Ch07-1 Jacobi & Gauss-Seidel Iterative Techniques I

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Outline

1 Introducing Iterative Techniques for Linear Systems

2 The Jacobi Iterative Method

3 **Converting A$\mathbf{x} = \mathbf{b}$ into an Equivalent System**

4 The Jacobi Iterative Algorithm

The Jacobi & Gauss-Seidel Methods

Intyroduction

- We will now describe the Jacobi and the Gauss-Seidel iterative methods, classic methods that date to the late eighteenth century.
- Iterative techniques are seldom used for solving linear systems of small dimension since the time required for sufficient accuracy exceeds that required for direct techniques such as Gaussian elimination.
- For large systems with a high percentage of 0 entries, however, these techniques are efficient in terms of both computer storage and computation.

The Jacobi & Gauss-Seidel Methods

Iterative Technique

An iterative technique to solve the $n \times n$ linear system

$$A\mathbf{x} = \mathbf{b}$$

starts with an initial approximation

$$\mathbf{x}^{(0)}$$

to the solution $\mathbf{x}$ and generates a sequence of vectors

$$\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$$

that converges to $\mathbf{x}$.

Jacobi's Method

The Jacobi iterative method is obtained by solving the ith equation in $A\mathbf{x} = \mathbf{b}$ for x_i to obtain (provided $a_{ii} \neq 0$)

$$x_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(-\frac{a_{ij}x_j}{a_{ii}} \right) + \frac{b_i}{a_{ii}}, \qquad \text{for } i = 1, 2, \ldots, n$$

For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$ from the components of $\mathbf{x}^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j\neq i}}^{n} \left(-a_{ij}x_j^{(k-1)} \right) + b_i \right], \qquad \text{for } i = 1, 2, \ldots, n$$

Jacobi's Method

Example

The linear system $A\mathbf{x} = \mathbf{b}$ given by

$$
\begin{aligned}
E_1: &\quad 10x_1 - x_2 + 2x_3 = 6\\
E_2: &\quad -x_1 + 11x_2 - x_3 + 3x_4 = 25\\
E_3: &\quad 2x_1 - x_2 + 10x_3 - x_4 = -11,\\
E_4: &\quad 3x_2 - x_3 + 8x_4 = 15
\end{aligned}
$$

has the unique solution $\mathbf{x} = (1, 2, -1, 1)^t$. Use Jacobi's iterative technique to find approximations $\mathbf{x}^{(k)}$ to $\mathbf{x}$ starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}$$

Jacobi's Method: Example

Solution (1/4)

We first solve equation E_i for x_i, for each $i = 1, 2, 3, 4$, to obtain

$$
\begin{aligned}
x_1 &= \phantom{-\frac{1}{5}x_1 +} \frac{1}{10}x_2 - \frac{1}{5}x_3 \phantom{+\frac{1}{10}x_4} + \frac{3}{5} \\
x_2 &= \frac{1}{11}x_1 \phantom{+\frac{1}{10}x_2} + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11} \\
x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 \phantom{+\frac{1}{11}x_3} + \frac{1}{10}x_4 - \frac{11}{10} \\
x_4 &= \phantom{-\frac{1}{5}x_1} - \frac{3}{8}x_2 + \frac{1}{8}x_3 \phantom{+\frac{1}{10}x_4} + \frac{15}{8}
\end{aligned}
$$

Jacobi's Method: Example

Solution (2/4)

From the initial approximation $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ we have $\mathbf{x}^{(1)}$ given by

$$
\begin{aligned}
x_1^{(1)} &= \quad \frac{1}{10}x_2^{(0)} - \frac{1}{5}x_3^{(0)} + \frac{3}{5} = 0.6000 \\
x_2^{(1)} &= \frac{1}{11}x_1^{(0)} + \frac{1}{11}x_3^{(0)} - \frac{3}{11}x_4^{(0)} + \frac{25}{11} = 2.2727 \\
x_3^{(1)} &= -\frac{1}{5}x_1^{(0)} + \frac{1}{10}x_2^{(0)} + \frac{1}{10}x_4^{(0)} - \frac{11}{10} = -1.1000 \\
x_4^{(1)} &= -\frac{3}{8}x_2^{(0)} + \frac{1}{8}x_3^{(0)} + \frac{15}{8} = 1.8750
\end{aligned}
$$

Jacobi's Method: Example

Solution (3/4)

