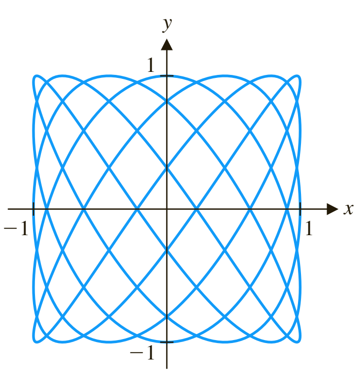


FIGURE 9.14
A Lissajous curve

The arc length formula (3.1) should seem familiar to you. Parametric equations for a curve $y = f(x)$ are $x = t$, $y = f(t)$ and from (3.1), the arc length of this curve for $a \le x \le b$ is then

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{1 + [f'(t)]^2}\, dt,$$

which is the arc length formula derived in section 5.4. Thus, the formula developed in section 5.4 is a special case of (3.1).

Observe that the speed of the Scrambler calculated in example 2.3 and the length of the Scrambler curve found in example 3.1 both depend on the same quantity: $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Observe that if the parameter t represents time, then $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ represents speed and from Theorem 3.1, the arc length (i.e., the distance traveled) is the integral of the speed with respect to time.

FIGURE 9.15
Downhill skier

We can use our notion of arc length to address a famous problem called the **brachistochrone problem.** We state this problem in the context of a downhill skier. Consider a ski slope consisting of a tilted plane, where a skier wishes to get from a point A at the top of the slope to a point B down the slope (but *not* directly beneath A) in the least time possible (see Figure 9.15). Suppose the path taken by the skier is given by the parametric equations $x = x(u)$ and $y = y(u)$, $0 \le u \le 1$, where x and y determine the position of the skier in the plane of the ski slope. (For simplicity, we orient the positive y-axis so that it points down. Also, we name the parameter u since u will, in general, *not* represent time.)

To derive a formula for the time required to get from point A to point B, start with the simple formula $d = r \cdot t$ relating the distance to the time and the rate. As seen in the derivation of the arc length formula (3.1), for a small section of the curve, the distance is approximately $\sqrt{[x'(u)]^2 + [y'(u)]^2}$. The rate is harder to identify since we aren't given position as a function of time. For simplicity, we assume that the only effect of friction is to keep the skier on the path and that $y(t) \ge 0$. In this case, using the principle of conservation of energy, it can be shown that the skier's speed is given by $\frac{\sqrt{y(u)}}{k}$ for some constant $k \ge 0$. Putting the pieces together, the total time from point A to point B is given by

$$\text{Time} = \int_0^1 k\sqrt{\frac{[x'(u)]^2 + [y'(u)]^2}{y(u)}}\,du. \tag{3.2}$$

Your first thought might be that the shortest path from point A to point B is along a straight line. If you're thinking of *short* in terms of *distance,* you're right, of course. However, if you think of *short* in terms of *time* (how most skiers would think of it), this is not true. In example 3.3, we show that the fastest path from point A to point B is, in fact, *not* along a straight line, by exhibiting a faster path.

EXAMPLE 3.3 Skiing a Curved Path that Is Faster Than Skiing a Straight Line

If point A in our skiing example is $(0, 0)$ and point B is $(\pi, 2)$, show that the **cycloid** defined by

$$x = \pi u - \sin \pi u, \quad y = 1 - \cos \pi u$$

is faster than the line segment connecting the points. Explain the result in physical terms.

Solution First, note that the line segment connecting the points is given by $x = \pi u$, $y = 2u$, for $0 \le u \le 1$. Further, both curves meet the endpoint requirements that $(x(0), y(0)) = (0, 0)$ and $(x(1), y(1)) = (\pi, 2)$. For the cycloid, we have from (3.2) that

$$\begin{aligned}
\text{Time} &= \int_0^1 k\sqrt{\frac{[x'(u)]^2 + [y'(u)]^2}{y(u)}}\,du \\
&= k\int_0^1 \sqrt{\frac{(\pi - \pi\cos\pi u)^2 + (\pi\sin\pi u)^2}{1 - \cos\pi u}}\,du \\
&= k\sqrt{2}\pi \int_0^1 \sqrt{\frac{1 - \cos\pi u}{1 - \cos\pi u}}\,du \\
&= k\sqrt{2}\pi.
\end{aligned}$$

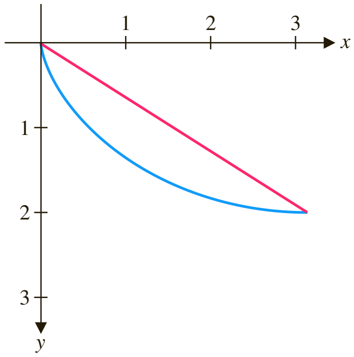


FIGURE 9.16
Two skiing paths

Similarly, for the line segment, we have that

$$\text{Time} = \int_0^1 k\sqrt{\frac{[x'(u)]^2 + [y'(u)]^2}{y(u)}}\,du$$

$$= k\int_0^1 \sqrt{\frac{\pi^2 + 2^2}{2u}}\,du$$

$$= k\sqrt{2}\sqrt{\pi^2 + 4}.$$

Notice that the cycloid route is faster since $\pi < \sqrt{\pi^2 + 4}$. The two paths are shown in Figure 9.16. Observe that the cycloid is very steep at the beginning, which would allow a skier to go faster following the cycloid than following the straight line. As it turns out, the greater speed of the cycloid more than compensates for the longer distance of the cycloid path. ■

We will ask you to construct some skiing paths of your own in the exercises. However, it has been proved that the cycloid is the plane curve with the shortest time (which is what the Greek root words for **brachistochrone** mean). In addition, we will give you an opportunity to discover another remarkable property of the cycloid, relating to another famous problem, the **tautochrone problem.** Both problems have an interesting history focused on brothers Jacob and Johann Bernoulli, who solved the problem in 1697 (along with Newton, Leibniz and l'Hôpital) and argued incessantly about who deserved credit.

Much as we did in section 5.4, we can use our arc length formula to find a formula for the surface area of a surface of revolution. Recall that if the curve $y = f(x)$ for $c \le x \le d$ is revolved about the x-axis (see Figure 9.17), the surface area is given by

$$\text{Surface Area} = \int_c^d 2\pi \underbrace{|f(x)|}_{\text{radius}} \underbrace{\sqrt{1 + [f'(x)]^2}}_{\text{arc length}}\,dx.$$

Let C be the curve defined by the parametric equations $x = x(t)$ and $y = y(t)$ with $a \le t \le b$, where x, x', y and y' are continuous and where the curve does not intersect itself for $a \le t \le b$. We leave it as an exercise to derive the corresponding formula for parametric equations:

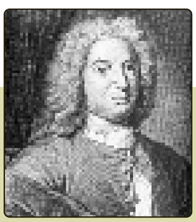
$$\text{Surface Area} = \int_a^b 2\pi \underbrace{|y(t)|}_{\text{radius}} \underbrace{\sqrt{[x'(t)]^2 + [y'(t)]^2}}_{\text{arc length}}\,dt.$$

More generally, we have that if the curve is revolved about the line $y = c$, the surface area is given by

$$\text{Surface Area} = \int_a^b 2\pi \underbrace{|y(t) - c|}_{\text{radius}} \underbrace{\sqrt{[x'(t)]^2 + [y'(t)]^2}}_{\text{arc length}}\,dt. \tag{3.3}$$

Likewise, if we revolve the curve about the line $x = d$, the surface area is given by

$$\text{Surface Area} = \int_a^b 2\pi \underbrace{|x(t) - d|}_{\text{radius}} \underbrace{\sqrt{[x'(t)]^2 + [y'(t)]^2}}_{\text{arc length}}\,dt. \tag{3.4}$$



HISTORICAL NOTES

Jacob Bernoulli (1654–1705) and Johann Bernoulli (1667–1748) Swiss mathematicians who were instrumental in the development of the calculus. Jacob was the first of several generations of Bernoullis to make important contributions to mathematics. He was active in probability, series and the calculus of variations and introduced the term "integral." Johann followed his brother into mathematics while also earning a doctorate in medicine. Johann first stated l'Hôpital's Rule, one of many results over which he fought bitterly (usually with his brother, but, after Jacob's death, also with his own son Daniel) to receive credit. Both brothers were sensitive, irritable, egotistical (Johann had his tombstone inscribed, "The Archimedes of his age") and quick to criticize others. Their competitive spirit accelerated the development of calculus.

Look carefully at what all of the surface area formulas have in common. That is, in each case, the surface area is given by

SURFACE AREA

$$\text{Surface Area} = \int_a^b 2\pi(\text{radius})(\text{arc length})\,dt. \tag{3.5}$$

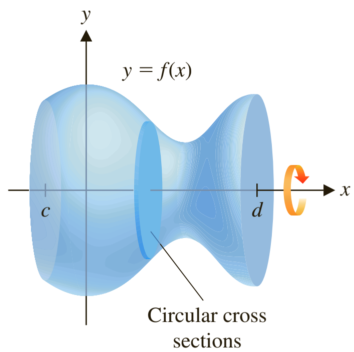


FIGURE 9.17
Surface of revolution

Look carefully at the graph of the curve and the axis about which you are revolving, to see how to fill in the blanks in (3.5). As we observed in section 5.4, it is very important that you draw a picture here.

EXAMPLE 3.4 Finding Surface Area with Parametric Equations

Find the surface area of the surface formed by revolving the half-ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, $y \geq 0$, about the x-axis (see Figure 9.18).

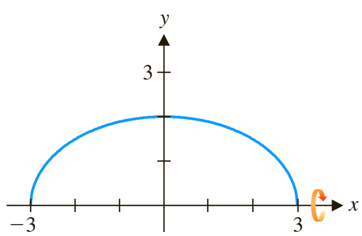


FIGURE 9.18
$y = 2\sqrt{\frac{1-x^2}{9}}$

Solution It would truly be a mess to set up the integral for $y = f(x) = 2\sqrt{1 - x^2/9}$. (Think about this!) Instead, notice that you can represent the curve by the parametric equations $x = 3\cos t$, $y = 2\sin t$, for $0 \leq t \leq \pi$. From (3.3), the surface area is then given by

$$\begin{aligned}
\text{Surface Area} &= \int_0^\pi 2\pi\, \underbrace{(2\sin t)}_{\text{radius}}\, \underbrace{\sqrt{(-3\sin t)^2 + (2\cos t)^2}\,dt}_{\text{arc length}} \\
&= 4\pi \int_0^\pi \sin t\sqrt{9\sin^2 t + 4\cos^2 t}\,dt \\
&= 4\pi\,\frac{9\sqrt{5}\sin^{-1}(\sqrt{5}/3) + 10}{5} \approx 67.7,
\end{aligned}$$

where we used a CAS to evaluate the integral. ■

EXAMPLE 3.5 Revolving about a Line Other Than a Coordinate Axis

Find the surface area of the surface formed by revolving the curve $x = \sin 2t$, $y = \cos 3t$, for $0 \leq t \leq \pi/3$, about the line $x = 2$.

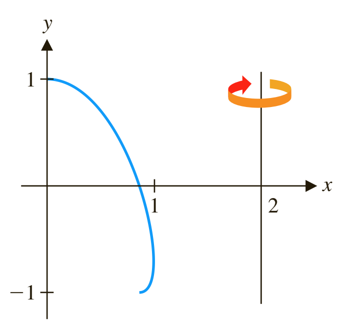


FIGURE 9.19
$x = \sin 2t$, $y = \cos 3t$

Solution A sketch of the curve is shown in Figure 9.19. Since the x-values on the curve are all less than 2, the radius of the solid of revolution is $2 - x = 2 - \sin 2t$ and so, from (3.4), the surface area is given by

$$\text{Surface Area} = \int_0^{\pi/3} 2\pi\, \underbrace{(2 - \sin 2t)}_{\text{radius}}\, \underbrace{\sqrt{[2\cos 2t]^2 + [-3\sin 3t]^2}\,dt}_{\text{arc length}} \approx 20.1,$$

where we have approximated the value of the integral numerically. ■

In example 3.6, we model a physical process with parametric equations. Since the modeling process is itself of great importance, be sure that you understand all of the steps. See if you can find an alternative approach to this problem.

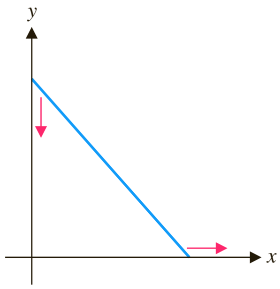


FIGURE 9.20
Ladder sliding down a wall

EXAMPLE 3.6 Arc Length for a Falling Ladder

An 8-foot-tall ladder stands vertically against a wall. The bottom of the ladder is pulled along the floor, with the top remaining in contact with the wall, until the ladder rests flat on the floor. Find the distance traveled by the midpoint of the ladder.

Solution We first find parametric equations for the position of the midpoint of the ladder. We orient the x- and y-axes as shown in Figure 9.20.

Let x denote the distance from the wall to the bottom of the ladder and let y denote the distance from the floor to the top of the ladder. Since the ladder is 8 feet long, observe that $x^2 + y^2 = 64$. Defining the parameter $t = x$, we have $y = \sqrt{64 - t^2}$. The midpoint of the ladder has coordinates $\left(\frac{x}{2}, \frac{y}{2}\right)$ and so, parametric equations for the midpoint are

$$\begin{cases} x(t) = \frac{1}{2}t \\ y(t) = \frac{1}{2}\sqrt{64 - t^2} \end{cases}.$$

When the ladder stands vertically against the wall, we have $x = 0$ and when it lies flat on the floor, $x = 8$. So, $0 \le t \le 8$. From (3.1), the arc length is then given by

$$s = \int_0^8 \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\frac{-t}{\sqrt{64 - t^2}}\right)^2}\,dt = \int_0^8 \sqrt{\frac{1}{4}\left(1 + \frac{t^2}{64 - t^2}\right)}\,dt$$
$$= \int_0^8 \frac{1}{2}\sqrt{\frac{64}{64 - t^2}}\,dt = \int_0^8 \frac{1}{2}\sqrt{\frac{1}{1 - (t/8)^2}}\,dt.$$

Substituting $u = \frac{t}{8}$ gives us $du = \frac{1}{8}\,dt$ or $dt = 8\,du$. For the limits of integration, note that when $t = 0$, $u = 0$ and when $t = 8$, $u = 1$. The arc length is then

$$s = \int_0^8 \frac{1}{2}\sqrt{\frac{1}{1 - (t/8)^2}}\,dt = \int_0^1 \frac{1}{2}\sqrt{\frac{1}{1 - u^2}}\,8\,du = 4\sin^{-1} u\Big|_{u=0}^{u=1}$$
$$= 4\left(\frac{\pi}{2} - 0\right) = 2\pi.$$

Since this is a rare arc length integral that can be evaluated exactly, you might be suspicious that there is an easier way to find the arc length. We explore this in the exercises. ■

EXERCISES 9.3

WRITING EXERCISES

1. In the derivation preceding Theorem 3.1, we justified the equation
$$g(t_i) - g(t_{i-1}) = g'(c_i)\,\Delta t.$$
Thinking of $g(t)$ as position and $g'(t)$ as velocity, explain why this makes sense.

2. The curve in example 3.2 was a long curve contained within a small rectangle. What would you guess would be the maximum length for a curve contained in such a rectangle? Briefly explain.

3. In example 3.3, we noted that the steeper initial slope of the cycloid would allow the skier to build up more speed than the straight-line path. The cycloid takes this idea to the limit by having a vertical tangent line at the origin. Explain why, despite the vertical tangent line, it is still physically possible for the skier to stay on this slope. (Hint: How do the two dimensions of the path relate to the three dimensions of the ski slope?)

4. The tautochrone problem discussed in exploratory exercise 2 involves starting on the same curve at two different places and

comparing the times required to reach the end. For the cycloid, compare the speed of a skier starting at the origin versus one starting halfway to the bottom. Explain why it is not clear whether starting halfway down would get you to the bottom faster.

In exercises 1–12, find the arc length of the curve; approximate numerically, if needed.

1. $\begin{cases} x = 2\cos t \\ y = 4\sin t \end{cases}$

2. $\begin{cases} x = 1 - 2\cos t \\ y = 2 + 3\sin t \end{cases}$

3. $\begin{cases} x = t^3 - 4t \\ y = t^2 - 3 \end{cases}, -2 \le t \le 2$

4. $\begin{cases} x = t^3 - 4t \\ y = t^2 - 3t \end{cases}, -2 \le t \le 2$

5. $\begin{cases} x = \cos 4t \\ y = \sin 4t \end{cases}$

6. $\begin{cases} x = \cos 7t \\ y = \sin 11t \end{cases}$

7. $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}, -1 \le t \le 1$

8. $\begin{cases} x = t^2\cos t \\ y = t^2\sin t \end{cases}, -1 \le t \le 1$

9. $\begin{cases} x = \sin 2t\cos t \\ y = \sin 2t\sin t \end{cases}, 0 \le t \le \pi/2$

10. $\begin{cases} x = \sin 4t\cos t \\ y = \sin 4t\sin t \end{cases}, 0 \le t \le \pi/2$

11. $\begin{cases} x = \sin t \\ y = \sin \pi t \end{cases}, 0 \le t \le \pi$

12. $\begin{cases} x = \sin t \\ y = \sin \sqrt{2}t \end{cases}, 0 \le t \le \pi$

In exercises 13–16, show that the curve starts at the origin at $t = 0$ and reaches the point $(\pi, 2)$ at $t = 1$. Then use the time formula (3.2) to determine how long it would take a skier to take the given path.

13. $\begin{cases} x = \pi t \\ y = 2\sqrt{t} \end{cases}$

14. $\begin{cases} x = \pi t \\ y = 2\sqrt[4]{t} \end{cases}$

15. $\begin{cases} x = -\frac{1}{2}\pi(\cos \pi t - 1) \\ y = 2t + \frac{7}{10}\sin \pi t \end{cases}$

16. $\begin{cases} x = \pi t - 0.6\sin \pi t \\ y = 2t + 0.4\sin \pi t \end{cases}$

In exercises 17–20, find the slope at the origin and the arc length for the curve in the indicated exercise. Compare to the cycloid from example 3.3.

17. exercise 13

18. exercise 14

19. exercise 15

20. exercise 16

In exercises 21–26, compute the surface area of the surface obtained by revolving the given curve about the indicated axis.

21. $\begin{cases} x = t^2 - 1 \\ y = t^3 - 4t \end{cases}, -2 \le t \le 0$, about the x-axis

22. $\begin{cases} x = t^2 - 1 \\ y = t^3 - 4t \end{cases}, 0 \le t \le 2$, about the x-axis

23. $\begin{cases} x = t^2 - 1 \\ y = t^3 - 4t \end{cases}, -1 \le t \le 1$, about the y-axis

24. $\begin{cases} x = t^2 - 1 \\ y = t^3 - 4t \end{cases}, -2 \le t \le 0$, about $x = -1$

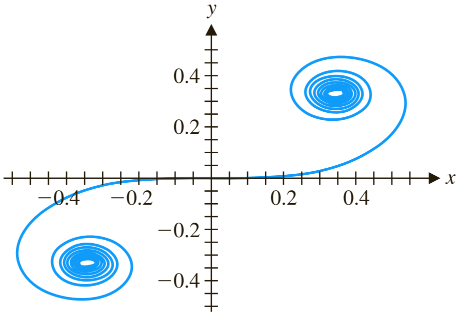
25. $\begin{cases} x = t^3 - 4t \\ y = t^2 - 3 \end{cases}, 0 \le t \le 2$, about the y-axis

26. $\begin{cases} x = t^3 - 4t \\ y = t^2 - 3 \end{cases}, 0 \le t \le 2$, about $y = 2$

27. An 8-foot-tall ladder stands vertically against a wall. The top of the ladder is pulled directly away from the wall, with the bottom remaining in contact with the wall, until the ladder rests on the floor. Find parametric equations for the position of the midpoint of the ladder. Find the distance traveled by the midpoint of the ladder.

28. The answer in exercise 27 equals the circumference of a quarter-circle of radius 4. Discuss whether this is a coincidence or not. Compare this value to the arc length in example 3.6. Discuss whether or not this is a coincidence.

29. The figure shown here is called **Cornu's spiral.** It is defined by the parametric equations $x = \int_0^t \cos \pi s^2 ds$ and $y = \int_0^t \sin \pi s^2 ds$. Each of these integrals is important in the study of Fresnel diffraction. Find the arc length of the spiral for (a) $-2\pi \le t \le 2\pi$ and (b) general $a \le t \le b$. Use this result to discuss the rate at which the spiralling occurs.



30. A cycloid is the curve traced out by a point on a circle as the circle rolls along the x-axis. Suppose the circle has radius 4, the point we are following starts at (0, 8) and the circle rolls from left to right. Find parametric equations for the cycloid and find the arc length as the circle completes one rotation.

EXPLORATORY EXERCISES

1. For the brachistochrone problem, two criteria for the fastest curve are: (1) steep slope at the origin and (2) concave down (note in Figure 9.16 that the positive y-axis points downward). Explain why these criteria make sense and identify other criteria. Then find parametric equations for a curve (different from the cycloid or those of exercises 13–16) that meet all the criteria. Use the formula of example 3.3 to find out how fast your curve is. You can't beat the cycloid, but get as close as you can!

2. The **tautochrone** problem is another surprising problem that was studied and solved by the same seventeenth-century mathematicians as the brachistochrone problem. (See *Journey Through Genius* by William Dunham for a description of this interesting piece of history, featuring the brilliant yet combative Bernoulli brothers.) Recall that the cycloid of example 3.3 runs from (0, 0) to $(\pi, 2)$. It takes the skier $k\sqrt{2}\pi = \pi/g$ seconds to ski the path. How long would it take the skier starting partway down the path, for instance, at $(\pi/2 - 1, 1)$? Find the slope of the cycloid at this point and compare it to the slope at (0, 0). Explain why the skier would build up less speed starting at this new point. Graph the speed function for the cycloid with $0 \le u \le 1$ and explain why the farther down the slope you start, the less speed you'll have. To see how speed and distance balance, use the time formula

$$T = \frac{\pi}{g}\int_a^1 \frac{\sqrt{1-\cos \pi u}}{\sqrt{\cos \pi a - \cos \pi u}}\, du$$

for the time it takes to ski the cycloid starting at the point $(\pi a - \sin \pi a, 1 - \cos \pi a)$, $0 < a < 1$. What is the remarkable property that the cycloid has?