Additional iterates, $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t$, are generated in a similar manner and are summarized as follows:

k	0	1	2	3	4		10
$x_1^{(k)}$	0.0	0.6000	1.0473	0.9326	1.0152	$\cdots$	1.0001
$x_2^{(k)}$	0.0	2.2727	1.7159	2.053	1.9537	$\cdots$	1.9998
$x_3^{(k)}$	0.0	-1.1000	-0.8052	-1.0493	-0.9681	$\cdots$	-0.9998
$x_4^{(k)}$	0.0	1.8750	0.8852	1.1309	0.9739	$\cdots$	0.9998

Jacobi's Method: Example

Solution (4/4)

The process was stopped after 10 iterations because

$$\frac{\|\mathbf{x}^{(10)} - \mathbf{x}^{(9)}\|_\infty}{\|\mathbf{x}^{(10)}\|_\infty} = \frac{8.0 \times 10^{-4}}{1.9998} < 10^{-3}$$

In fact, $\|\mathbf{x}^{(10)} - \mathbf{x}\|_\infty = 0.0002$.

Jacobi's Method in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$

A More General Representation

- In general, iterative techniques for solving linear systems involve a process that converts the system $A\mathbf{x} = \mathbf{b}$ into an equivalent system of the form

$$\mathbf{x} = T\mathbf{x} + \mathbf{c}$$

  for some fixed matrix T and vector $\mathbf{c}$.

- After the initial vector $\mathbf{x}^{(0)}$ is selected, the sequence of approximate solution vectors is generated by computing

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

  for each $k = 1, 2, 3, \ldots$ (reminiscent of the fixed-point iteration for solving nonlinear equations).

Jacobi's Method in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$

A More General Representation (Cont'd)

- The Jacobi method can be written in the form

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

  by splitting A into its diagonal and off-diagonal parts.

- To see this, let D be the diagonal matrix whose diagonal entries are those of A, $-L$ be the strictly lower-triangular part of A, and $-U$ be the strictly upper-triangular part of A where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Jacobi's Method in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$

A More General Representation (Cont'd)

We then write $A = D - L - U$ where

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \qquad L = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ -a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Jacobi's Method in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$

A More General Representation (Cont'd)

The equation $A\mathbf{x} = \mathbf{b}$, or $(D - L - U)\mathbf{x} = \mathbf{b}$, is then transformed into

$$D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b}$$

and, if D^{-1} exists, that is, if $a_{ii} \neq 0$ for each i, then

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}$$

This results in the matrix form of the Jacobi iterative technique:

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}, \quad k = 1, 2, \ldots$$

Jacobi's Method in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$

A More General Representation (Cont'd)

Introducing the notation $T_j = D^{-1}(L+U)$ and $\mathbf{c}_j = D^{-1}\mathbf{b}$ gives the Jacobi technique the form

$$\mathbf{x}^{(k)} = T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j$$

In practice, this form is only used for theoretical purposes while

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1\\ j\neq i}}^{n}\left(-a_{ij}x_j^{(k-1)}\right) + b_i\right], \qquad \text{for } i = 1, 2, \ldots, n$$

is used in computation.

Jacobi's Method in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$

Example

Express the Jacobi iteration method for the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned}
E_1: &\quad 10x_1 - x_2 + 2x_3 = 6\\
E_2: &\quad -x_1 + 11x_2 - x_3 + 3x_4 = 25\\
E_3: &\quad 2x_1 - x_2 + 10x_3 - x_4 = -11\\
E_4: &\quad 3x_2 - x_3 + 8x_4 = 15
\end{aligned}$$

in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.

Jacobi's Method in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$

Solution (1/2)

We saw earlier that the Jacobi method for this system has the form

$$\begin{aligned}
x_1 &= \phantom{-\frac{1}{5}x_1 +} \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5} \\
x_2 &= \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11} \\
x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10} \\
x_4 &= -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}
\end{aligned}$$

Jacobi's Method in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$

Solution (2/2)

Hence, we have

$$T = \begin{bmatrix} 0 & \frac{1}{10} & -\frac{1}{5} & 0 \\ \frac{1}{11} & 0 & \frac{1}{11} & -\frac{3}{11} \\ -\frac{1}{5} & \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & -\frac{3}{8} & \frac{1}{8} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} \frac{3}{5} \\ \frac{25}{11} \\ -\frac{11}{10} \\ \frac{15}{8} \end{bmatrix}$$

Discuss Algorithm