9.4 POLAR COORDINATES

You've probably heard the cliche about how difficult it is to try to fit a round peg into a square hole. In some sense, we have faced this problem on several occasions so far in our study of calculus. For instance, if we were to use an integral to calculate the area of the circle $x^2 + y^2 = 9$, we would have

$$A = \int_{-3}^{3}\left[\sqrt{9-x^2} - \left(-\sqrt{9-x^2}\right)\right] dx = 2\int_{-3}^{3}\sqrt{9-x^2}\, dx. \tag{4.1}$$

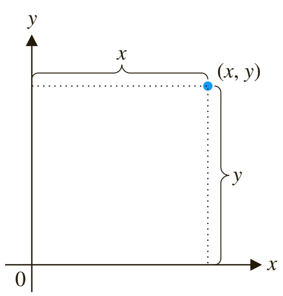


FIGURE 9.21
Rectangular coordinates

Note that you can evaluate this integral by making the trigonometric substitution $x = 3\sin\theta$. (It's a good thing that we already know a simple formula for the area of a circle!) A better plan might be to use parametric equations, such as $x = 3\cos t$, $y = 3\sin t$, for $0 \le t \le 2\pi$, to describe the circle. In section 9.2, we saw that the area is given by

$$\int_0^{2\pi} x(t)y'(t)\, dt = \int_0^{2\pi} (3\cos t)(3\cos t)\, dt$$
$$= 9\int_0^{2\pi} \cos^2 t\, dt.$$

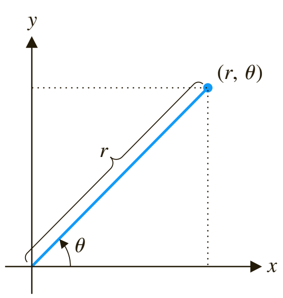


FIGURE 9.22
Polar coordinates

This is certainly better than the integral in (4.1), but it still requires some effort to evaluate this. The basic problem is that circles do not translate well into the usual x-y coordinate system. We often refer to this system as a system of **rectangular coordinates,** because a point is described in terms of the horizontal and vertical distances from the origin (see Figure 9.21).

An alternative description of a point in the xy-plane consists of specifying the distance r from the point to the origin and an angle θ (in radians) measured from the positive x-axis counterclockwise to the ray connecting the point and the origin (see Figure 9.22). We describe the point by the ordered pair (r, θ) and refer to r and θ as **polar coordinates** for the point.

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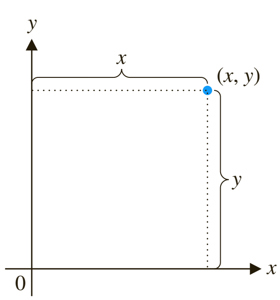

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Rectangular coordinates

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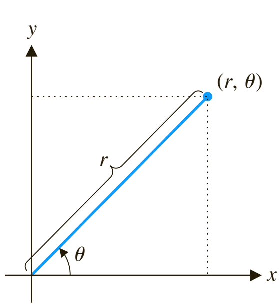

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EXAMPLE 4.1 Converting from Polar to Rectangular Coordinates

Plot the points with the indicated polar coordinates and determine the corresponding rectangular coordinates (x, y) for: (a) $(2, 0)$, (b) $(3, \frac{\pi}{2})$, (c) $(-3, \frac{\pi}{2})$ and (d) $(2, \pi)$.

Solution (a) Notice that the angle $\theta = 0$ locates the point on the positive x-axis. At a distance of $r = 2$ units from the origin, this corresponds to the point $(2, 0)$ in rectangular coordinates (see Figure 9.23a).

(b) The angle $\theta = \frac{\pi}{2}$ locates points on the positive y-axis. At a distance of $r = 3$ units from the origin, this corresponds to the point $(0, 3)$ in rectangular coordinates (see Figure 9.23b).

(c) The angle is the same as in (b), but a negative value of r indicates that the point is located 3 units in the opposite direction, at the point $(0, -3)$ in rectangular coordinates (see Figure 9.23b).

(d) The angle $\theta = \pi$ corresponds to the negative x-axis. The distance of $r = 2$ units from the origin gives us the point $(-2, 0)$ in rectangular coordinates (see Figure 9.23c).

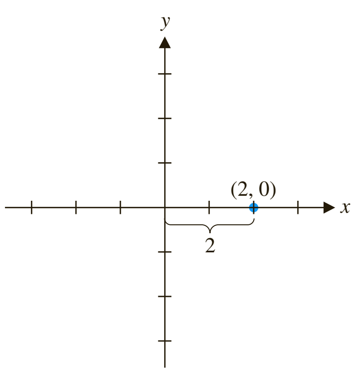


FIGURE 9.23a
The point $(2, 0)$ in polar coordinates

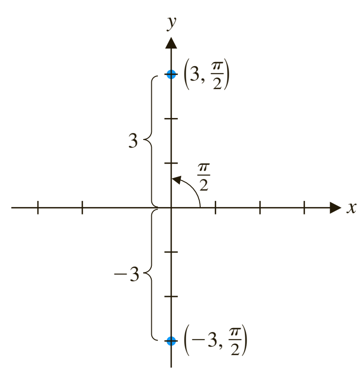


FIGURE 9.23b
The points $\left(3, \dfrac{\pi}{2}\right)$ and $\left(-3, \dfrac{\pi}{2}\right)$ in polar coordinates

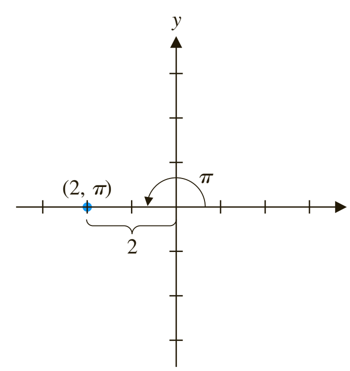


FIGURE 9.23c
The point $(2, \pi)$ in polar coordinates

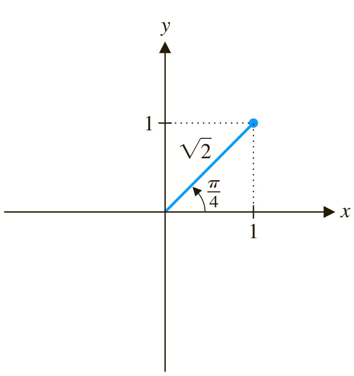


FIGURE 9.24a
Polar coordinates for the point $(1, 1)$

EXAMPLE 4.2 Converting from Rectangular to Polar Coordinates

Find a polar coordinate representation of the rectangular point $(1, 1)$.

Solution From Figure 9.24a, notice that the point lies on the line $y = x$, which makes an angle of $\frac{\pi}{4}$ with the positive x-axis. From the distance formula, we get that $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. This says that we can write the point as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates. Referring to Figure 9.24b (on the following page), notice that we can specify the same point by using a negative value of r, $r = -\sqrt{2}$, with the angle $\frac{5\pi}{4}$. (Think about this some.) Notice further, that the angle $\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi$ corresponds to the

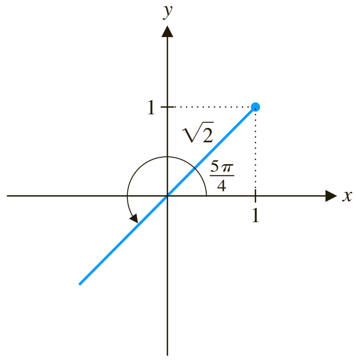


FIGURE 9.24b
An alternative polar representation of (1, 1)

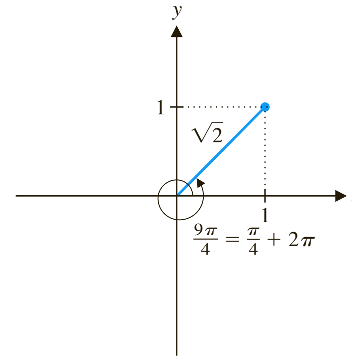


FIGURE 9.24c
Another polar representation of the point (1, 1)

same ray shown in Figure 9.24a (see Figure 9.24c). In fact, all of the polar points $(\sqrt{2}, \frac{\pi}{4} + 2n\pi)$ and $(-\sqrt{2}, \frac{5\pi}{4} + 2n\pi)$ for any integer n correspond to the same point in the xy-plane. ■

REMARK 4.1

As we see in example 4.2, each point (x, y) in the plane has infinitely many polar coordinate representations. For a given angle θ, the angles $\theta \pm 2\pi$, $\theta \pm 4\pi$ and so on, all correspond to the same ray. For convenience, we use the notation $\theta + 2n\pi$ (for any integer n) to represent all of these possible angles.

Referring to Figure 9.25, notice that it is a simple matter to find the rectangular coordinates (x, y) of a point specified in polar coordinates as (r, θ). From the usual definitions for $\sin\theta$ and $\cos\theta$, we get

$$\boxed{x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.} \tag{4.2}$$

From equations (4.2), notice that for a point (x, y) in the plane,

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

and for $x \neq 0$,
$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

That is, every polar coordinate representation (r, θ) of the point (x, y), where $x \neq 0$ must satisfy

$$\boxed{r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}.} \tag{4.3}$$

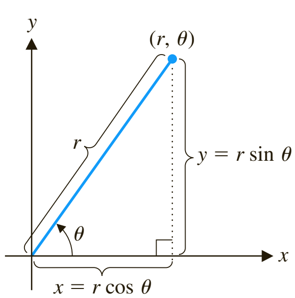


FIGURE 9.25
Converting from polar to rectangular coordinates

Notice that since there's more than one choice of r and θ, we cannot actually solve equations (4.3) to produce formulas for r and θ. In particular, while you might be tempted to write $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, this is not the only possible choice. Remember that for (r, θ) to be a polar representation of the point (x, y), θ can be *any* angle for which $\tan\theta = \frac{y}{x}$, while $\tan^{-1}\left(\frac{y}{x}\right)$ gives you an angle θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Finding polar coordinates for a given point is typically a process involving some graphing and some thought.

EXAMPLE 4.3 Converting from Rectangular to Polar Coordinates

Find all polar coordinate representations for the rectangular points (a) (2, 3) and (b) (−3, 1).

Solution (a) With $x = 2$ and $y = 3$, we have from (4.3) that

$$r^2 = x^2 + y^2 = 2^2 + 3^2 = 13,$$

REMARK 4.2

Notice that for any point (x, y) specified in rectangular coordinates $(x \neq 0)$, we can always write the point in polar coordinates using either of the polar angles $\tan^{-1}\left(\frac{y}{x}\right)$ or $\tan^{-1}\left(\frac{y}{x}\right)+\pi$. You can determine which angle corresponds to $r=\sqrt{x^2+y^2}$ and which corresponds to $r=-\sqrt{x^2+y^2}$ by looking at the quadrant in which the point lies.

so that $r=\pm\sqrt{13}$. Also,

$$\tan\theta=\frac{y}{x}=\frac{3}{2}.$$

One angle is then $\theta=\tan^{-1}\left(\frac{3}{2}\right)\approx 0.98$ radian. To determine which choice of r corresponds to this angle, note that (2, 3) is located in the first quadrant (see Figure 9.26a). Since 0.98 radian also puts you in the first quadrant, this angle corresponds to the positive value of r, so that $\left(\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right)\right)$ is one polar representation of the point. The negative choice of r corresponds to an angle one half-circle (i.e., π radians) away (see Figure 9.26b), so that another representation is $\left(-\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right)+\pi\right)$. Every other polar representation is found by adding multiples of 2π to the two angles used above. That is, every polar representation of the point (2, 3) must have the form $\left(\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right)+2n\pi\right)$ or $\left(-\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right)+\pi+2n\pi\right)$, for some integer choice of n.

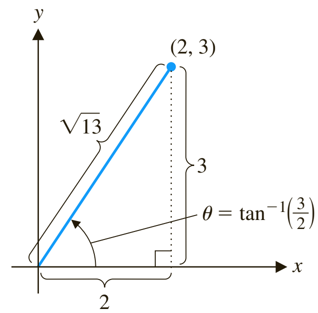


FIGURE 9.26a
The point (2, 3)

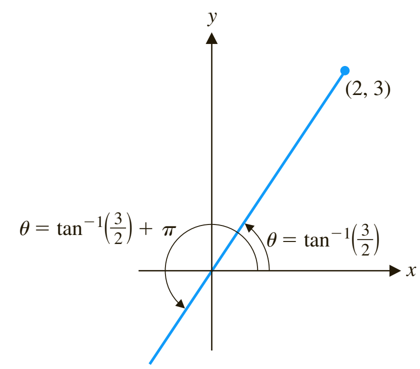


FIGURE 9.26b
Negative value of r

(b) For the point $(-3, 1)$, we have $x=-3$ and $y=1$. From (4.3), we have

$$r^2=x^2+y^2=(-3)^2+1^2=10,$$

so that $r=\pm\sqrt{10}$. Further,

$$\tan\theta=\frac{y}{x}=\frac{1}{-3},$$

so that the most obvious choice for the polar angle is $\theta=\tan^{-1}\left(-\frac{1}{3}\right)\approx -0.32$, which lies in the fourth quadrant. Since the point $(-3, 1)$ is in the second quadrant, this choice of the angle corresponds to the negative value of r (see Figure 9.27). The positive value of r then corresponds to the angle $\theta=\tan^{-1}\left(-\frac{1}{3}\right)+\pi$. Observe that all polar coordinate representations must then be of the form $\left(-\sqrt{10}, \tan^{-1}\left(-\frac{1}{3}\right)+2n\pi\right)$ or $\left(\sqrt{10}, \tan^{-1}\left(-\frac{1}{3}\right)+\pi+2n\pi\right)$, for some integer choice of n. ■

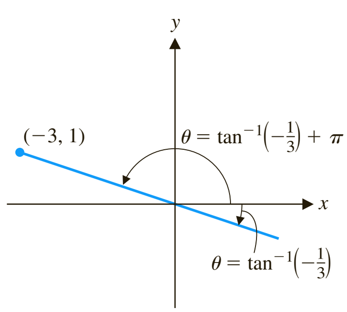


FIGURE 9.27
The point $(-3, 1)$

Observe that the conversion from polar coordinates to rectangular coordinates is completely straightforward, as in example 4.4.

EXAMPLE 4.4 Converting from Polar to Rectangular Coordinates

Find the rectangular coordinates for the polar points (a) $\left(3, \frac{\pi}{6}\right)$ and (b) $(-2, 3)$.

Solution For (a), we have from (4.2) that

$$x=r\cos\theta=3\cos\frac{\pi}{6}=\frac{3\sqrt{3}}{2}$$

and
$$y = r\sin\theta = 3\sin\frac{\pi}{6} = \frac{3}{2}.$$

The rectangular point is then $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. For (b), we have
$$x = r\cos\theta = -2\cos 3 \approx 1.98$$
and
$$y = r\sin\theta = -2\sin 3 \approx -0.28.$$

The rectangular point is $(-2\cos 3, -2\sin 3)$, which is located at approximately $(1.98, -0.28)$. ■

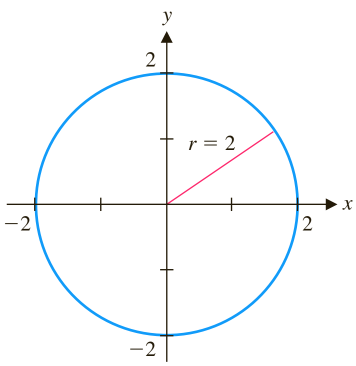


FIGURE 9.28a
The circle $r = 2$

The **graph** of a polar equation $r = f(\theta)$ is the set of all points (x, y) for which $x = r\cos\theta$, $y = r\sin\theta$ and $r = f(\theta)$. In other words, the graph of a polar equation is a graph in the xy-plane of all those points whose polar coordinates satisfy the given equation. We begin by sketching two very simple (and familiar) graphs. The key to drawing the graph of a polar equation is to always keep in mind what the polar coordinates represent.

EXAMPLE 4.5 Some Simple Graphs in Polar Coordinates

Sketch the graphs of (a) $r = 2$ and (b) $\theta = \pi/3$.

Solution For (a), notice that $2 = r = \sqrt{x^2 + y^2}$ and so, we want all points whose distance from the origin is 2 (with *any* polar angle θ). Of course, this is the definition of a circle of radius 2 with center at the origin (see Figure 9.28a). For (b), notice that $\theta = \pi/3$ specifies all points with a polar angle of $\pi/3$ from the positive x-axis (at *any* distance r from the origin). Including negative values for r, this defines a line with slope $\tan\pi/3 = \sqrt{3}$ (see Figure 9.28b). ■

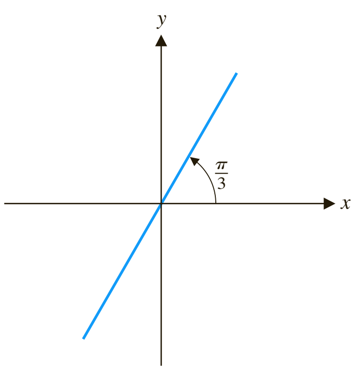


FIGURE 9.28b
The line $\theta = \frac{\pi}{3}$

It turns out that many familiar curves have simple polar equations.

EXAMPLE 4.6 Converting an Equation from Rectangular to Polar Coordinates

Find the polar equation(s) corresponding to the hyperbola $x^2 - y^2 = 9$ (see Figure 9.29).

Solution From (4.2), we have
$$\begin{aligned} 9 = x^2 - y^2 &= r^2\cos^2\theta - r^2\sin^2\theta \\ &= r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta. \end{aligned}$$

Solving for r, we get
$$r^2 = \frac{9}{\cos 2\theta} = 9\sec 2\theta,$$
so that
$$r = \pm 3\sqrt{\sec 2\theta}.$$

Notice that in order to keep $\sec 2\theta > 0$, we can restrict 2θ to lie in the interval $-\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$, so that $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Observe that with this range of values of θ, the hyperbola is drawn exactly once, where $r = 3\sqrt{\sec 2\theta}$ corresponds to the right branch of the hyperbola and $r = -3\sqrt{\sec 2\theta}$ corresponds to the left branch. ■

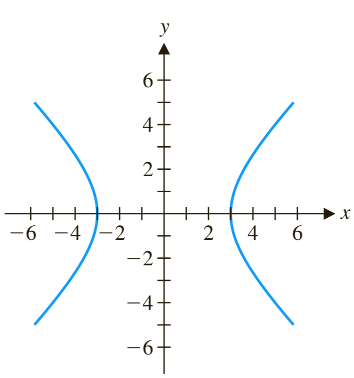


FIGURE 9.29
$x^2 - y^2 = 9$

EXAMPLE 4.7 A Surprisingly Simple Polar Graph

Sketch the graph of the polar equation $r = \sin\theta$.

Solution For reference, we first sketch a graph of the sine function in *rectangular* coordinates on the interval $[0, 2\pi]$ (see Figure 9.30a). Notice that on the interval $0 \le \theta \le \frac{\pi}{2}$, $\sin\theta$ increases from 0 to its maximum value of 1. This corresponds to a polar arc in the first quadrant from the origin ($r = 0$) to 1 unit up on the y-axis. Then, on the interval $\frac{\pi}{2} \le \theta \le \pi$, $\sin\theta$ decreases from 1 to 0. This corresponds to an arc in the second quadrant, from 1 unit up on the y-axis back to the origin. Next, on the interval $\pi \le \theta \le \frac{3\pi}{2}$, $\sin\theta$ decreases from 0 to its minimum value of -1. Since the values of r are negative, remember that this means that the points plotted are in the *opposite* quadrant (i.e., the first quadrant). Notice that this traces out the same curve in the first quadrant as we've already drawn for $0 \le \theta \le \frac{\pi}{2}$. Likewise, taking θ in the interval $\frac{3\pi}{2} \le \theta \le 2\pi$ retraces the portion of the curve in the second quadrant. Since $\sin\theta$ is periodic of period 2π, taking further values of θ simply retraces portions of the curve that we have already drawn. A sketch of the polar graph is shown in Figure 9.30b. We now verify that this curve is actually a circle. Notice that if we multiply the equation $r = \sin\theta$ through by r, we get

$$r^2 = r\sin\theta.$$

You should immediately recognize from (4.2) and (4.3) that $y = r\sin\theta$ and $r^2 = x^2 + y^2$. This gives us the rectangular equation

$$x^2 + y^2 = y$$

or

$$0 = x^2 + y^2 - y.$$

Completing the square, we get

$$0 = x^2 + \left(y^2 - y + \frac{1}{4}\right) - \frac{1}{4}$$

or, adding $\frac{1}{4}$ to both sides,

$$\left(\frac{1}{2}\right)^2 = x^2 + \left(y - \frac{1}{2}\right)^2.$$

This is the rectangular equation for the circle of radius $\frac{1}{2}$ centered at the point $\left(0, \frac{1}{2}\right)$, which is what we see in Figure 9.30b. ■

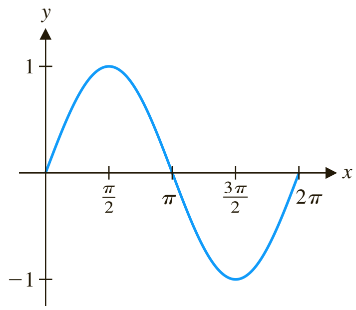


FIGURE 9.30a
$y = \sin x$ plotted in *rectangular* coordinates

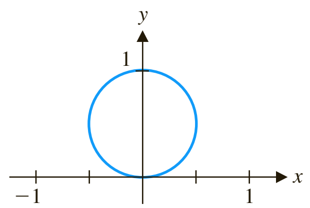


FIGURE 9.30b
The circle $r = \sin\theta$

The graphs of many polar equations are not the graphs of *any* functions of the form $y = f(x)$, as in example 4.8.

EXAMPLE 4.8 An Archimedian Spiral

Sketch the graph of the polar equation $r = \theta$, for $\theta \ge 0$.

Solution Notice that here, as θ increases, so too does r. That is, as the polar angle increases, the distance from the origin also increases accordingly. This produces the spiral (an example of an **Archimedian spiral**) seen in Figure 9.31. ■

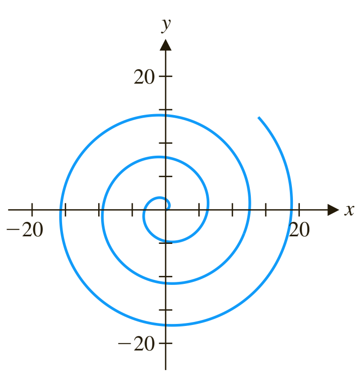


FIGURE 9.31
The spiral $r = \theta, \theta \ge 0$

The graphs shown in examples 4.9, 4.10 and 4.11 are all in the general class known as **limaçons.** This class of graphs is defined by $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$, for positive constants a and b. If $a = b$, the graphs are called **cardioids.**

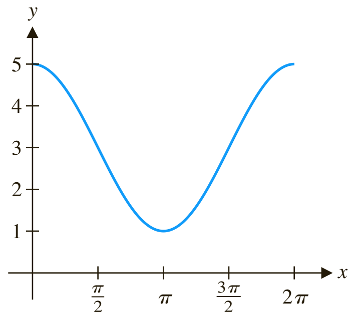


FIGURE 9.32
$y = 3 + 2\cos x$ in rectangular coordinates

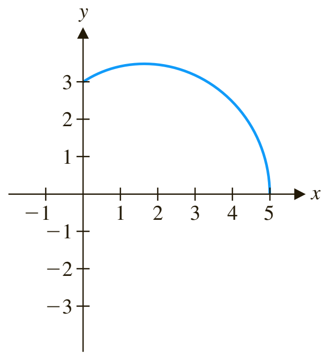


FIGURE 9.33a
$0 \le \theta \le \frac{\pi}{2}$

EXAMPLE 4.9 A Limaçon

Sketch the graph of the polar equation $r = 3 + 2\cos\theta$.

Solution We begin by sketching the graph of $y = 3 + 2\cos x$ in rectangular coordinates on the interval $[0, 2\pi]$, to use as a reference (see Figure 9.32). Notice that in this case, we have $r = 3 + 2\cos\theta > 0$ for all values of θ. Further, the maximum value of r is 5 (corresponding to when $\cos\theta = 1$ at $\theta = 0, 2\pi$, etc.) and the minimum value of r is 1 (corresponding to when $\cos\theta = -1$ at $\theta = \pi, 3\pi$, etc.). In this case, the polar graph is traced out with $0 \le \theta \le 2\pi$. We summarize the intervals of increase and decrease for r in the following table.

Interval	$\cos\theta$	$r = 3 + 2\cos\theta$
$\left[0, \frac{\pi}{2}\right]$	Decreases from 1 to 0	Decreases from 5 to 3
$\left[\frac{\pi}{2}, \pi\right]$	Decreases from 0 to -1	Decreases from 3 to 1
$\left[\pi, \frac{3\pi}{2}\right]$	Increases from -1 to 0	Increases from 1 to 3
$\left[\frac{3\pi}{2}, 2\pi\right]$	Increases from 0 to 1	Increases from 3 to 5

In Figures 9.33a–9.33d, we show how the sketch progresses through each interval indicated in the table, with the completed figure (called a **limaçon**) shown in Figure 9.33d.

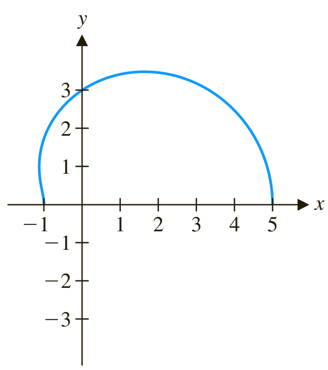


FIGURE 9.33b
$0 \le \theta \le \pi$

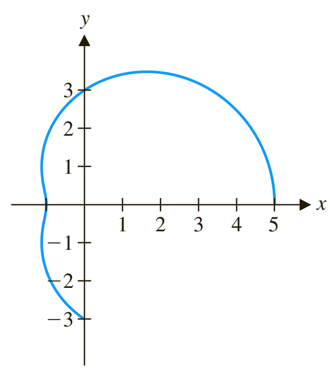


FIGURE 9.33c
$0 \le \theta \le \frac{3\pi}{2}$

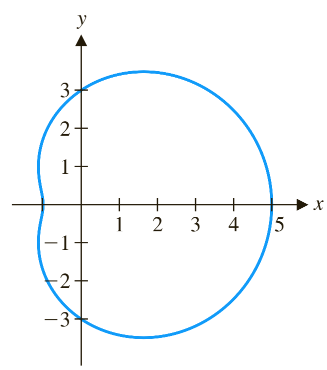


FIGURE 9.33d
$0 \le \theta \le 2\pi$

EXAMPLE 4.10 The Graph of a Cardioid

Sketch the graph of the polar equation $r = 2 - 2\sin\theta$.

Solution As we have done several times now, we first sketch a graph of $y = 2 - 2\sin x$ in rectangular coordinates, on the interval $[0, 2\pi]$, as in Figure 9.34. We summarize the intervals of increase and decrease in the following table.

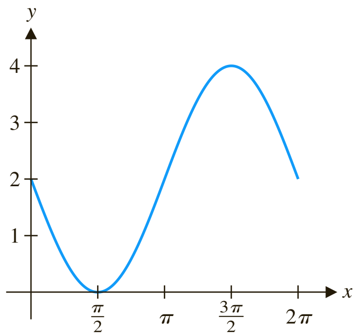


FIGURE 9.34
$y = 2 - 2\sin x$ in rectangular coordinates

Interval	$\sin\theta$	$r = 2 - 2\sin\theta$
$\left[0, \frac{\pi}{2}\right]$	Increases from 0 to 1	Decreases from 2 to 0
$\left[\frac{\pi}{2}, \pi\right]$	Decreases from 1 to 0	Increases from 0 to 2
$\left[\pi, \frac{3\pi}{2}\right]$	Decreases from 0 to -1	Increases from 2 to 4
$\left[\frac{3\pi}{2}, 2\pi\right]$	Increases from -1 to 0	Decreases from 4 to 2

Again, we sketch the graph in stages, corresponding to each of the intervals indicated in the table, as seen in Figures 9.35a–9.35d.

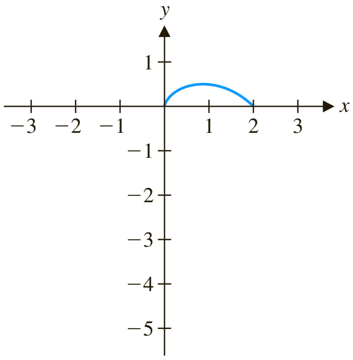


FIGURE 9.35a
$0 \le \theta \le \frac{\pi}{2}$

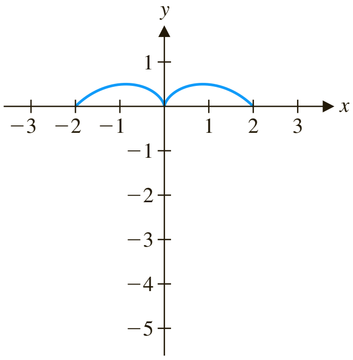


FIGURE 9.35b
$0 \le \theta \le \pi$

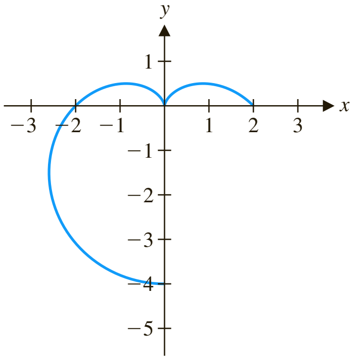


FIGURE 9.35c
$0 \le \theta \le \frac{3\pi}{2}$

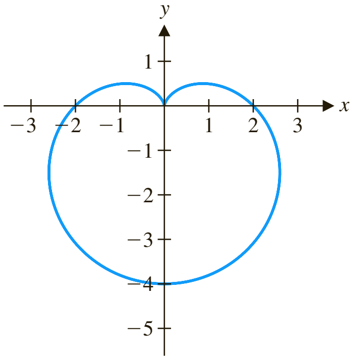


FIGURE 9.35d
$0 \le \theta \le 2\pi$

The completed graph appears in Figure 9.35d and is sketched out for $0 \le \theta \le 2\pi$. You can see why this figure is called a **cardioid** ("heartlike"). ■

EXAMPLE 4.11 A Limaçon with a Loop

Sketch the graph of the polar equation $r = 1 - 2\sin\theta$.

Solution We again begin by sketching a graph of $y = 1 - 2\sin x$ in rectangular coordinates, as in Figure 9.36. We summarize the intervals of increase and decrease in the following table.

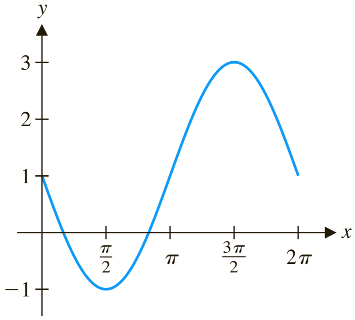


FIGURE 9.36
$y = 1 - 2\sin x$ in rectangular coordinates

Interval	$\sin\theta$	$r = 1 - 2\sin\theta$
$\left[0, \frac{\pi}{2}\right]$	Increases from 0 to 1	Decreases from 1 to −1
$\left[\frac{\pi}{2}, \pi\right]$	Decreases from 1 to 0	Increases from −1 to 1
$\left[\pi, \frac{3\pi}{2}\right]$	Decreases from 0 to −1	Increases from 1 to 3
$\left[\frac{3\pi}{2}, 2\pi\right]$	Increases from −1 to 0	Decreases from 3 to 1

Notice that since r assumes both positive and negative values in this case, we need to exercise a bit more caution, as negative values for r cause us to draw that portion of the graph in the *opposite* quadrant. Observe that $r = 0$ when $1 - 2\sin\theta = 0$, that is, when $\sin\theta = \frac{1}{2}$. This will occur when $\theta = \frac{\pi}{6}$ and when $\theta = \frac{5\pi}{6}$. For this reason, we expand the above table, to include more intervals and where we also indicate the quadrant where the graph is to be drawn, as follows:

Interval	$\sin\theta$	$r = 1 - 2\sin\theta$	*Quadrant*
$\left[0, \frac{\pi}{6}\right]$	Increases from 0 to $\frac{1}{2}$	Decreases from 1 to 0	First
$\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$	Increases from $\frac{1}{2}$ to 1	Decreases from 0 to −1	Third
$\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$	Decreases from 1 to $\frac{1}{2}$	Increases from −1 to 0	Fourth
$\left[\frac{5\pi}{6}, \pi\right]$	Decreases from $\frac{1}{2}$ to 0	Increases from 0 to 1	Second
$\left[\pi, \frac{3\pi}{2}\right]$	Decreases from 0 to −1	Increases from 1 to 3	Third
$\left[\frac{3\pi}{2}, 2\pi\right]$	Increases from −1 to 0	Decreases from 3 to 1	Fourth

We sketch the graph in stages in Figures 9.37a–9.37f, corresponding to each of the intervals indicated in the table.

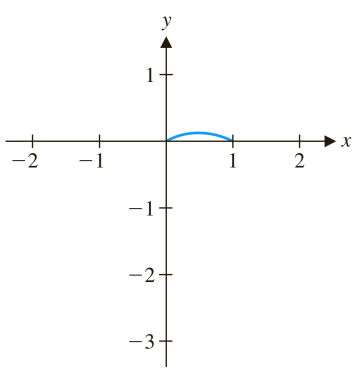


FIGURE 9.37a
$0 \le \theta \le \frac{\pi}{6}$

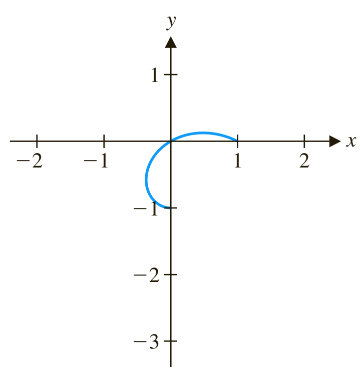


FIGURE 9.37b
$0 \le \theta \le \frac{\pi}{2}$

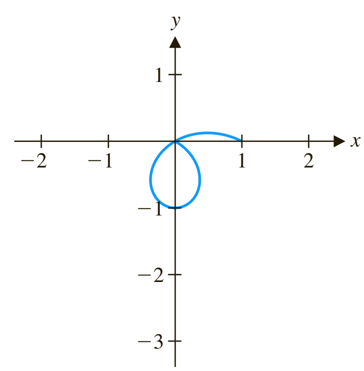


FIGURE 9.37c
$0 \le \theta \le \frac{5\pi}{6}$

The completed graph appears in Figure 9.37f and is sketched out for $0 \le \theta \le 2\pi$. You should observe from this the importance of determining where $r = 0$, as well as where r is increasing and decreasing.

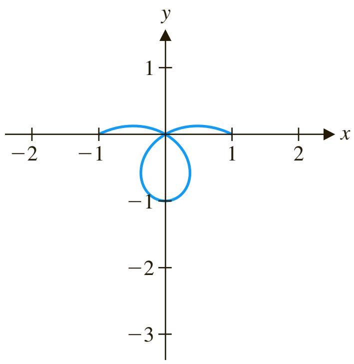


FIGURE 9.37d
$0 \leq \theta \leq \pi$

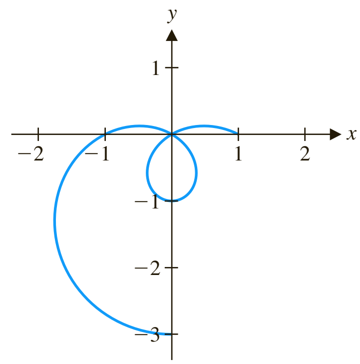


FIGURE 9.37e
$0 \leq \theta \leq \frac{3\pi}{2}$

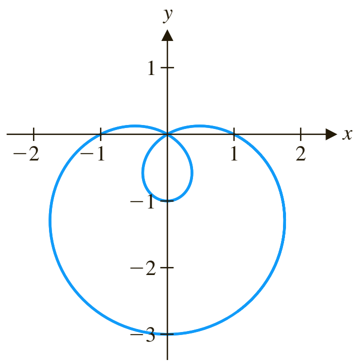


FIGURE 9.37f
$0 \leq \theta \leq 2\pi$

EXAMPLE 4.12 A Four-Leaf Rose

Sketch the graph of the polar equation $r = \sin 2\theta$.

Solution As usual, we will first draw a graph of $y = \sin 2x$ in rectangular coordinates on the interval $[0, 2\pi]$, as seen in Figure 9.38. Notice that the period of $\sin 2\theta$ is only π. We summarize the intervals on which the function is increasing and decreasing in the following table.

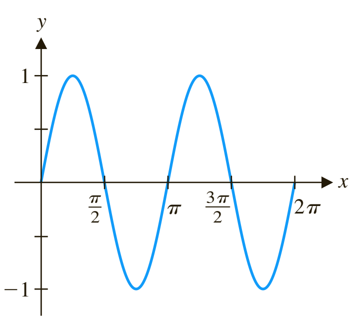


FIGURE 9.38
$y = \sin 2x$ in rectangular coordinates

Interval	$r = \sin 2\theta$	*Quadrant*
$\left[0, \frac{\pi}{4}\right]$	Increases from 0 to 1	First
$\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$	Decreases from 1 to 0	First
$\left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$	Decreases from 0 to −1	Fourth
$\left[\frac{3\pi}{4}, \pi\right]$	Increases from −1 to 0	Fourth
$\left[\pi, \frac{5\pi}{4}\right]$	Increases from 0 to 1	Third
$\left[\frac{5\pi}{4}, \frac{3\pi}{2}\right]$	Decreases from 1 to 0	Third
$\left[\frac{3\pi}{2}, \frac{7\pi}{4}\right]$	Decreases from 0 to −1	Second
$\left[\frac{7\pi}{4}, 2\pi\right]$	Increases from −1 to 0	Second

We sketch the graph in stages in Figures 9.39a–9.39h, each one corresponding to the intervals indicated in the table, where we have also indicated the lines $y = \pm x$, as a guide.

This is an interesting curve known as a **four-leaf rose.** Notice again the significance of the points corresponding to $r = 0$, or $\sin 2\theta = 0$. Also, notice that r reaches a maximum of 1 when $2\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$ or $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$ and r reaches a minimum of −1 when $2\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$ or $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$. Again, you must keep in mind that when the value of r is negative, this causes us to draw the graph in the opposite quadrant.

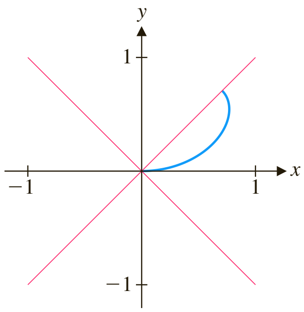


FIGURE 9.39a
$0 \le \theta \le \frac{\pi}{4}$

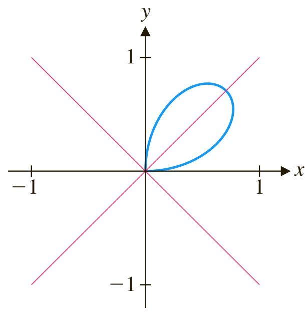


FIGURE 9.39b
$0 \le \theta \le \frac{\pi}{2}$

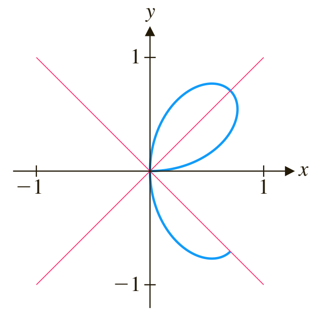


FIGURE 9.39c
$0 \le \theta \le \frac{3\pi}{4}$

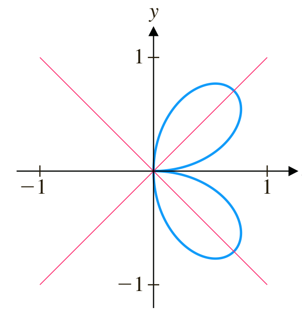


FIGURE 9.39d
$0 \le \theta \le \pi$

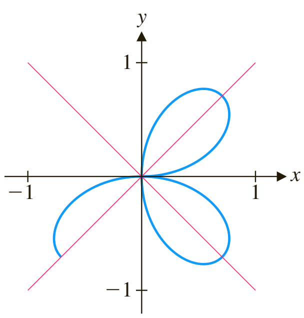


FIGURE 9.39e
$0 \le \theta \le \frac{5\pi}{4}$

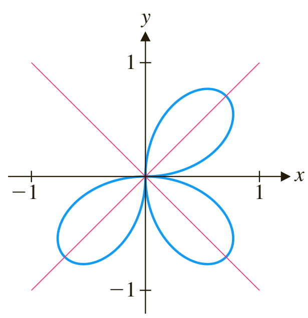


FIGURE 9.39f
$0 \le \theta \le \frac{3\pi}{2}$

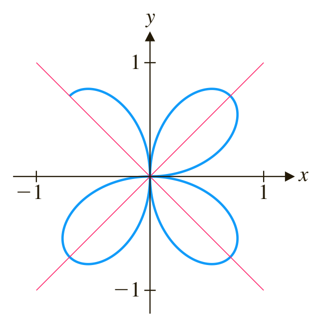


FIGURE 9.39g
$0 \le \theta \le \frac{7\pi}{4}$

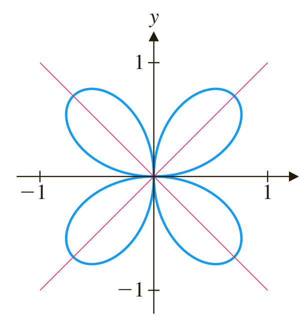


FIGURE 9.39h
$0 \le \theta \le 2\pi$

Note that in example 4.12, even though the period of the function $\sin 2\theta$ is π, it took θ-values ranging from 0 to 2π to sketch the entire curve $r = \sin 2\theta$. By contrast, the period of the function $\sin\theta$ is 2π, but the circle $r = \sin\theta$ was completed with $0 \le \theta \le \pi$. To determine the range of values of θ that produces a graph, you need to carefully identify important points as we did in example 4.12. The Trace feature found on graphing calculators can be very helpful for getting an idea of the θ-range, but remember that such Trace values are only approximate.

You will explore a variety of other interesting graphs in the exercises.

BEYOND FORMULAS

The graphics in Figures 9.35, 9.37 and 9.39 provide a good visual model of how to think of polar graphs. Most polar graphs $r = f(\theta)$ can be sketched as a sequence of connected arcs, where the arcs start and stop at places where $r = 0$ or where a new quadrant is entered. By breaking the larger graph into small arcs, you can use the properties of f to quickly determine where each arc starts and stops.

EXERCISES 9.4

WRITING EXERCISES

1. Suppose a point has polar representation (r, θ). Explain why another polar representation of the same point is $(-r, \theta + \pi)$.
2. After working with rectangular coordinates for so long, the idea of polar representations may seem slightly awkward. However, polar representations are entirely natural in many settings. For instance, if you were on a ship at sea and another ship was approaching you, explain whether you would use a polar representation (distance and bearing) or a rectangular representation (distance east-west and distance north-south).
3. In example 4.7, the graph (a circle) of $r = \sin\theta$ is completely traced out with $0 \le \theta \le \pi$. Explain why graphing $r = \sin\theta$ with $\pi \le \theta \le 2\pi$ would produce the same full circle.
4. Two possible advantages of introducing a new coordinate system are making previous problems easier to solve and allowing new problems to be solved. Give two examples of graphs for which the polar equation is simpler than the rectangular equation. Give two examples of polar graphs for which you have not seen a rectangular equation.

In exercises 1–6, plot the given polar points (r, θ) and find their rectangular representation.

1. $(2, 0)$ **2.** $(2, \pi)$ **3.** $(-2, \pi)$
4. $\left(-3, \frac{3\pi}{2}\right)$ **5.** $(3, -\pi)$ **6.** $\left(5, -\frac{\pi}{2}\right)$

In exercises 7–12, find all polar coordinate representations of the given rectangular point.

7. $(2, -2)$ **8.** $(-1, 1)$ **9.** $(0, 3)$
10. $(2, -1)$ **11.** $(3, 4)$ **12.** $(-2, -\sqrt{5})$

In exercises 13–18, find rectangular coordinates for the given polar point.

13. $\left(2, -\frac{\pi}{3}\right)$ **14.** $\left(-1, \frac{\pi}{3}\right)$ **15.** $(0, 3)$
16. $\left(3, \frac{\pi}{8}\right)$ **17.** $\left(4, \frac{\pi}{10}\right)$ **18.** $(-3, 1)$

In exercises 19–26, sketch the graph of the polar equation and find a corresponding x-y equation.

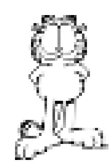
19. $r = 4$ **20.** $r = \sqrt{3}$ **21.** $\theta = \pi/6$
22. $\theta = 3\pi/4$ **23.** $r = \cos\theta$ **24.** $r = 2\cos\theta$
25. $r = 3\sin\theta$ **26.** $r = 2\sin\theta$

In exercises 27–50, sketch the graph and identify all values of θ where $r = 0$ and a range of values of θ that produces one copy of the graph.

27. $r = \cos 2\theta$ **28.** $r = \cos 3\theta$
29. $r = \sin 3\theta$ **30.** $r = \sin 2\theta$
31. $r = 3 + 2\sin\theta$ **32.** $r = 2 - 2\cos\theta$
33. $r = 2 - 4\sin\theta$ **34.** $r = 2 + 4\cos\theta$
35. $r = 2 + 2\sin\theta$ **36.** $r = 3 - 6\cos\theta$
37. $r = \frac{1}{4}\theta$ **38.** $r = e^{\theta/4}$
39. $r = 2\cos(\theta - \pi/4)$ **40.** $r = 2\sin(3\theta - \pi)$
41. $r = \cos\theta + \sin\theta$ **42.** $r = \cos\theta + \sin 2\theta$
43. $r = \tan^{-1} 2\theta$ **44.** $r = \theta/\sqrt{\theta^2 + 1}$
45. $r = 2 + 4\cos 3\theta$ **46.** $r = 2 - 4\sin 4\theta$
47. $r = \dfrac{2}{1 + \sin\theta}$ **48.** $r = \dfrac{3}{1 - \sin\theta}$
49. $r = \dfrac{2}{1 + \cos\theta}$ **50.** $r = \dfrac{3}{1 - \cos\theta}$

51. Graph $r = 4\cos\theta\sin^2\theta$ and explain why there is no curve to the left of the y-axis.

52. Graph $r = \theta\cos\theta$ for $-2\pi \le \theta \le 2\pi$. Explain why this is called the Garfield curve.



53. Based on your graphs in exercises 23 and 24, conjecture the graph of $r = a\cos\theta$ for any positive constant a.

54. Based on your graphs in exercises 25 and 26, conjecture the graph of $r = a\sin\theta$ for any positive constant a.

55. Based on the graphs in exercises 27 and 28 and others (try $r = \cos 4\theta$ and $r = \cos 5\theta$), conjecture the graph of $r = \cos n\theta$ for any positive integer n.

56. Based on the graphs in exercises 29 and 30 and others (try $r = \sin 4\theta$ and $r = \sin 5\theta$), conjecture the graph of $r = \sin n\theta$ for any positive integer n.

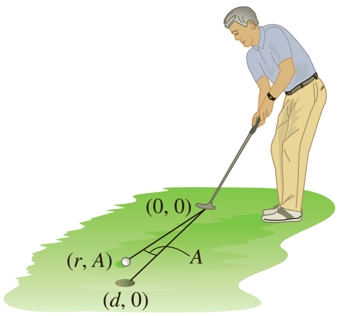
In exercises 57–62, find a polar equation corresponding to the given rectangular equation.

57. $y^2 - x^2 = 4$ **58.** $x^2 + y^2 = 9$
59. $x^2 + y^2 = 16$ **60.** $x^2 + y^2 = x$
61. $y = 3$ **62.** $x = 2$

63. Sketch the graph of $r = \cos \frac{11}{12}\theta$ first for $0 \le \theta \le \pi$, then for $0 \le \theta \le 2\pi$, then for $0 \le \theta \le 3\pi, \ldots,$ and finally for $0 \le \theta \le 24\pi$. Discuss any patterns that you find and predict what will happen for larger domains.

64. Sketch the graph of $r = \cos \pi\theta$ first for $0 \le \theta \le 1$, then for $0 \le \theta \le 2$, then for $0 \le \theta \le 3, \ldots$ and finally for $0 \le \theta \le 20$. Discuss any patterns that you find and predict what will happen for larger domains.

65. One situation where polar coordinates apply directly to sports is in making a golf putt. The two factors that the golfer tries to control are distance (determined by speed) and direction (usually called the "line"). Suppose a putter is d feet from the hole, which has radius $h = \frac{1}{6}'$. Show that the path of the ball will intersect the hole if the angle A in the figure satisfies $-\sin^{-1}(h/d) < A < \sin^{-1}(h/d)$.



66. The distance r that the golf ball in exercise 65 travels also needs to be controlled. The ball must reach the front of the hole. In rectangular coordinates, the hole has equation $(x - d)^2 + y^2 = h^2$, so the left side of the hole is $x = d - \sqrt{h^2 - y^2}$. Show that this converts in polar coordinates to $r = d\cos\theta - \sqrt{d^2\cos^2\theta - (d^2 - h^2)}$. (Hint: Substitute for x and y, isolate the square root term, square both sides, combine r^2 terms and use the quadratic formula.)

67. The golf putt in exercises 65 and 66 will not go in the hole if it is hit too hard. Suppose that the putt would go $r = d + c$ feet if it did not go in the hole ($c > 0$). For a putt hit toward the center of the hole, define b to be the largest value of c such that the putt goes in (i.e., if the ball is hit more than b feet past the hole, it is hit too hard). Experimental evidence (see Dave Pelz's *Putt Like the Pros*) shows that at other angles A, the distance r must be less than $d + b\left(1 - \left[\dfrac{A}{\sin^{-1}(h/d)}\right]^2\right)$. The results of exercises 65 and 66 define limits for the angle A and distance r of a successful putt. Identify the functions $r_1(A)$ and $r_2(A)$ such that $r_1(A) < r < r_2(A)$ and constants A_1 and A_2 such that $A_1 < A < A_2$.

68. Take the general result of exercise 67 and apply it to a putt of $d = 15$ feet with a value of $b = 4$ feet. Visualize this by graphing the region

$$15\cos\theta - \sqrt{225\cos^2\theta - (225 - 1/36)}$$
$$< r < 15 + 4\left(1 - \left[\frac{\theta}{\sin^{-1}(1/90)}\right]^2\right)$$

with $-\sin^{-1}(1/90) < \theta < \sin^{-1}(1/90)$. A good choice of graphing windows is $13.8 \le x \le 19$ and $-0.5 \le y \le 0.5$.

EXPLORATORY EXERCISES

1. In this exercise, you will explore the roles of the constants a, b and c in the graph of $r = af(b\theta + c)$. To start, sketch $r = \sin\theta$ followed by $r = 2\sin\theta$ and $r = 3\sin\theta$. What does the constant a affect? Then sketch $r = \sin(\theta + \pi/2)$ and $r = \sin(\theta - \pi/4)$. What does the constant c affect? Now for the tough one. Sketch $r = \sin 2\theta$ and $r = \sin 3\theta$. What does the constant b seem to affect? Test all of your hypotheses on the base function $r = 1 + 2\cos\theta$ and several functions of your choice.

2. The polar curve $r = ae^{b\theta}$ is sometimes called an **equiangular curve.** To see why, sketch the curve and then show that $\dfrac{dr}{d\theta} = br$. A somewhat complicated geometric argument shows that $\dfrac{dr}{d\theta} = r\cot\alpha$, where α is the angle between the tangent line and the line connecting the point on the curve to the origin. Comparing equations, conclude that the angle α is constant (hence "equiangular"). To illustrate this property, compute α for the points at $\theta = 0$ and $\theta = \pi$ for $r = e^\theta$. This type of spiral shows up often in nature, possibly because the equal-angle property can be easily achieved. Spirals can be found among shellfish (the picture shown here is of an ammonite fossil from about 350 million years ago) and the florets of the common daisy. Other examples, including the connection to sunflowers, the Fibonacci sequence and the musical scale, can be found in H. E. Huntley's *The Divine Proportion*.

9.5 CALCULUS AND POLAR COORDINATES

Having introduced polar coordinates and looked at a variety of polar graphs, our next step is to extend the techniques of calculus to the case of polar coordinates. In this section, we focus on tangent lines, area and arc length. Surface area and other applications will be examined in the exercises.

Notice that you can think of the graph of the polar equation $r = f(\theta)$ as the graph of the parametric equations $x = f(t)\cos t$, $y(t) = f(t)\sin t$ (where we have used the parameter $t = \theta$), since from (4.2)

$$x = r\cos\theta = f(\theta)\cos\theta \tag{5.1}$$

and

$$y = r\sin\theta = f(\theta)\sin\theta. \tag{5.2}$$

In view of this, we can now take any results already derived for parametric equations and extend these to the special case of polar coordinates.

In section 9.2, we showed that the slope of the tangent line at the point corresponding to $\theta = a$ is given [from (2.1)] to be

$$\boxed{\left.\frac{dy}{dx}\right|_{\theta=a} = \frac{\frac{dy}{d\theta}(a)}{\frac{dx}{d\theta}(a)}.} \tag{5.3}$$

From the product rule, (5.1) and (5.2), we have

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

and

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta.$$

Putting these together with (5.3), we get

$$\left.\frac{dy}{dx}\right|_{\theta=a} = \frac{f'(a)\sin a + f(a)\cos a}{f'(a)\cos a - f(a)\sin a}. \tag{5.4}$$

EXAMPLE 5.1 Finding the Slope of the Tangent Line to a Three-Leaf Rose

Find the slope of the tangent line to the three-leaf rose $r = \sin 3\theta$ at $\theta = 0$ and $\theta = \frac{\pi}{4}$.

Solution A sketch of the curve is shown in Figure 9.40a. From (4.2), we have

$$y = r\sin\theta = \sin 3\theta\sin\theta$$

and

$$x = r\cos\theta = \sin 3\theta\cos\theta.$$

Using (5.3), we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{(3\cos 3\theta)\sin\theta + \sin 3\theta(\cos\theta)}{(3\cos 3\theta)\cos\theta - \sin 3\theta(\sin\theta)}. \tag{5.5}$$

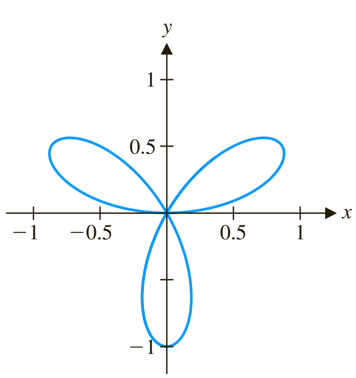


FIGURE 9.40a
Three-leaf rose

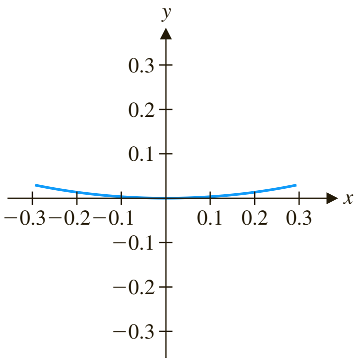


FIGURE 9.40b
$-0.1 \le \theta \le 0.1$

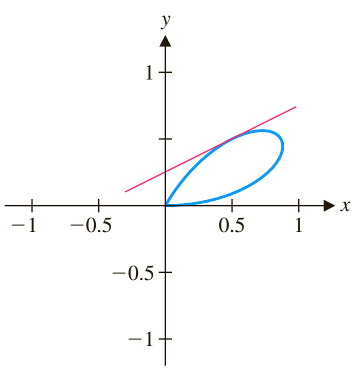


FIGURE 9.40c
The tangent line at $\theta = \frac{\pi}{4}$

At $\theta = 0$, this gives us

$$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{(3\cos 0)\sin 0 + \sin 0(\cos 0)}{(3\cos 0)\cos 0 - \sin 0(\sin 0)} = \frac{0}{3} = 0.$$

In Figure 9.40b, we sketch $r = \sin 3\theta$ for $-0.1 \le \theta \le 0.1$, in order to isolate the portion of the curve around $\theta = 0$. Notice that from this figure, a slope of 0 seems reasonable.

Similarly, at $\theta = \frac{\pi}{4}$, we have from (5.5) that

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{\left(3\cos\frac{3\pi}{4}\right)\sin\frac{\pi}{4} + \sin\frac{3\pi}{4}\left(\cos\frac{\pi}{4}\right)}{\left(3\cos\frac{3\pi}{4}\right)\cos\frac{\pi}{4} - \sin\frac{3\pi}{4}\left(\sin\frac{\pi}{4}\right)} = \frac{-\frac{3}{2} + \frac{1}{2}}{-\frac{3}{2} - \frac{1}{2}} = \frac{1}{2}.$$

In Figure 9.40c, we show the section of $r = \sin 3\theta$ for $0 \le \theta \le \frac{\pi}{3}$, along with the tangent line at $\theta = \frac{\pi}{4}$. ■

Recall that for functions $y = f(x)$, horizontal tangents were especially significant for locating maximum and minimum points. For polar graphs, the significant points are often places where r has reached a maximum or minimum, which may or may not correspond to a horizontal tangent. We explore this idea further in example 5.2.

EXAMPLE 5.2 Polar Graphs and Horizontal Tangent Lines

For the three-leaf rose $r = \sin 3\theta$, find the locations of all horizontal tangent lines and interpret the significance of these points. Further, at the three points where $|r|$ is a maximum, show that the tangent line is perpendicular to the line segment connecting the point to the origin.

Solution From (5.3) and (5.4), we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Here, $f(\theta) = \sin 3\theta$ and so, to have $\frac{dy}{dx} = 0$, we must have

$$0 = \frac{dy}{d\theta} = 3\cos 3\theta \sin\theta + \sin 3\theta \cos\theta.$$

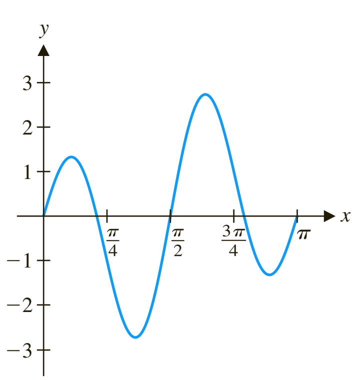


FIGURE 9.41a
$y = 3\cos 3x \sin x + \sin 3x \cos x$

Solving this equation is not an easy matter. As a start, we graph $f(x) = 3\cos 3x \sin x + \sin 3x \cos x$ with $0 \le x \le \pi$ (see Figure 9.41a). You should observe that there appear to be five solutions. Three of the solutions can be found exactly: $\theta = 0$, $\theta = \frac{\pi}{2}$ and $\theta = \pi$. You can find the remaining two numerically: $\theta \approx 0.659$ and $\theta \approx 2.48$. (You can also use trig identities to arrive at $\sin^2\theta = \frac{3}{8}$.) The corresponding points on the curve $r = \sin 3\theta$ (specified in rectangular coordinates) are (0, 0), (0.73, 0.56), (0, −1), (−0.73, 0.56) and (0, 0). The point (0, −1) lies at the bottom of a leaf. This is the familiar situation of a horizontal tangent line at a local (and in fact, absolute) minimum. The point (0, 0) is a little more tricky to interpret. As seen in Figure 9.40b, if we graph a small piece of the curve with θ near 0 (or π), the point (0, 0) is a minimum point. However, this is not true for other values of θ (e.g., $\frac{\pi}{3}$) where the curve passes through the point (0, 0). The tangent lines at the points (±0.73, 0.56) are shown in Figure 9.41b. Note that these points correspond to points where the y-coordinate is a maximum. However, referring to the graph, these points do not appear to be of particular

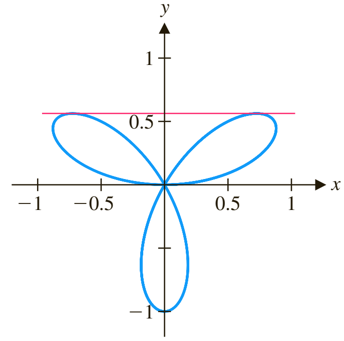


FIGURE 9.41b
Horizontal tangent lines

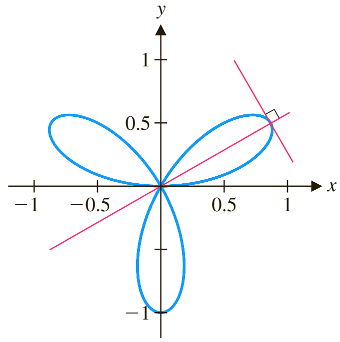


FIGURE 9.41c
The tangent line at the tip of a leaf

interest. Rather, the tips of the leaves represent the extreme points of most interest. Notice that the tips are where $|r|$ is a maximum. For $r = \sin 3\theta$, this occurs when $\sin 3\theta = \pm 1$, that is, where $3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$, or $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \ldots$. From (5.4), the slope of the tangent line to the curve at $\theta = \frac{\pi}{6}$ is given by

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{\left(3\cos\frac{3\pi}{6}\right)\sin\frac{\pi}{6} + \sin\frac{3\pi}{6}\left(\cos\frac{\pi}{6}\right)}{\left(3\cos\frac{3\pi}{6}\right)\cos\frac{\pi}{6} - \sin\frac{3\pi}{6}\left(\sin\frac{\pi}{6}\right)} = \frac{0 + \frac{\sqrt{3}}{2}}{0 - \frac{1}{2}} = -\sqrt{3}.$$

The rectangular point corresponding to $\theta = \frac{\pi}{6}$ is given by

$$\left(1\cos\frac{\pi}{6}, 1\sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

The slope of the line segment joining this point to the origin is then $\frac{1}{\sqrt{3}}$. Observe that the line segment from the origin to the point is perpendicular to the tangent line since the product of the slopes ($-\sqrt{3}$ and $\frac{1}{\sqrt{3}}$) is -1. This is illustrated in Figure 9.41c. Similarly, the slope of the tangent line at $\theta = \frac{5\pi}{6}$ is $\sqrt{3}$, which again makes the tangent line at that point perpendicular to the line segment from the origin to the point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$. Finally, we have already shown that the slope of the tangent line at $\theta = \frac{\pi}{2}$ is 0 and a horizontal tangent line is perpendicular to the vertical line from the origin to the point $(0, -1)$. ■

Next, for polar curves like the three-leaf rose seen in Figure 9.40a, we would like to compute the area enclosed by the curve. Since such a graph is *not* the graph of a function of the form $y = f(x)$, we cannot use the usual area formulas developed in Chapter 5. While we can convert our area formulas for parametric equations (from Theorem 2.2) into polar coordinates, a simpler approach uses the following geometric argument.

Observe that a sector of a circle of radius r and central angle θ, measured in radians (see Figure 9.42) contains a fraction $\left(\frac{\theta}{2\pi}\right)$ of the area of the entire circle. So, the area of the sector is given by

$$A = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2}r^2\theta.$$

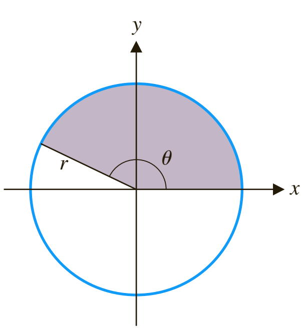


FIGURE 9.42
Circular sector

Now, consider the area enclosed by the polar curve defined by the equation $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$ (see Figure 9.43a), where f is continuous and positive on the interval $a \le \theta \le b$. As we did when we defined the definite integral, we begin by partitioning the

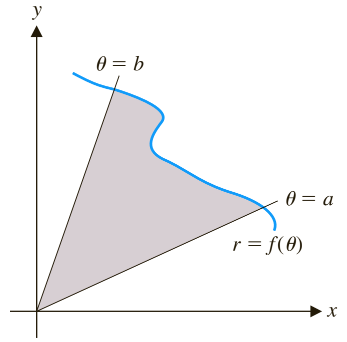


FIGURE 9.43a
Area of a polar region

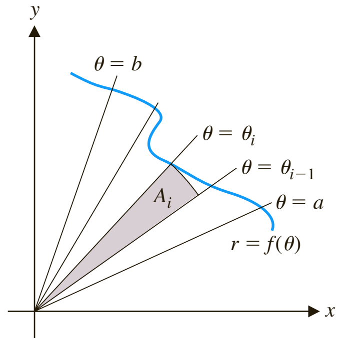


FIGURE 9.43b
Approximating the area of a polar region

θ-interval into n equal pieces:

$$a = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = b.$$

The width of each of these subintervals is then $\Delta\theta = \theta_i - \theta_{i-1} = \dfrac{b-a}{n}$. (Does this look familiar?) On each subinterval $[\theta_{i-1}, \theta_i]$ $(i = 1, 2, \ldots, n)$, we approximate the curve with the circular arc $r = f(\theta_i)$ (see Figure 9.43b). The area A_i enclosed by the curve on this subinterval is then approximately the same as the area of the circular sector of radius $f(\theta_i)$ and central angle $\Delta\theta$:

$$A_i \approx \frac{1}{2} r^2 \Delta\theta = \frac{1}{2}[f(\theta_i)]^2 \Delta\theta.$$

The total area A enclosed by the curve is then approximately the same as the sum of the areas of all such circular sectors:

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2 \Delta\theta.$$

As we have done numerous times now, we can improve the approximation by making n larger. Taking the limit as $n \to \infty$ gives us a definite integral:

Area in polar coordinates

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2 \Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta.} \tag{5.6}$$

EXAMPLE 5.3 The Area of One Leaf of a Three-Leaf Rose

Find the area of one leaf of the rose $r = \sin 3\theta$.

Solution Notice that one leaf of the rose is traced out with $0 \le \theta \le \frac{\pi}{3}$ (see Figure 9.44). From (5.6), the area is given by

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2}(\sin 3\theta)^2\, d\theta = \frac{1}{2}\int_0^{\pi/3} \sin^2 3\theta\, d\theta \\ &= \frac{1}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta = \frac{1}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\Bigg|_0^{\pi/3} = \frac{\pi}{12}, \end{aligned}$$

where we have used the half-angle formula $\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$ to simplify the integrand. ■

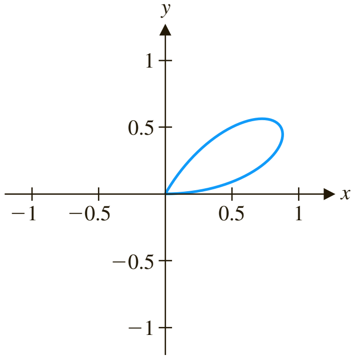


FIGURE 9.44
One leaf of $r = \sin 3\theta$

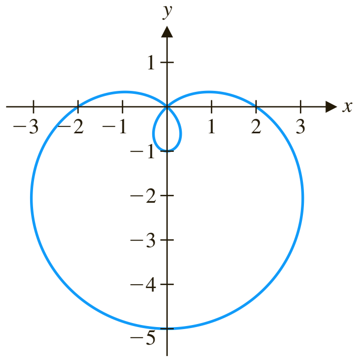


FIGURE 9.45
$r = 2 - 3\sin\theta$

Often, the most challenging part of finding the area of a polar region is determining the limits of integration.

EXAMPLE 5.4 The Area of the Inner Loop of a Limaçon

Find the area of the inner loop of the limaçon $r = 2 - 3\sin\theta$.

Solution A sketch of the limaçon is shown in Figure 9.45. Starting at $\theta = 0$, the curve starts at the point $(2, 0)$, passes through the origin, traces out the inner loop, passes back through the origin and finally traces out the outer loop. Thus, the inner loop is formed by θ-values between the first and second occurrences of $r = 0$ with $\theta > 0$. Solving $r = 0$, we get $\sin\theta = \frac{2}{3}$. The two smallest positive solutions are $\theta = \sin^{-1}\left(\frac{2}{3}\right)$ and $\theta = \pi - \sin^{-1}\left(\frac{2}{3}\right)$. Numerically, these are approximately equal to $\theta = 0.73$ and $\theta = 2.41$. From (5.6), the area is approximately

$$A \approx \int_{0.73}^{2.41} \frac{1}{2}(2 - 3\sin\theta)^2\, d\theta = \frac{1}{2}\int_{0.73}^{2.41} (4 - 12\sin\theta + 9\sin^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{0.73}^{2.41} \left[4 - 12\sin\theta + \frac{9}{2}(1 - \cos 2\theta)\right] d\theta \approx 0.44,$$

where we have used the half-angle formula $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ to simplify the integrand. (Here the area is approximate, owing only to the approximate limits of integration.) ■

When finding the area lying between two polar graphs, we use the familiar device of subtracting one area from another. Although the calculations in example 5.5 aren't too messy, finding the points of intersection of two polar curves often provides the greatest challenge.

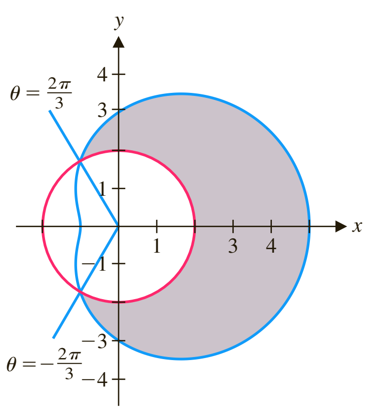


FIGURE 9.46a
$r = 3 + 2\cos\theta$ and $r = 2$

EXAMPLE 5.5 Finding the Area between Two Polar Graphs

Find the area inside the limaçon $r = 3 + 2\cos\theta$ and outside the circle $r = 2$.

Solution We show a sketch of the two curves in Figure 9.46a. Notice that the limits of integration correspond to the values of θ where the two curves intersect. So, we must first solve the equation $3 + 2\cos\theta = 2$. Notice that since $\cos\theta$ is periodic, there are infinitely many solutions of this equation. Consequently, it is essential to consult the graph to determine which solutions you are interested in. In this case, we want the least negative and the smallest positive solutions. (Look carefully at Figure 9.46b, where we have shaded the area between the graphs corresponding to θ between $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, the first two positive solutions. This portion of the graphs corresponds to the area *outside* the limaçon and *inside* the circle!) With $3 + 2\cos\theta = 2$, we have $\cos\theta = -\frac{1}{2}$, which occurs at $\theta = -\frac{2\pi}{3}$ and $\theta = \frac{2\pi}{3}$. From (5.6), the area enclosed by the portion of the limaçon on this interval is given by

$$\int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(3 + 2\cos\theta)^2\, d\theta = \frac{33\sqrt{3} + 44\pi}{6}.$$

Similarly, the area enclosed by the circle on this interval is given by

$$\int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(2)^2\, d\theta = \frac{8\pi}{3}.$$

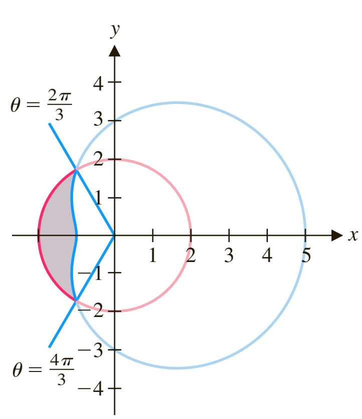


FIGURE 9.46b
$\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$

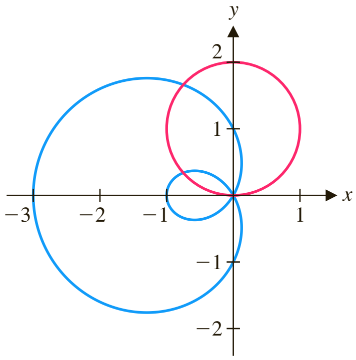


FIGURE 9.47a
$r = 1 - 2\cos\theta$ and $r = 2\sin\theta$

The area *inside* the limaçon and *outside* the circle is then given by

$$A = \int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(3 + 2\cos\theta)^2\, d\theta - \int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(2)^2\, d\theta$$
$$= \frac{33\sqrt{3} + 44\pi}{6} - \frac{8\pi}{3} = \frac{33\sqrt{3} + 28\pi}{6} \approx 24.2.$$

Here, we have left the (routine) details of the integrations to you. ■

In cases where r takes on both positive and negative values, finding the intersection points of two curves is more complicated.

EXAMPLE 5.6 Finding Intersections of Polar Curves Where r Can Be Negative

Find all intersections of the limaçon $r = 1 - 2\cos\theta$ and the circle $r = 2\sin\theta$.

Solution We show a sketch of the two curves in Figure 9.47a. Notice from the sketch that there are three intersections of the two curves. Since $r = 2\sin\theta$ is completely traced with $0 \le \theta \le \pi$, you might reasonably expect to find three solutions of the equation $1 - 2\cos\theta = 2\sin\theta$ on the interval $0 \le \theta \le \pi$. However, if we draw a rectangular plot of the two curves $y = 1 - 2\cos x$ and $y = 2\sin x$, on the interval $0 \le x \le \pi$ (see Figure 9.47b), we can clearly see that there is only one solution in this range, at approximately $\theta \approx 1.99$. (Use Newton's method or your calculator's solver to obtain an accurate approximation.) The corresponding rectangular point is $(r\cos\theta, r\sin\theta) \approx (-0.74, 1.67)$. Looking at Figure 9.47a, observe that there is another intersection located below this point. One way to find this point is to look at a rectangular plot of the two curves corresponding to an expanded range of values of θ (see Figure 9.47c). Notice that there is a second solution of the equation $1 - 2\cos\theta = 2\sin\theta$, near $\theta = 5.86$, which corresponds to the point $(-0.74, 0.34)$. Note that this point is on the inner loop of $r = 1 - 2\cos\theta$ and corresponds to a negative value of r. Finally, there appears to be a third intersection at or near the origin. Notice that this does not arise from any solution of the equation $1 - 2\cos\theta = 2\sin\theta$. This is because, while both curves pass through the origin (You should verify this!), they each do so for *different* values of θ. (Keep in mind that the origin corresponds to the point $(0, \theta)$, in polar coordinates, for *any* angle θ.) Notice that $1 - 2\cos\theta = 0$ for $\theta = \frac{\pi}{3}$ and $2\sin\theta = 0$ for $\theta = 0$. So, although the curves intersect at the origin, they each pass through the origin for different values of θ. ■

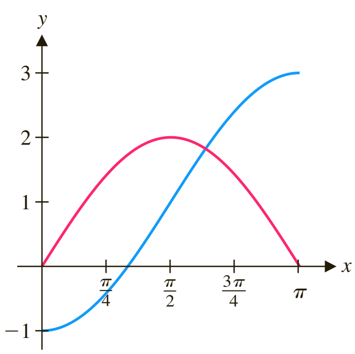


FIGURE 9.47b
Rectangular plot:
$y = 1 - 2\cos x$, $y = 2\sin x$,
$0 \le x \le \pi$

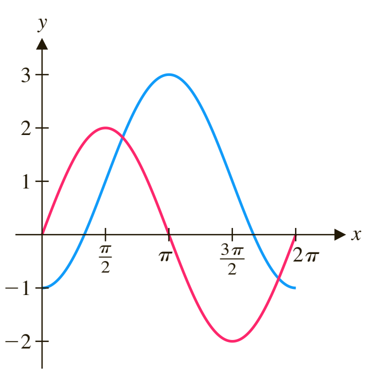


FIGURE 9.47c
Rectangular plot: $y = 1 - 2\cos x$,
$y = 2\sin x$, $0 \le x \le 2\pi$

REMARK 5.1

To find points of intersection of two polar curves $r = f(\theta)$ and $r = g(\theta)$, you must keep in mind that points have more than one representation in polar coordinates. In particular, this says that points of intersection need not correspond to solutions of $f(\theta) = g(\theta)$.

In example 5.7, we see an application that is far simpler to set up in polar coordinates than in rectangular coordinates.

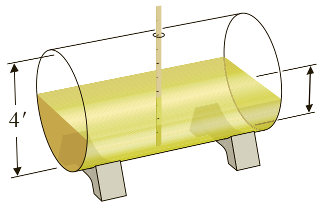


FIGURE 9.48a
A cylindrical oil tank

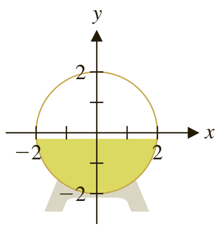


FIGURE 9.48b
Cross section of tank

EXAMPLE 5.7 Finding the Volume of a Partially Filled Cylinder

A cylindrical oil tank with a radius of 2 feet is lying on its side. A measuring stick shows that the oil is 1.8 feet deep (see Figure 9.48a). What percentage of a full tank is left?

Solution Notice that since we wish to find the *percentage* of oil remaining in the tank, the length of the tank has no bearing on this problem. (Think about this some.) We need only consider a cross section of the tank, which we represent as a circle of radius 2 centered at the origin. The proportion of oil remaining is given by the area of that portion of the circle lying beneath the line $y = -0.2$, divided by the total area of the circle. The area of the circle is 4π, so we need only find the area of the shaded region in Figure 9.48b. Computing this area in rectangular coordinates is a mess (try it!), but it is straightforward in polar coordinates. First, notice that the line $y = -0.2$ corresponds to $r \sin\theta = -0.2$ or $r = -0.2 \csc\theta$. The area beneath the line and inside the circle is then given by (5.6) as

$$\text{Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2}(2)^2\, d\theta - \int_{\theta_1}^{\theta_2} \frac{1}{2}(-0.2\csc\theta)^2\, d\theta,$$

where θ_1 and θ_2 are the appropriate intersections of $r = 2$ and $r = -0.2\csc\theta$. Using Newton's method, the first two positive solutions of $2 = -0.2\csc\theta$ are $\theta_1 \approx 3.242$ and $\theta_2 \approx 6.183$. The area is then

$$\begin{aligned}\text{Area} &= \int_{\theta_1}^{\theta_2} \frac{1}{2}(2)^2\, d\theta - \int_{\theta_1}^{\theta_2} \frac{1}{2}(-0.2\csc\theta)^2\, d\theta \\ &= (2\theta + 0.02\cot\theta)\Big|_{\theta_1}^{\theta_2} \approx 5.485.\end{aligned}$$

The fraction of oil remaining in the tank is then approximately $5.485/4\pi \approx 0.43648$ or about 43.6% of the total capacity of the tank. ■

We close this section with a brief discussion of arc length for polar curves. Recall that from (3.1), the arc length of a curve defined parametrically by $x = x(t)$, $y = y(t)$, for $a \le t \le b$, is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{5.7}$$

Once again thinking of a polar curve as a parametric representation (where the parameter is θ), we have that for the polar curve $r = f(\theta)$,

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

This gives us

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= [f'(\theta)\cos\theta - f(\theta)\sin\theta]^2 + [f'(\theta)\sin\theta + f(\theta)\cos\theta]^2 \\ &= [f'(\theta)]^2(\cos^2\theta + \sin^2\theta) + f'(\theta)f(\theta)(-2\cos\theta\sin\theta + 2\sin\theta\cos\theta) \\ &\quad + [f(\theta)]^2(\cos^2\theta + \sin^2\theta) \\ &= [f'(\theta)]^2 + [f(\theta)]^2.\end{aligned}$$

From (5.7), the arc length is then

Arc length in polar coordinates

$$s = \int_a^b \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta. \tag{5.8}$$

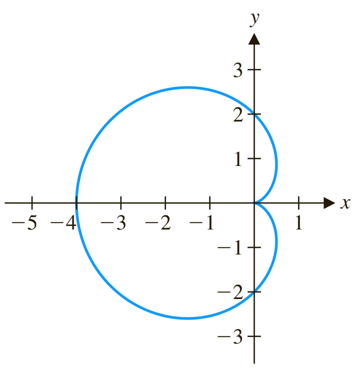


FIGURE 9.49
$r = 2 - 2\cos\theta$

EXAMPLE 5.8 Arc Length of a Polar Curve

Find the arc length of the cardioid $r = 2 - 2\cos\theta$.

Solution A sketch of the cardioid is shown in Figure 9.49. First, notice that the curve is traced out with $0 \le \theta \le 2\pi$. From (5.8), the arc length is given by

$$s = \int_a^b \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta = \int_0^{2\pi} \sqrt{(2\sin\theta)^2 + (2 - 2\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\theta + 4 - 8\cos\theta + 4\cos^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{8 - 8\cos\theta}\, d\theta = 16,$$

where we leave the details of the integration as an exercise. (Hint: Use the half-angle formula $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ to simplify the integrand. Be careful: remember that $\sqrt{x^2} = |x|$!) ■

EXERCISES 9.5

WRITING EXERCISES

1. Explain why the tangent line is perpendicular to the radius line at any point at which r is a local maximum. (See example 5.2.) In particular, if the tangent and radius are not perpendicular at (r, θ), explain why r is not a local maximum.

2. In example 5.5, explain why integrating from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$ would give the area shown in Figure 9.46b and not the desired area.

3. Referring to example 5.6, explain why intersections can occur in each of the cases $f(\theta) = g(\theta)$, $f(\theta) = -g(\theta + \pi)$ and $f(\theta_1) = g(\theta_2) = 0$.

4. In example 5.7, explain why the length of the tank doesn't matter. If the problem were to compute the *amount* of oil left, would the length matter?

In exercises 1–10, find the slope of the tangent line to the polar curve at the given point.

1. $r = \sin 3\theta$ at $\theta = \frac{\pi}{3}$
2. $r = \sin 3\theta$ at $\theta = \frac{\pi}{2}$
3. $r = \cos 2\theta$ at $\theta = 0$
4. $r = \cos 2\theta$ at $\theta = \frac{\pi}{4}$
5. $r = 3\sin\theta$ at $\theta = 0$
6. $r = 3\sin\theta$ at $\theta = \frac{\pi}{2}$
7. $r = \sin 4\theta$ at $\theta = \frac{\pi}{4}$
8. $r = \sin 4\theta$ at $\theta = \frac{\pi}{16}$
9. $r = \cos 3\theta$ at $\theta = \frac{\pi}{6}$
10. $r = \cos 3\theta$ at $\theta = \frac{\pi}{3}$

In exercises 11–14, find all points at which $|r|$ is a maximum and show that the tangent line is perpendicular to the radius connecting the point to the origin.

11. $r = \sin 3\theta$
12. $r = \cos 4\theta$
13. $r = 2 - 4\sin 2\theta$
14. $r = 2 + 4\sin 2\theta$

In exercises 15–30, find the area of the indicated region.

15. One leaf of $r = \cos 3\theta$
16. One leaf of $r = \sin 4\theta$
17. Inner loop of $r = 3 - 4\sin\theta$
18. Inner loop of $r = 1 - 2\cos\theta$
19. Bounded by $r = 2\cos\theta$
20. Bounded by $r = 2 - 2\cos\theta$
21. Small loop of $r = 1 + 2\sin 2\theta$
22. Large loop of $r = 1 + 2\sin 2\theta$
23. Inner loop of $r = 2 + 3\sin 3\theta$
24. Outer loop of $r = 2 + 3\sin 3\theta$
25. Inside of $r = 3 + 2\sin\theta$ and outside of $r = 2$
26. Inside of $r = 2$ and outside of $r = 2 - 2\sin\theta$

27. Inside of $r = 2$ and outside of both loops of $r = 1 + 2\sin\theta$

28. Inside of $r = 2\sin 2\theta$ and outside $r = 1$

29. Inside of both $r = 1 + \cos\theta$ and $r = 1$

30. Inside of both $r = 1 + \sin\theta$ and $r = 1 + \cos\theta$

In exercises 31–34, find all points at which the two curves intersect.

31. $r = 1 - 2\sin\theta$ and $r = 2\cos\theta$

32. $r = 1 + 3\cos\theta$ and $r = -2 + 5\sin\theta$

33. $r = 1 + \sin\theta$ and $r = 1 + \cos\theta$

34. $r = 1 + \sqrt{3}\sin\theta$ and $r = 1 + \cos\theta$

In exercises 35–40, find the arc length of the given curve.

35. $r = 2 - 2\sin\theta$

36. $r = 3 - 3\cos\theta$

37. $r = \sin 3\theta$

38. $r = 2\cos 3\theta$

39. $r = 1 + 2\sin 2\theta$

40. $r = 2 + 3\sin 3\theta$

41. Repeat example 5.7 for the case where the oil stick shows a depth of 1.4.

42. Repeat example 5.7 for the case where the oil stick shows a depth of 1.0.

43. Repeat example 5.7 for the case where the oil stick shows a depth of 2.4.

44. Repeat example 5.7 for the case where the oil stick shows a depth of 2.6.

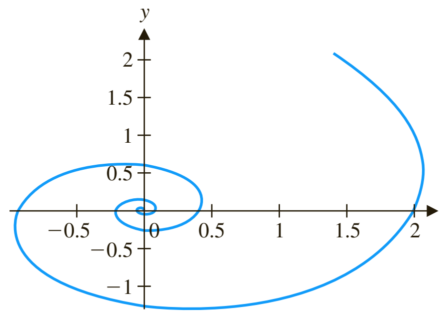
45. The problem of finding the slope of $r = \sin 3\theta$ at the point (0, 0) is not a well-defined problem. To see what we mean, show that the curve passes through the origin at $\theta = 0$, $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$, and find the slopes at these angles. Briefly explain why they are different even though the point is the same.

46. For each of the three slopes found in exercise 45, illustrate with a sketch of $r = \sin 3\theta$ for θ-values near the given values (e.g., $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$ to see the slope at $\theta = 0$).

47. If the polar curve $r = f(\theta)$, $a \le \theta \le b$, has length L, show that $r = cf(\theta)$, $a \le \theta \le b$, has length $|c|L$ for any constant c.

48. If the polar curve $r = f(\theta)$, $a \le \theta \le b$, encloses area A, show that for any constant c, $r = cf(\theta)$, $a \le \theta \le b$, encloses area c^2A.

49. A logarithmic spiral is the graph of $r = ae^{b\theta}$ for positive constants a and b. The accompanying figure shows the case where $a = 2$ and $b = \frac{1}{4}$ with $\theta \le 1$. Although the graph never reaches the origin, the limit of the arc length from $\theta = d$ to a given point with $\theta = c$, as d decreases to $-\infty$, exists. Show that this total arc length equals $\frac{\sqrt{b^2+1}}{b}R$, where R is the distance from the starting point to the origin.



50. For the logarithmic spiral of exercise 49, if the starting point P is on the x-axis, show that the total arc length to the origin equals the distance from P to the y-axis along the tangent line to the curve at P.

EXPLORATORY EXERCISES

1. In this exercise, you will discover a remarkable property about the area underneath the graph of $y = \frac{1}{x}$. First, show that a polar representation of this curve is $r^2 = \frac{1}{\sin\theta\cos\theta}$. We will find the area bounded by $y = \frac{1}{x}$, $y = mx$ and $y = 2mx$ for $x > 0$, where m is a positive constant. Sketch graphs for $m = 1$ (the area bounded by $y = \frac{1}{x}$, $y = x$ and $y = 2x$) and $m = 2$ (the area bounded by $y = \frac{1}{x}$, $y = 2x$ and $y = 4x$). Which area looks larger? To find out, you should integrate. Explain why this would be a very difficult integration in rectangular coordinates. Then convert all curves to polar coordinates and compute the polar area. You should discover that the area equals $\frac{1}{2}\ln 2$ for any value of m. (Are you surprised?)

2. In the study of biological oscillations (e.g., the beating of heart cells), an important mathematical term is **limit cycle.** A simple example of a limit cycle is produced by the polar coordinates initial value problem $\frac{dr}{dt} = ar(1-r)$, $r(0) = r_0$ and $\frac{d\theta}{dt} = 2\pi$, $\theta(0) = \theta_0$. Here, a is a positive constant. In section 7.2, we showed that the solution of the initial value problem $\frac{dr}{dt} = ar(1-r)$, $r(0) = r_0$ is

$$r(t) = \frac{r_0}{r_0 - (r_0 - 1)e^{-at}}$$

and it is not hard to show that the solution of the initial value problem $\frac{d\theta}{dt} = 2\pi$, $\theta(0) = \theta_0$ is $\theta(t) = 2\pi t + \theta_0$. In rectangular coordinates, the solution of the combined initial value problem has parametric equations $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. Graph the solution in the cases (a) $a = 1, r_0 = \frac{1}{2}, \theta_0 = 0$; (b) $a = 1, r_0 = \frac{3}{2}, \theta_0 = 0$; (c) your choice of $a > 0$, your choice of r_0 with $0 < r_0 < 1$, your choice of θ_0; (d) your choice of $a > 0$, your choice of r_0 with $r_0 > 1$, your choice of θ_0. As t increases, what is the limiting behavior of the solution? Explain what is meant by saying that this system has a limit cycle of $r = 1$.

9.6 CONIC SECTIONS

So far in this chapter, we have introduced a variety of interesting curves, many of which are not graphs of a function $y = f(x)$ in rectangular coordinates. Among the most important curves are the **conic sections,** which we explore here. The conic sections include parabolas, ellipses and hyperbolas, which are undoubtedly already familiar to you. In this section, we focus on geometric properties that are most easily determined in rectangular coordinates.

We visualize each conic section as the intersection of a plane with a right circular cone (see Figures 9.50a–9.50c).

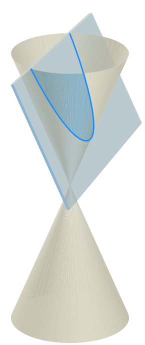


FIGURE 9.50a
Parabola

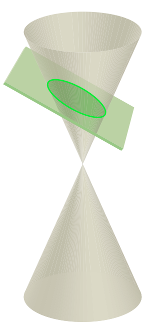


FIGURE 9.50b
Ellipse

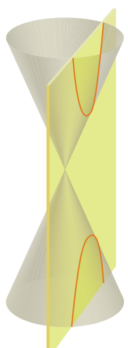


FIGURE 9.50c
Hyperbola

Depending on the orientation of the plane, the resulting curve can be a parabola, an ellipse or a hyperbola.

Parabolas

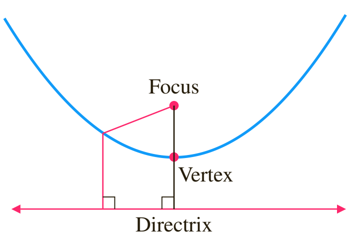


FIGURE 9.51
Parabola

We define a **parabola** (see Figure 9.51) to be the set of all points that are equidistant from a fixed point (called the **focus**) and a line (called the **directrix**). A special point on the parabola is the **vertex,** the midpoint of the perpendicular line segment from the focus to the directrix.

A parabola whose directrix is a horizontal line has a simple rectangular equation.

EXAMPLE 6.1 Finding the Equation of a Parabola

Find an equation of the parabola with focus at the point (0, 2) whose directrix is the line $y = -2$.

Solution By definition, any point (x, y) on the parabola must be equidistant from the focus and the directrix (see Figure 9.52). From the distance formula, the distance from problem has parametric equations $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. Graph the solution in the cases (a) $a = 1, r_0 = \frac{1}{2}, \theta_0 = 0$; (b) $a = 1, r_0 = \frac{3}{2}, \theta_0 = 0$; (c) your choice of $a > 0$, your choice of r_0 with $0 < r_0 < 1$, your choice of θ_0; (d) your choice of $a > 0$, your choice of r_0 with $r_0 > 1$, your choice of θ_0. As t increases, what is the limiting behavior of the solution? Explain what is meant by saying that this system has a limit cycle of $r = 1$.

9.6 CONIC SECTIONS

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We visualize each conic section as the intersection of a plane with a right circular cone (see Figures 9.50a–9.50c).

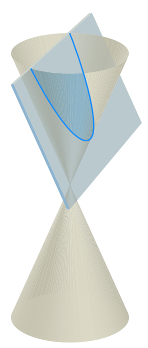

FIGURE 9.50a
Parabola

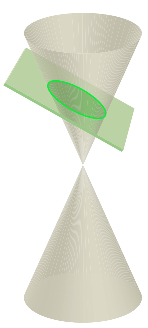

FIGURE 9.50b
Ellipse

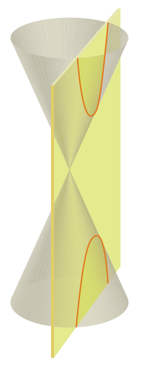

FIGURE 9.50c
Hyperbola

Depending on the orientation of the plane, the resulting curve can be a parabola, an ellipse or a hyperbola.

Parabolas

We define a **parabola** (see Figure 9.51) to be the set of all points that are equidistant from a fixed point (called the **focus**) and a line (called the **directrix**). A special point on the parabola is the **vertex,** the midpoint of the perpendicular line segment from the focus to the directrix.

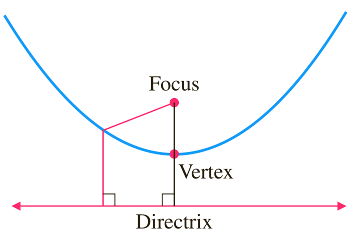

FIGURE 9.51
Parabola

A parabola whose directrix is a horizontal line has a simple rectangular equation.

EXAMPLE 6.1 Finding the Equation of a Parabola

Find an equation of the parabola with focus at the point (0, 2) whose directrix is the line $y = -2$.

Solution By definition, any point (x, y) on the parabola must be equidistant from the focus and the directrix (see Figure 9.52). From the distance formula, the distance from

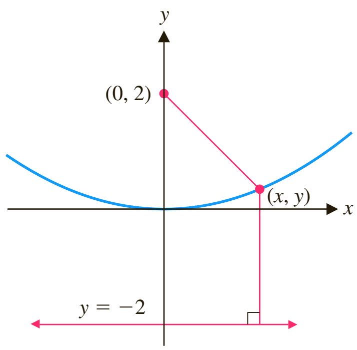

FIGURE 9.52
The parabola with focus at (0, 2) and directrix $y = -2$

(x, y) to the focus is given by $\sqrt{x^2 + (y-2)^2}$ and the distance to the directrix is $|y - (-2)|$. Since these distances must be equal, the parabola is defined by the equation

$$\sqrt{x^2 + (y-2)^2} = |y + 2|.$$

Squaring both sides, we get

$$x^2 + (y-2)^2 = (y+2)^2.$$

Expanding this out and simplifying, we get

$$x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$$

or

$$x^2 = 8y.$$

Solving for y gives us $y = \frac{1}{8}x^2$. ■

In general, the following relationship holds.

THEOREM 6.1

The parabola with vertex at the point (b, c), focus at $(b, c + \frac{1}{4a})$ and directrix given by the line $y = c - \frac{1}{4a}$ is described by the equation $y = a(x-b)^2 + c$.

PROOF

Given the focus $(b, c + \frac{1}{4a})$ and directrix $y = c - \frac{1}{4a}$, the vertex is the midpoint (b, c) (see Figure 9.53). For any point (x, y) on the parabola, its distance to the focus is given by $\sqrt{(x-b)^2 + (y - c - \frac{1}{4a})^2}$, while its distance to the directrix is given by $|y - c + \frac{1}{4a}|$. Setting these equal and squaring as in example 6.1, we have

$$(x-b)^2 + \left(y - c - \frac{1}{4a}\right)^2 = \left(y - c + \frac{1}{4a}\right)^2.$$

Expanding this out and simplifying, we get the more familiar form of the equation: $y = a(x-b)^2 + c$, as desired. ■

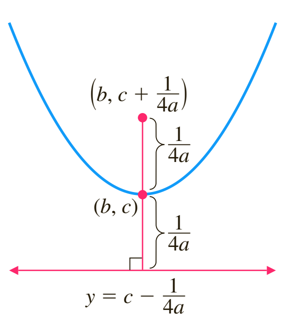

FIGURE 9.53
Parabola

Notice that the roles of x and y can be reversed. We leave the proof of the following result as an exercise.

THEOREM 6.2

The parabola with vertex at the point (c, b), focus at $(c + \frac{1}{4a}, b)$ and directrix given by the line $x = c - \frac{1}{4a}$ is described by the equation $x = a(y-b)^2 + c$.

We illustrate Theorem 6.2 in example 6.2.

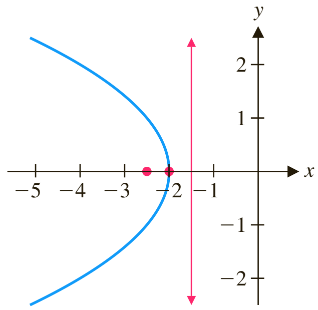

FIGURE 9.54
Parabola with focus at $(-\frac{5}{2}, 0)$ and directrix $x = -\frac{3}{2}$

EXAMPLE 6.2 A Parabola Opening to the Left

For the parabola $4x + 2y^2 + 8 = 0$, find the vertex, focus and directrix.

Solution To put this into the form of the equation given in Theorem 6.2, we must first solve for x. We have $x = -\frac{1}{2}y^2 - 2$. The vertex is then at $(-2, 0)$. The focus and directrix are shifted left and right, respectively from the vertex by $\frac{1}{4a} = -\frac{1}{2}$. This puts the focus at $(-2 - \frac{1}{2}, 0) = (-\frac{5}{2}, 0)$ and the directrix at $x = -2 - (-\frac{1}{2}) = -\frac{3}{2}$. We show a sketch of the parabola in Figure 9.54. ■

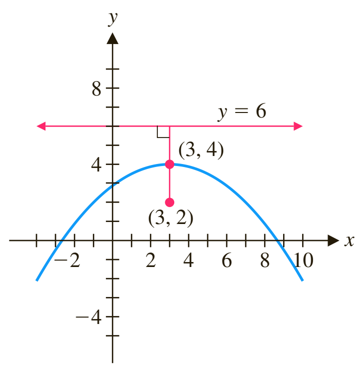

FIGURE 9.55
Parabola with focus at $(3, 2)$ and directrix $y = 6$

EXAMPLE 6.3 Finding the Equation of a Parabola

Find an equation relating all points that are equidistant from the point $(3, 2)$ and the line $y = 6$.

Solution Referring to Figure 9.55, notice that the vertex must be at the point $(3, 4)$ (i.e., the midpoint of the perpendicular line segment connecting the focus to the directrix) and the parabola opens down. From the vertex, the focus is shifted vertically by $\frac{1}{4a} = -2$ units, so $a = \frac{1}{(-2)4} = -\frac{1}{8}$. An equation is then

$$y = -\frac{1}{8}(x - 3)^2 + 4. \quad ■$$

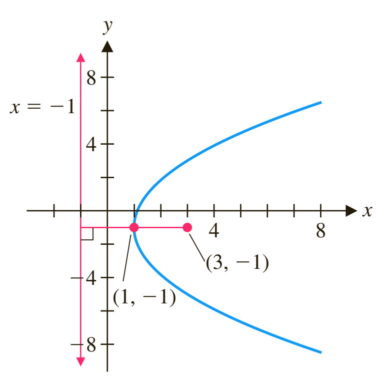

FIGURE 9.56
Parabola with focus at $(3, -1)$ and directrix at $x = -1$

EXAMPLE 6.4 A Parabola Opening to the Right

Find an equation relating all points that are equidistant from the point $(3, -1)$ and the line $x = -1$.

Solution Referring to Figure 9.56, notice that the vertex must be halfway between the focus $(3, -1)$ and the directrix $x = -1$, that is, at the point $(1, -1)$, and the parabola opens to the right. From the vertex, the focus is shifted horizontally by $\frac{1}{4a} = 2$ units, so that $a = \frac{1}{8}$. An equation is then

$$x = \frac{1}{8}(y + 1)^2 + 1. \quad ■$$

You see parabolas nearly every day. As we discussed in section 5.5, the motion of many projectiles is approximately parabolic. In addition, parabolas have a **reflective property** that is extremely useful in many important applications. This can be seen as follows. For the parabola $x = ay^2$ indicated in Figure 9.57a, draw a horizontal line that intersects the parabola at the point A. Then, one can show that the acute angle α between the horizontal line and the tangent line at A is the same as the acute angle β between the tangent line and the line segment joining A to the focus. You may already have recognized that light rays are reflected from a surface in exactly the same fashion (since the angle of incidence must equal the angle of reflection). In Figure 9.57b, we indicate a number of rays (you can think of them as light rays, although they could represent other forms of energy) traveling horizontally until they strike the parabola. As indicated, *all* rays striking the parabola are reflected *through the focus* of the parabola.

Due to this reflective property, satellite dishes are usually built with a parabolic shape and have a microphone located at the focus to receive all signals. This reflective property works in both directions. That is, energy emitted from the focus will reflect off the parabola

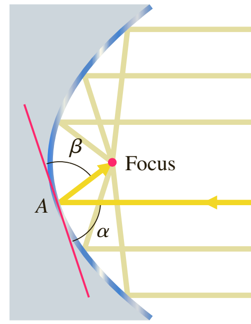

FIGURE 9.57a
Reflection of rays

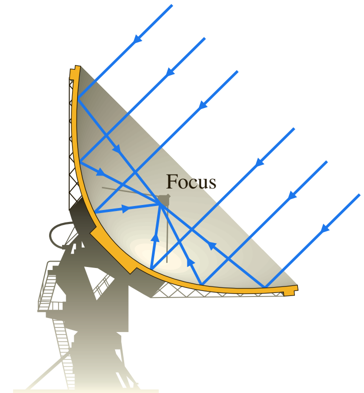

FIGURE 9.57b
The reflective property

and travel in parallel rays. For this reason, flashlights utilize parabolic reflectors to direct their light in a beam of parallel rays.

EXAMPLE 6.5 Design of a Flashlight

A parabolic reflector for a flashlight has the shape $x = 2y^2$. Where should the lightbulb be located?

Solution Based on the reflective property of parabolas, the lightbulb should be located at the focus of the parabola. The vertex is at (0, 0) and the focus is shifted to the right from the vertex $\frac{1}{4a} = \frac{1}{8}$ units, so the lightbulb should be located at the point $\left(\frac{1}{8}, 0\right)$. ■

Ellipses

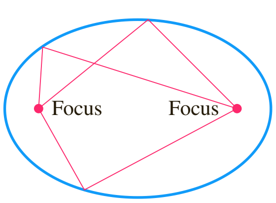

FIGURE 9.58a
Definition of ellipse

The second conic section we study is the **ellipse.** We define an ellipse to be the set of all points for which the sum of the distances to two fixed points (called **foci,** the plural of focus) is constant. This definition is illustrated in Figure 9.58a. We define the **center** of an ellipse to be the midpoint of the line segment connecting the foci.

The familiar equation of an ellipse can be derived from this definition. For convenience, we assume that the foci lie at the points $(c, 0)$ and $(-c, 0)$, for some positive constant c (i.e., they lie on the x-axis, at the same distance from the origin). For any point (x, y) on the ellipse, the distance from (x, y) to the focus $(c, 0)$ is $\sqrt{(x-c)^2 + y^2}$ and the distance to the focus $(-c, 0)$ is $\sqrt{(x+c)^2 + y^2}$. The sum of these distances must equal a constant that we'll call k. We then have

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = k.$$

Subtracting the first square root from both sides and then squaring, we get

$$\left(\sqrt{(x+c)^2 + y^2}\right)^2 = \left(k - \sqrt{(x-c)^2 + y^2}\right)^2$$

or $$x^2 + 2cx + c^2 + y^2 = k^2 - 2k\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2.$$

Now, solving for the remaining term with the radical and squaring gives us

$$\left[2k\sqrt{(x-c)^2+y^2}\right]^2 = (k^2-4cx)^2,$$

so that
$$4k^2x^2 - 8k^2cx + 4k^2c^2 + 4k^2y^2 = k^4 - 8k^2cx + 16c^2x^2$$

or
$$(4k^2-16c^2)x^2 + 4k^2y^2 = k^4 - 4k^2c^2.$$

To simplify this expression, we set $k = 2a$, to obtain

$$(16a^2-16c^2)x^2 + 16a^2y^2 = 16a^4 - 16a^2c^2.$$

Notice that since $2a$ is the sum of the distances from (x, y) to $(c, 0)$ and from (x, y) to $(-c, 0)$ and the distance from $(c, 0)$ to $(-c, 0)$ is $2c$, we must have $2a > 2c$, so that $a > c > 0$. Dividing both sides of the equation by 16 and defining $b^2 = a^2 - c^2$, we get

$$b^2x^2 + a^2y^2 = a^2b^2.$$

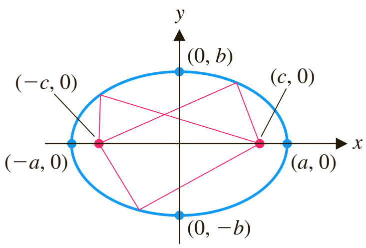

FIGURE 9.58b
Ellipse with foci at $(c, 0)$ and $(-c, 0)$

Finally, dividing by a^2b^2 leaves us with the familiar equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In this equation, notice that x can assume values from $-a$ to a and y can assume values from $-b$ to b. The points $(a, 0)$ and $(-a, 0)$ are called the **vertices** of the ellipse (see Figure 9.58b). Since $a > b$, we call the line segment joining the vertices the **major axis** and we call the line segment joining the points $(0, b)$ and $(0, -b)$ the **minor axis.** Notice that the length of the major axis is $2a$ and the length of the minor axis is $2b$.

We state the general case in Theorem 6.3.

THEOREM 6.3

The equation

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \tag{6.1}$$

with $a > b > 0$ describes an ellipse with foci at $(x_0 - c, y_0)$ and $(x_0 + c, y_0)$, where $c = \sqrt{a^2-b^2}$. The **center** of the ellipse is at the point (x_0, y_0) and the **vertices** are located at $(x_0 \pm a, y_0)$ on the major axis. The endpoints of the minor axis are located at $(x_0, y_0 \pm b)$.

The equation

$$\frac{(x-x_0)^2}{b^2} + \frac{(y-y_0)^2}{a^2} = 1 \tag{6.2}$$

with $a > b > 0$ describes an ellipse with foci at $(x_0, y_0 - c)$ and $(x_0, y_0 + c)$ where $c = \sqrt{a^2-b^2}$. The **center** of the ellipse is at the point (x_0, y_0) and the **vertices** are located at $(x_0, y_0 \pm a)$ on the major axis. The endpoints of the minor axis are located at $(x_0 \pm b, y_0)$.

In example 6.6, we use Theorem 6.3 to identify the features of an ellipse whose major axis lies along the x-axis.

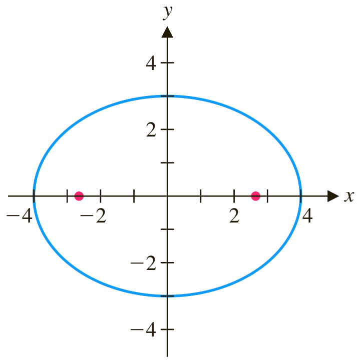

FIGURE 9.59
$\frac{x^2}{16} + \frac{y^2}{9} = 1$

EXAMPLE 6.6 Identifying the Features of an Ellipse

Identify the center, foci and vertices of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Solution From (6.1), the equation describes an ellipse with center at the origin. The values of a^2 and b^2 are 16 and 9, respectively, and so, $c = \sqrt{a^2 - b^2} = \sqrt{7}$. Since the major axis is parallel to the x-axis, the foci are shifted c units to the left and right of the center. That is, the foci are located at $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$. The vertices here are the x-intercepts (i.e., the intersections of the ellipse with the major axis). With $y = 0$, we have $x^2 = 16$ and so, the vertices are at $(\pm 4, 0)$. Taking $x = 0$, we get $y^2 = 9$ so that $y = \pm 3$. The y-intercepts are then $(0, -3)$ and $(0, 3)$. The ellipse is sketched in Figure 9.59. ■

Theorem 6.3 can also be used to identify the features of an ellipse whose major axis runs parallel to the y-axis.

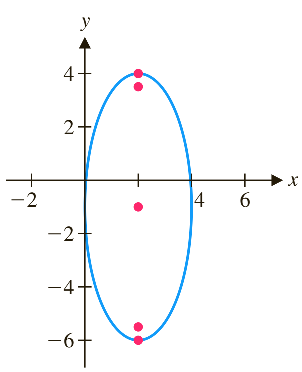

FIGURE 9.60
$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{25} = 1$

EXAMPLE 6.7 An Ellipse with Major Axis Parallel to the y-axis

Identify the center, foci and vertices of the ellipse $\frac{(x-2)^2}{4} + \frac{(y+1)^2}{25} = 1$.

Solution From (6.2), the center is at $(2, -1)$. The values of a^2 and b^2 are 25 and 4, respectively, so that $c = \sqrt{21}$. Since the major axis is parallel to the y-axis, the foci are shifted c units above and below the center, at $(2, -1 - \sqrt{21})$ and $(2, -1 + \sqrt{21})$. Notice that in this case, the vertices are the intersections of the ellipse with the line $x = 2$. With $x = 2$, we have $(y + 1)^2 = 25$, so that $y = -1 \pm 5$ and the vertices are $(2, -6)$ and $(2, 4)$. Finally, the endpoints of the minor axis are found by setting $y = -1$. We have $(x - 2)^2 = 4$, so that $x = 2 \pm 2$ and these endpoints are $(0, -1)$ and $(4, -1)$. The ellipse is sketched in Figure 9.60. ■

EXAMPLE 6.8 Finding an Equation of an Ellipse

Find an equation of the ellipse with foci at $(2, 3)$ and $(2, 5)$ and vertices $(2, 2)$ and $(2, 6)$.

Solution Recall that the center is the midpoint of the foci, in this case $(2, 4)$. You can now see that the foci are shifted $c = 1$ unit from the center. The vertices are shifted $a = 2$ units from the center. From $c^2 = a^2 - b^2$, we get $b^2 = 4 - 1 = 3$. Notice that the major axis is parallel to the y-axis, so that $a^2 = 4$ is the divisor of the y-term. From (6.2), the ellipse has the equation

$$\frac{(x-2)^2}{3} + \frac{(y-4)^2}{4} = 1. \quad ■$$

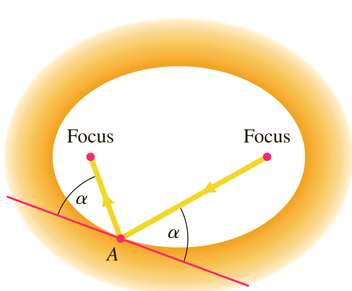

FIGURE 9.61
The reflective property of ellipses

Much like parabolas, ellipses have some useful reflective properties. As illustrated in Figure 9.61, a line segment joining one focus to a point A on the ellipse makes the same acute angle with the tangent line at A as does the line segment joining the other focus to A. Again, this is the same way in which light and sound reflect off a surface, so that a ray originating at one focus will always reflect off the ellipse toward the other focus. A surprising application of this principle is found in the so-called "whispering gallery" of the U.S. Senate. The ceiling of this room is elliptical, so that by standing at one focus you can hear everything said on the other side of the room at the other focus. (You probably never imagined how much of a role mathematics could play in political intrigue.)

EXAMPLE 6.9 A Medical Application of the Reflective Property of Ellipses

A medical procedure called **shockwave lithotripsy** is used to break up kidney stones that are too large or irregular to be passed. In this procedure, shockwaves emanating from a transducer located at one focus are bounced off of an elliptical reflector to the kidney stone located at the other focus. Suppose that the reflector is described by the equation $\frac{x^2}{112} + \frac{y^2}{48} = 1$ (in units of inches). Where should the transducer be placed?

Solution In this case,

$$c = \sqrt{a^2 - b^2} = \sqrt{112 - 48} = 8,$$

so that the foci are 16 inches apart. Since the transducer must be located at one focus, it should be placed 16 inches away from the kidney stone and aligned so that the line segment from the kidney stone to the transducer lies along the major axis of the elliptical reflector. ■

Hyperbolas

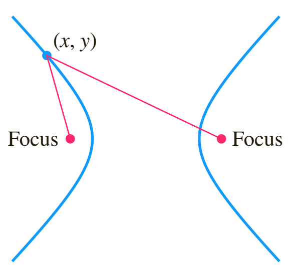

FIGURE 9.62
Definition of hyperbola

The third type of conic section is the **hyperbola.** We define a hyperbola to be the set of all points such that the *difference* of the distances between two fixed points (called the **foci**) is a constant. This definition is illustrated in Figure 9.62. Notice that it is nearly identical to the definition of the ellipse, except that we subtract the distances instead of add them.

The familiar equation of the hyperbola can be derived from the definition. The derivation is almost identical to that of the ellipse, except that the quantity $a^2 - c^2$ is now negative. We leave the details of the derivation of this as an exercise. An equation of the hyperbola with foci at $(\pm c, 0)$ and parameter $2a$ (equal to the difference of the distances) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b^2 = c^2 - a^2$. An important feature of hyperbolas that is not shared by ellipses is the presence of asymptotes. For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have $\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$ or $y^2 = \frac{b^2}{a^2}x^2 - b^2$. Notice that

$$\lim_{x\to\pm\infty} \frac{y^2}{x^2} = \lim_{x\to\pm\infty} \left(\frac{b^2}{a^2} - \frac{b^2}{x^2}\right) = \frac{b^2}{a^2}.$$

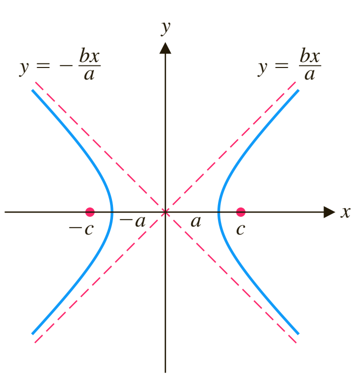

FIGURE 9.63
Hyperbola, shown with its asymptotes

That is, as $x \to \pm\infty$, $\frac{y^2}{x^2} \to \frac{b^2}{a^2}$, so that $\frac{y}{x} \to \pm\frac{b}{a}$ and so, $y = \pm\frac{b}{a}x$ are the (slant) asymptotes, as shown in Figure 9.63.

We state the general case in Theorem 6.4.

THEOREM 6.4

The equation

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1 \quad (6.3)$$

describes a hyperbola with foci at the points $(x_0 - c, y_0)$ and $(x_0 + c, y_0)$, where $c = \sqrt{a^2+b^2}$. The **center** of the hyperbola is at the point (x_0, y_0) and the **vertices** are located at $(x_0 \pm a, y_0)$. The asymptotes are $y = \pm\frac{b}{a}(x - x_0) + y_0$.

The equation

$$\frac{(y-y_0)^2}{a^2} - \frac{(x-x_0)^2}{b^2} = 1 \quad (6.4)$$

describes a hyperbola with foci at the points $(x_0, y_0 - c)$ and $(x_0, y_0 + c)$, where $c = \sqrt{a^2+b^2}$. The **center** of the hyperbola is at the point (x_0, y_0) and the **vertices** are located at $(x_0, y_0 \pm a)$. The asymptotes are $y = \pm\frac{a}{b}(x - x_0) + y_0$.

In example 6.10, we use Theorem 6.4 to identify the features of a hyperbola.

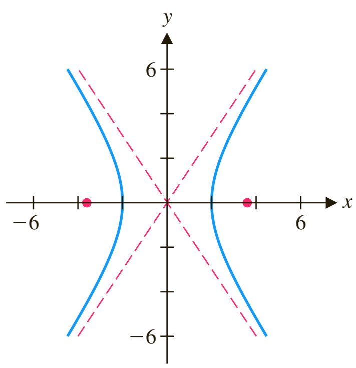

FIGURE 9.64
$\frac{x^2}{4} - \frac{y^2}{9} = 1$

EXAMPLE 6.10 Identifying the Features of a Hyperbola

For the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$, find the center, vertices, foci and asymptotes.

Solution From (6.3), we can see that the center is at (0, 0). Further, the vertices lie on the x-axis, where $\frac{x^2}{4} = 1$ (set $y = 0$), so that $x = \pm 2$. The vertices are then located at (2, 0) and (−2, 0). The foci are shifted by $c = \sqrt{a^2+b^2} = \sqrt{4+9} = \sqrt{13}$ units from the center, to $(\pm\sqrt{13}, 0)$. Finally, the asymptotes are $y = \pm\frac{3}{2}x$. A sketch of the hyperbola is shown in Figure 9.64. ■

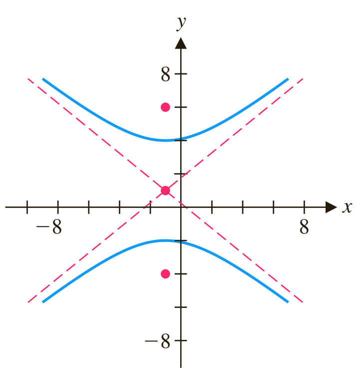

FIGURE 9.65
$\frac{(y-1)^2}{9} - \frac{(x+1)^2}{16} = 1$

EXAMPLE 6.11 Identifying the Features of a Hyperbola

For the hyperbola $\frac{(y-1)^2}{9} - \frac{(x+1)^2}{16} = 1$, find the center, vertices, foci and asymptotes.

Solution Notice that from (6.4), the center is at (−1, 1). Setting $x = -1$, we find that the vertices are shifted vertically by $a = 3$ units from the center, to (−1, −2) and (−1, 4). The foci are shifted vertically by $c = \sqrt{a^2+b^2} = \sqrt{25} = 5$ units from the center, to (−1, −4) and (−1, 6). The asymptotes are $y = \pm\frac{3}{4}(x+1) + 1$. A sketch of the hyperbola is shown in Figure 9.65. ■

EXAMPLE 6.12 Finding the Equation of a Hyperbola

Find an equation of the hyperbola with center at (−2, 0), vertices at (−4, 0) and (0, 0) and foci at (−5, 0) and (1, 0).

Solution Notice that since the center, vertices and foci all lie on the x-axis, the hyperbola must have an equation of the form of (6.3). Here, the vertices are shifted

$a = 2$ units from the center and the foci are shifted $c = 3$ units from the center. Then, we have $b^2 = c^2 - a^2 = 5$. Following (6.3), we have the equation

$$\frac{(x+2)^2}{4} - \frac{y^2}{5} = 1.$$

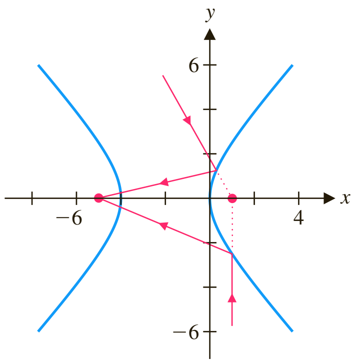

FIGURE 9.66
The reflective property of hyperbolas

Much like parabolas and ellipses, hyperbolas have a reflective property that is useful in applications. It can be shown that a ray directed toward one focus will reflect off the hyperbola toward the other focus. We illustrate this in Figure 9.66.

EXAMPLE 6.13 An Application to Hyperbolic Mirrors

A hyperbolic mirror is constructed in the shape of the top half of the hyperbola $(y+2)^2 - \frac{x^2}{3} = 1$. Toward what point will light rays following the paths $y = kx$ reflect (where k is a constant)?

Solution For the given hyperbola, we have $c = \sqrt{a^2 + b^2} = \sqrt{1+3} = 2$. Notice that the center is at $(0, -2)$ and the foci are at $(0, 0)$ and $(0, -4)$. Since rays of the form $y = kx$ will pass through the focus at $(0, 0)$, they will be reflected toward the focus at $(0, -4)$.

As a final note on the reflective properties of the conic sections, we briefly discuss a clever use of parabolic and hyperbolic mirrors in telescope design. In Figure 9.67, a parabolic mirror to the left and a hyperbolic mirror to the right are arranged so that they have a common focus at the point F. The vertex of the parabola is located at the other focus of the hyperbola, at the point E, where there is an opening for the eye or a camera. Notice that light entering the telescope from the right (and passing around the hyperbolic mirror) will reflect off the parabola directly toward its focus at F. Since F is also a focus of the hyperbola, the light will reflect off the hyperbola toward its other focus at E. In combination, the mirrors focus all incoming light at the point E.

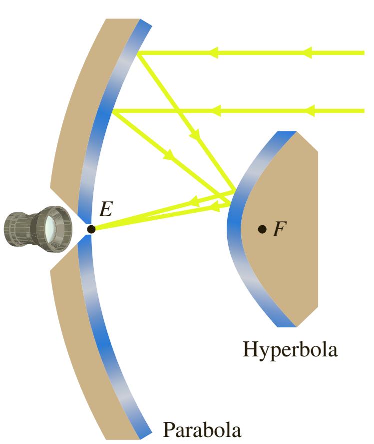

FIGURE 9.67
A combination of parabolic and hyperbolic mirrors

EXERCISES 9.6

WRITING EXERCISES

1. Each fixed point referred to in the definitions of the conic sections is called a **focus.** Use the reflective properties of the conic sections to explain why this is an appropriate name.
2. A hyperbola looks somewhat like a pair of parabolas facing opposite directions. Discuss the differences between a parabola and one half of a hyperbola (recall that hyperbolas have asymptotes).
3. Carefully explain why in example 6.8 (or for any other ellipse) the sum of the distances from a point on the ellipse to the two foci equals $2a$.
4. Imagine playing a game of pool on an elliptical pool table with a single hole located at one focus. If a ball rests near the other focus, which is clearly marked, describe an easy way to hit the ball into the hole.

In exercises 1–12, find an equation for the indicated conic section.

1. Parabola with focus $(0, -1)$ and directrix $y = 1$
2. Parabola with focus $(1, 2)$ and directrix $y = -2$
3. Parabola with focus $(3, 0)$ and directrix $x = 1$
4. Parabola with focus $(2, 0)$ and directrix $x = -2$
5. Ellipse with foci $(0, 1)$ and $(0, 5)$ and vertices $(0, -1)$ and $(0, 7)$
6. Ellipse with foci $(1, 2)$ and $(1, 4)$ and vertices $(1, 1)$ and $(1, 5)$
7. Ellipse with foci $(2, 1)$ and $(6, 1)$ and vertices $(0, 1)$ and $(8, 1)$
8. Ellipse with foci $(3, 2)$ and $(5, 2)$ and vertices $(2, 2)$ and $(6, 2)$
9. Hyperbola with foci $(0, 0)$ and $(4, 0)$ and vertices $(1, 0)$ and $(3, 0)$
10. Hyperbola with foci $(-2, 2)$ and $(6, 2)$ and vertices $(0, 2)$ and $(4, 2)$
11. Hyperbola with foci $(2, 2)$ and $(2, 6)$ and vertices $(2, 3)$ and $(2, 5)$
12. Hyperbola with foci $(0, -2)$ and $(0, 4)$ and vertices $(0, 0)$ and $(0, 2)$

In exercises 13–24, identify the conic section and find each vertex, focus and directrix.

13. $y = 2(x + 1)^2 - 1$
14. $y = -2(x + 2)^2 - 1$
15. $\dfrac{(x-1)^2}{4} + \dfrac{(y-2)^2}{9} = 1$
16. $\dfrac{(x+2)^2}{16} + \dfrac{y^2}{4} = 1$
17. $\dfrac{(x-1)^2}{9} - \dfrac{y^2}{4} = 1$
18. $\dfrac{(x+1)^2}{4} - \dfrac{(y-3)^2}{4} = 1$
19. $\dfrac{(y+1)^2}{16} - \dfrac{(x+2)^2}{4} = 1$
20. $\dfrac{y^2}{4} - \dfrac{(x+2)^2}{9} = 1$
21. $(x - 2)^2 + 9y^2 = 9$
22. $4x^2 + (y + 1)^2 = 16$
23. $(x + 1)^2 - 4(y - 2) = 16$
24. $4(x + 2) - (y - 1)^2 = -4$

In exercises 25–30, graph the conic section and find an equation.

25. All points equidistant from the point $(2, 1)$ and the line $y = -3$
26. All points equidistant from the point $(-1, 0)$ and the line $y = 4$
27. All points such that the sum of the distances to the points $(0, 2)$ and $(4, 2)$ equals 8
28. All points such that the sum of the distances to the points $(3, 1)$ and $(-1, 1)$ equals 6
29. All points such that the difference of the distances to the points $(0, 4)$ and $(0, -2)$ equals 4
30. All points such that the difference of the distances to the points $(2, 2)$ and $(6, 2)$ equals 2
31. A parabolic flashlight reflector has the shape $x = 4y^2$. Where should the lightbulb be placed?
32. A parabolic flashlight reflector has the shape $x = \frac{1}{2}y^2$. Where should the lightbulb be placed?
33. A parabolic satellite dish has the shape $y = 2x^2$. Where should the microphone be placed?
34. A parabolic satellite dish has the shape $y = 4x^2$. Where should the microphone be placed?

35. In example 6.9, if the shape of the reflector is $\frac{x^2}{124}+\frac{y^2}{24}=1$, how far from the kidney stone should the transducer be placed?

36. In example 6.9, if the shape of the reflector is $\frac{x^2}{44}+\frac{y^2}{125}=1$, how far from the kidney stone should the transducer be placed?

37. If a hyperbolic mirror is in the shape of the top half of $(y+4)^2-\frac{x^2}{15}=1$, to which point will light rays following the path $y=cx$ $(y<0)$ reflect?

38. If a hyperbolic mirror is in the shape of the bottom half of $(y-3)^2-\frac{x^2}{8}=1$, to which point will light rays following the path $y=cx$ $(y>0)$ reflect?

39. If a hyperbolic mirror is in the shape of the right half of $\frac{x^2}{3}-y^2=1$, to which point will light rays following the path $y=c(x-2)$ reflect?

40. If a hyperbolic mirror is in the shape of the left half of $\frac{x^2}{8}-y^2=1$, to which point will light rays following the path $y=c(x+3)$ reflect?

41. If the ceiling of a room has the shape $\frac{x^2}{400}+\frac{y^2}{100}=1$, where should you place the desks so that you can sit at one desk and hear everything said at the other desk?

42. If the ceiling of a room has the shape $\frac{x^2}{900}+\frac{y^2}{100}=1$, where should you place two desks so that you can sit at one desk and hear everything said at the other desk?

43. A spectator at the 2000 Summer Olympic Games throws an object. After 2 seconds, the object is 28 meters from the spectator. After 4 seconds, the object is 48 meters from the spectator. If the object's distance from the spectator is a quadratic function of time, find an equation for the position of the object. Sketch a graph of the path. What is the object?

44. Halley's comet follows an elliptical path with $a=17.79$ Au (astronomical units) and $b=4.53$ (Au). Compute the distance the comet travels in one orbit. Given that Halley's comet completes an orbit in approximately 76 years, what is the average speed of the comet?

EXPLORATORY EXERCISES

1. All of the equations of conic sections that we have seen so far have been of the form $Ax^2+Cy^2+Dx+Ey+F=0$. In this exercise, you will classify the conic sections for different values of the constants. First, assume that $A>0$ and $C>0$. Which conic section will you get? Next, try $A>0$ and $C<0$. Which conic section is it this time? How about $A<0$ and $C>0$? $A<0$ and $C<0$? Finally, suppose that either A or C (not both) equals 0; which conic section is it? In all cases, the values of the constants D, E and F do not affect the classification. Explain what effect these constants have.

2. In this exercise, you will generalize the results of exercise 1 by exploring the equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. (In exercise 1, the coefficient of xy was 0.) You will need to have software that will graph such equations. Make up several examples with $B^2-4AC=0$ (e.g., $B=2$, $A=1$ and $C=1$). Which conic section results? Now, make up several examples with $B^2-4AC<0$ (e.g., $B=1$, $A=1$ and $C=1$). Which conic section do you get? Finally, make up several examples with $B^2-4AC>0$ (e.g., $B=4$, $A=1$ and $C=1$). Which conic section is this?

9.7 CONIC SECTIONS IN POLAR COORDINATES

There are a variety of alternative definitions of the conic sections. One such alternative, utilizing an important quantity called *eccentricity,* is especially convenient for studying conic sections in polar coordinates. We introduce this concept in this section and review some options for parametric representations of conic sections.

For a fixed point P (the **focus**) and a fixed line l not containing P (the **directrix**), consider the set of all points whose distance to the focus is a constant multiple of their distance to the directrix. The constant multiple $e>0$ is called the **eccentricity.** Note that if $e=1$, this is the usual definition of a parabola. For other values of e, we get the other conic sections, as we see in Theorem 7.1.

35. In example 6.9, if the shape of the reflector is $\frac{x^2}{124}+\frac{y^2}{24}=1$, how far from the kidney stone should the transducer be placed?

36. In example 6.9, if the shape of the reflector is $\frac{x^2}{44}+\frac{y^2}{125}=1$, how far from the kidney stone should the transducer be placed?

37. If a hyperbolic mirror is in the shape of the top half of $(y+4)^2-\frac{x^2}{15}=1$, to which point will light rays following the path $y=cx$ $(y<0)$ reflect?

38. If a hyperbolic mirror is in the shape of the bottom half of $(y-3)^2-\frac{x^2}{8}=1$, to which point will light rays following the path $y=cx$ $(y>0)$ reflect?

39. If a hyperbolic mirror is in the shape of the right half of $\frac{x^2}{3}-y^2=1$, to which point will light rays following the path $y=c(x-2)$ reflect?

40. If a hyperbolic mirror is in the shape of the left half of $\frac{x^2}{8}-y^2=1$, to which point will light rays following the path $y=c(x+3)$ reflect?

41. If the ceiling of a room has the shape $\frac{x^2}{400}+\frac{y^2}{100}=1$, where should you place the desks so that you can sit at one desk and hear everything said at the other desk?

42. If the ceiling of a room has the shape $\frac{x^2}{900}+\frac{y^2}{100}=1$, where should you place two desks so that you can sit at one desk and hear everything said at the other desk?

43. A spectator at the 2000 Summer Olympic Games throws an object. After 2 seconds, the object is 28 meters from the spectator. After 4 seconds, the object is 48 meters from the spectator. If the object's distance from the spectator is a quadratic function of time, find an equation for the position of the object. Sketch a graph of the path. What is the object?

44. Halley's comet follows an elliptical path with $a=17.79$ Au (astronomical units) and $b=4.53$ (Au). Compute the distance the comet travels in one orbit. Given that Halley's comet completes an orbit in approximately 76 years, what is the average speed of the comet?

EXPLORATORY EXERCISES

1. All of the equations of conic sections that we have seen so far have been of the form $Ax^2+Cy^2+Dx+Ey+F=0$. In this exercise, you will classify the conic sections for different values of the constants. First, assume that $A>0$ and $C>0$. Which conic section will you get? Next, try $A>0$ and $C<0$. Which conic section is it this time? How about $A<0$ and $C>0$? $A<0$ and $C<0$? Finally, suppose that either A or C (not both) equals 0; which conic section is it? In all cases, the values of the constants D, E and F do not affect the classification. Explain what effect these constants have.

2. In this exercise, you will generalize the results of exercise 1 by exploring the equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. (In exercise 1, the coefficient of xy was 0.) You will need to have software that will graph such equations. Make up several examples with $B^2-4AC=0$ (e.g., $B=2$, $A=1$ and $C=1$). Which conic section results? Now, make up several examples with $B^2-4AC<0$ (e.g., $B=1$, $A=1$ and $C=1$). Which conic section do you get? Finally, make up several examples with $B^2-4AC>0$ (e.g., $B=4$, $A=1$ and $C=1$). Which conic section is this?

9.7 CONIC SECTIONS IN POLAR COORDINATES

There are a variety of alternative definitions of the conic sections. One such alternative, utilizing an important quantity called *eccentricity,* is especially convenient for studying conic sections in polar coordinates. We introduce this concept in this section and review some options for parametric representations of conic sections.

For a fixed point P (the **focus**) and a fixed line l not containing P (the **directrix**), consider the set of all points whose distance to the focus is a constant multiple of their distance to the directrix. The constant multiple $e>0$ is called the **eccentricity.** Note that if $e=1$, this is the usual definition of a parabola. For other values of e, we get the other conic sections, as we see in Theorem 7.1.

THEOREM 7.1

The set of all points whose distance to the focus is the product of the eccentricity e and the distance to the directrix is

(i) an ellipse if $0 < e < 1$,
(ii) a parabola if $e = 1$ or
(iii) a hyperbola if $e > 1$.

PROOF

We can simplify the algebra greatly by assuming that the focus is located at the origin and the directrix is the line $x = d > 0$. (We illustrate this in Figure 9.68 for the case of a parabola.) For any point (x, y) on the curve, observe that the distance to the focus is given by $\sqrt{x^2 + y^2}$ and the distance to the directrix is $d - x$. We then have

$$\sqrt{x^2 + y^2} = e(d - x). \tag{7.1}$$

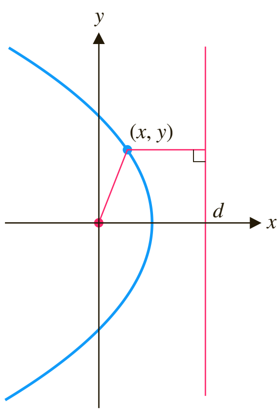

FIGURE 9.68
Focus and directrix

Squaring both sides gives us

$$x^2 + y^2 = e^2(d^2 - 2dx + x^2).$$

Finally, if we gather together the like terms, we get

$$x^2(1 - e^2) + 2de^2x + y^2 = e^2d^2. \tag{7.2}$$

Note that (7.2) has the form of the equation of a conic section. In particular, if $e = 1$, (7.2) becomes

$$2dx + y^2 = d^2,$$

which is the equation of a parabola. If $0 < e < 1$, notice that $(1 - e^2) > 0$ and so, (7.2) is the equation of an ellipse (with center shifted to the left by the x-term). Finally, if $e > 1$, then $(1 - e^2) < 0$ and so, (7.2) is the equation of a hyperbola. ■

Notice that the original form of the defining equation (7.1) of these conic sections includes the term $\sqrt{x^2 + y^2}$, which should make you think of polar coordinates. Recall that in polar coordinates, $r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$. Equation (7.1) now becomes

$$r = e(d - r\cos\theta)$$

or

$$r(e\cos\theta + 1) = ed.$$

Finally, solving for r, we have

$$\boxed{r = \frac{ed}{e\cos\theta + 1},}$$

which is the polar form of an equation for the conic sections with focus and directrix oriented as in Figure 9.68. As you will show in the exercises, different orientations of the focus and directrix can produce different forms of the polar equation. We summarize the possibilities in Theorem 7.2.

THEOREM 7.2

The conic section with eccentricity $e > 0$, focus at (0, 0) and the indicated directrix has the polar equation

(i) $r = \dfrac{ed}{e\cos\theta + 1}$, if the directrix is the line $x = d > 0$,

(ii) $r = \dfrac{ed}{e\cos\theta - 1}$, if the directrix is the line $x = d < 0$,

(iii) $r = \dfrac{ed}{e\sin\theta + 1}$, if the directrix is the line $y = d > 0$ or

(iv) $r = \dfrac{ed}{e\sin\theta - 1}$, if the directrix is the line $y = d < 0$.

Notice that we proved part (i) above. The remaining parts are derived in similar fashion and are left as exercises. In example 7.1, we illustrate how the eccentricity affects the graph of a conic section.

EXAMPLE 7.1 The Effect of Various Eccentricities

Find polar equations of the conic sections with focus (0, 0), directrix $x = 4$ and eccentricities (a) $e = 0.4$, (b) $e = 0.8$, (c) $e = 1$, (d) $e = 1.2$ and (e) $e = 2$.

Solution By Theorem 7.1, observe that (a) and (b) are ellipses, (c) is a parabola and (d) and (e) are hyperbolas. By Theorem 7.2, all have polar equations of the form $r = \dfrac{4e}{e\cos\theta + 1}$. The graphs of the ellipses $r = \dfrac{1.6}{0.4\cos\theta + 1}$ and $r = \dfrac{3.2}{0.8\cos\theta + 1}$ are shown in Figure 9.69a. Note that the ellipse with the smaller eccentricity is much more nearly circular than the ellipse with the larger eccentricity. Further, the ellipse with $e = 0.8$ opens up much farther to the left. In fact, as the value of e approaches 1, the ellipse will open up farther to the left, approaching the parabola with $e = 1$,

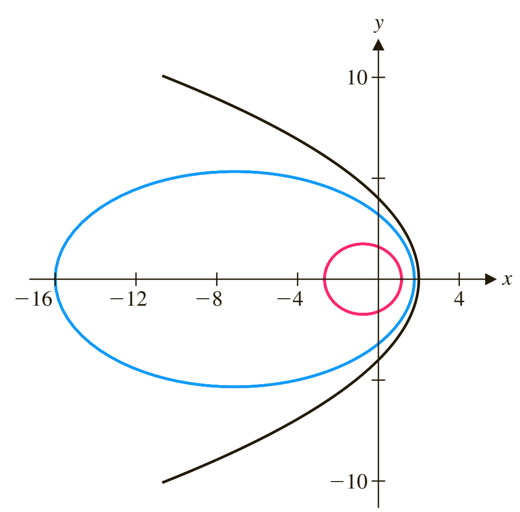

FIGURE 9.69a
$e = 0.4$, $e = 0.8$ and $e = 1.0$

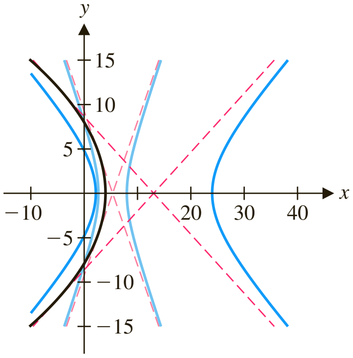

FIGURE 9.69b
$e = 1.0$, $e = 1.2$ and $e = 2.0$

$r = \dfrac{4}{\cos\theta + 1}$, also shown in Figure 9.69a. For values of $e > 1$, the graph is a hyperbola, opening up to the right and left. For instance, with $e = 1.2$ and $e = 2$, we have the hyperbolas $r = \dfrac{4.8}{1.2\cos\theta + 1}$ and $r = \dfrac{8}{2\cos\theta + 1}$ (shown in Figure 9.69b), where we also indicate the parabola with $e = 1$. Notice how the second hyperbola approaches its asymptotes much more rapidly than the first. ■

EXAMPLE 7.2 The Effect of Various Directrixes

Find polar equations of the conic sections with focus (0, 0), eccentricity $e = 0.5$ and directrix given by (a) $y = 2$, (b) $y = -3$ and (c) $x = -2$.

Solution First, note that with an eccentricity of $e = 0.5$, each of these conic sections is an ellipse. From Theorem 7.2, we know that (a) has the form $r = \dfrac{1}{0.5\sin\theta + 1}$. A sketch is shown in Figure 9.70a.

For (b), we have $r = \dfrac{-1.5}{0.5\sin\theta - 1}$ and show a sketch in Figure 9.70b. For (c), the directrix is parallel to the x-axis and so, from Theorem 7.2, we have $r = \dfrac{-1}{0.5\cos\theta - 1}$. A sketch is shown in Figure 9.70c.

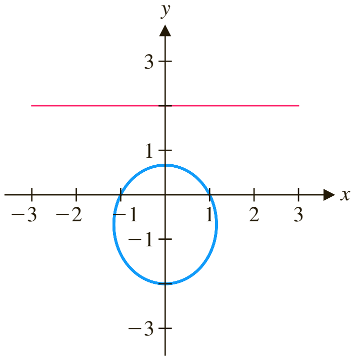

FIGURE 9.70a
Directrix: $y = 2$

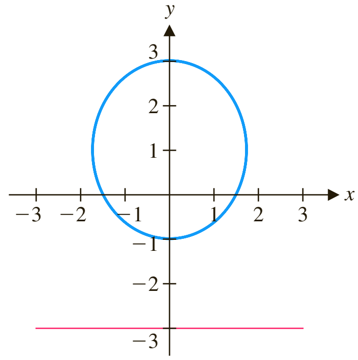

FIGURE 9.70b
Directrix: $y = -3$

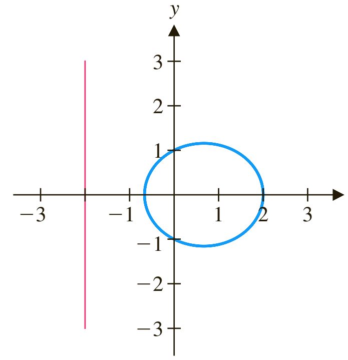

FIGURE 9.70c
Directrix: $x = -2$

■

The results of Theorem 7.2 apply only to conic sections with a focus at the origin. Recall that in rectangular coordinates, it's easy to translate the center of a conic section. Unfortunately, this is not true in polar coordinates.

In example 7.3, we see how to write some conic sections parametrically.

EXAMPLE 7.3 Parametric Equations for Some Conic Sections

Find parametric equations of the conic sections (a) $\dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{9} = 1$ and (b) $\dfrac{(x+2)^2}{9} - \dfrac{(y-3)^2}{16} = 1$.

Solution Notice that the curve in (a) is an ellipse with center at $(1, -2)$ and major axis parallel to the y-axis. Parametric equations for the ellipse are

$$\begin{cases} x = 2\cos t + 1 \\ y = 3\sin t - 2 \end{cases} \quad \text{with } 0 \le t \le 2\pi.$$

We show a sketch in Figure 9.71a.

You should recognize that the curve in (b) is a hyperbola. It is convenient to use hyperbolic functions in its parametric representation. The parameters are $a^2 = 9\,(a = 3)$ and $b^2 = 16\,(b = 4)$ and the center is $(-2, 3)$. Parametric equations are

$$\begin{cases} x = 3\cosh t - 2 \\ y = 4\sinh t + 3 \end{cases},$$

for the right half of the hyperbola and

$$\begin{cases} x = -3\cosh t - 2 \\ y = 4\sinh t + 3 \end{cases},$$

for the left half. We leave it as an exercise to verify that this is a correct parameterization. We sketch the hyperbola in Figure 9.71b. ■

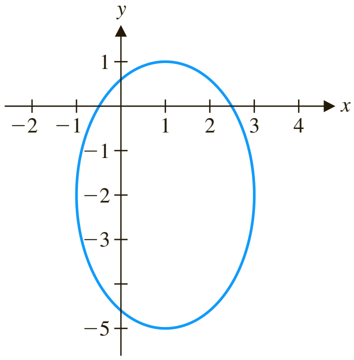

FIGURE 9.71a

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$

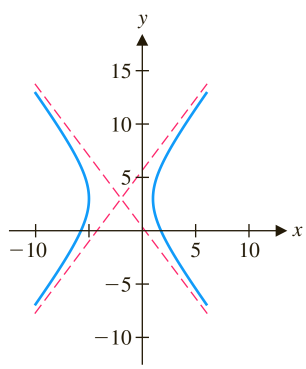

FIGURE 9.71b

$$\frac{(x+2)^2}{9} - \frac{(y-3)^2}{16} = 1$$

In 1543, the astronomer Copernicus shocked the world with the publication of his theory that the earth and the other planets revolve in circular orbits about the sun. This stood in sharp contrast to the age-old belief that the sun and other planets revolved around the earth. By the early part of the seventeenth century, Johannes Kepler had analyzed 20 years worth of painstaking observations of the known planets made by Tycho Brahe (before the invention of the telescope). He concluded that, in fact, each planet moves in an elliptical orbit, with the sun located at one focus. About 100 years later, Isaac Newton used his newly created calculus to show that Kepler's conclusions follow directly from Newton's **universal law of gravitation.** Although we must delay a more complete presentation of Kepler's laws until Chapter 11, we are now in a position to illustrate one of these. Kepler's second law states that, measuring from the sun to a planet, equal areas are swept out in equal times. As we see in example 7.4, this implies that planets speed up and slow down as they orbit the sun.

EXAMPLE 7.4 Kepler's Second Law of Planetary Motion

Suppose that a planet's orbit follows the elliptical path $r = \dfrac{2}{\sin\theta + 2}$ with the sun located at the origin (one of the foci), as illustrated in Figure 9.72a. Show that roughly equal areas are swept out from $\theta = 0$ to $\theta = \pi$ and from $\theta = \frac{3\pi}{2}$ to $\theta = 5.224895$. Then, find the corresponding arc lengths and compare the average speeds of the planet on these arcs.

Solution First, note that the area swept out by the planet from $\theta = 0$ to $\theta = \pi$ is the area bounded by the polar graphs $r = f(\theta) = \dfrac{2}{\sin\theta + 2}$, $\theta = 0$ and $\theta = \pi$ (see Figure 9.72b). From (5.6), this is given by

$$A = \frac{1}{2}\int_0^{\pi} [f(\theta)]^2\, d\theta = \frac{1}{2}\int_0^{\pi} \left(\frac{2}{\sin\theta + 2}\right)^2 d\theta \approx 0.9455994.$$

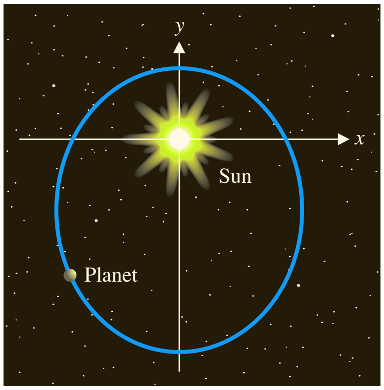

FIGURE 9.72a
Elliptical orbit

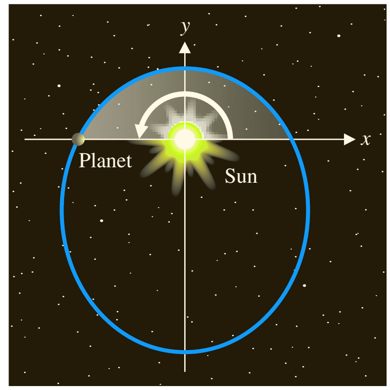

FIGURE 9.72b
Area swept out by the orbit from $\theta = 0$ to $\theta = \pi$

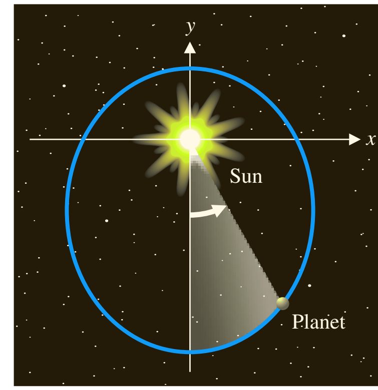

FIGURE 9.72c
Area swept out by the orbit from $\theta = \frac{3\pi}{2}$ to $\theta = 5.224895$

Similarly, the area swept out from $\theta = \frac{3\pi}{2}$ to $\theta = 5.224895$ (see Figure 9.72c) is given by

$$A = \frac{1}{2}\int_{3\pi/2}^{5.224895}\left(\frac{2}{\sin\theta + 2}\right)^2 d\theta \approx 0.9455995.$$

From (5.8), the arc length of the portion of the curve on the interval from $\theta = 0$ to $\theta = \pi$ is given by

$$\begin{aligned} s_1 &= \int_0^{\pi} \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta \\ &= \int_0^{\pi} \sqrt{\frac{4\cos^2\theta}{(\sin\theta + 2)^4} + \frac{4}{(\sin\theta + 2)^2}}\, d\theta \approx 2.53, \end{aligned}$$

while the arc length of the portion of the curve on the interval from $\theta = \frac{3\pi}{2}$ to $\theta = 5.224895$ is given by

$$s_2 = \int_{3\pi/2}^{5.224895} \sqrt{\frac{4\cos^2\theta}{(\sin\theta + 2)^4} + \frac{4}{(\sin\theta + 2)^2}}\, d\theta \approx 1.02.$$

Since these arcs are traversed in the same time, this says that the average speed on the portion of the orbit from $\theta = 0$ to $\theta = \pi$ is roughly two-and-a-half times the average speed on the portion of the orbit from $\theta = \frac{3\pi}{2}$ to $\theta = 5.224895$. ■

EXERCISES 9.7

WRITING EXERCISES

1. Based on Theorem 7.1, we might say that parabolas are the rarest of the conic sections, since they occur only for $e = 1$ exactly. Referring to Figure 9.50, explain why it takes a fairly precise cut of the cone to produce a parabola.

2. Describe how the ellipses in Figure 9.69 "open up" into a parabola as e increases to $e = 1$. What happens as e decreases to $e = 0$?

In exercises 1–16, find polar equations for and graph the conic section with focus (0, 0) and the given directrix and eccentricity.

1. Directrix $x = 2$, $e = 0.6$
2. Directrix $x = 2$, $e = 1.2$
3. Directrix $x = 2$, $e = 1$
4. Directrix $x = 2$, $e = 2$
5. Directrix $y = 2$, $e = 0.6$
6. Directrix $y = 2$, $e = 1.2$
7. Directrix $y = 2$, $e = 1$
8. Directrix $y = 2$, $e = 2$
9. Directrix $x = -2$, $e = 0.4$
10. Directrix $x = -2$, $e = 1$
11. Directrix $x = -2$, $e = 2$
12. Directrix $x = -2$, $e = 4$
13. Directrix $y = -2$, $e = 0.4$
14. Directrix $y = -2$, $e = 0.9$
15. Directrix $y = -2$, $e = 1$
16. Directrix $y = -2$, $e = 1.1$

In exercises 17–22, graph and interpret the conic section.

17. $r = \dfrac{4}{2\cos(\theta - \pi/6) + 1}$
18. $r = \dfrac{4}{4\sin(\theta - \pi/6) + 1}$
19. $r = \dfrac{-6}{\sin(\theta - \pi/4) - 2}$
20. $r = \dfrac{-4}{\cos(\theta - \pi/4) - 4}$
21. $r = \dfrac{-3}{2\cos(\theta + \pi/4) - 2}$
22. $r = \dfrac{3}{2\cos(\theta + \pi/4) + 2}$

In exercises 23–28, find parametric equations of the conic sections.

23. $\dfrac{(x+1)^2}{9} + \dfrac{(y-1)^2}{4} = 1$
24. $\dfrac{(x-2)^2}{9} - \dfrac{(y+1)^2}{16} = 1$
25. $\dfrac{(x+1)^2}{16} - \dfrac{y^2}{9} = 1$
26. $\dfrac{x^2}{4} + y^2 = 1$
27. $\dfrac{x^2}{4} + y = 1$
28. $x - \dfrac{y^2}{4} = 1$
29. Repeat example 7.4 with $0 \le \theta \le \frac{\pi}{2}$ and $\frac{3\pi}{2} \le \theta \le 4.953$.
30. Repeat example 7.4 with $\frac{\pi}{2} \le \theta \le \pi$ and $4.471 \le \theta \le \frac{3\pi}{2}$.
31. Prove Theorem 7.2 (ii).
32. Prove Theorem 7.2 (iii).
33. Prove Theorem 7.2 (iv).

EXPLORATORY EXERCISES

1. Earth's orbit is approximately elliptical with the sun at one focus, a minor axis of length 93 million miles and eccentricity $e = 0.017$. Find a polar equation for Earth's orbit.

2. If Neptune's orbit is given by
$$r = \frac{1.82 \times 10^{14}}{343\cos(\theta - 0.77) + 40{,}000}$$
and Pluto's orbit is given by
$$r = \frac{5.52 \times 10^{13}}{2481\cos(\theta - 3.91) + 10{,}000},$$
show that Pluto is sometimes closer and sometimes farther from the sun than Neptune. Based on these equations, will the planets ever collide?

3. Vision has proved to be one of the biggest challenges for building functional robots. Robot vision either can be designed to mimic human vision or can follow a different design. Two possibilities are analyzed here. In the diagram to the left, a camera follows an object directly from left to right. If the camera is at the origin, the object moves with speed 1 m/s and the line of motion is at $y = c$, find an expression for θ' as a function of the position of the object. In the diagram to the right, the camera looks down into a curved mirror and indirectly views the object. Assume that the mirror has equation $r = \dfrac{1 - \sin\theta}{2\cos^2\theta}$. Show that the mirror is parabolic and find its focus and directrix. With $x = r\cos\theta$, find an expression for θ' as a function of the position of the object. Compare values of θ' at $x = 0$ and other x-values. If a large value of θ' causes the image to blur, which camera system is better? Does the distance $y = c$ affect your preference?