# Ch05-1 Initial-Value Problems for ODEs
# Euler Method

Dr. Feras Fraige

# Outline

- Derivation of Euler’s Method
- Numerical Algorithm
- Geometric Interpretation
- Numerical Example

# Euler's Method: Derivation

## Obtaining Approximations

- The object of Euler's method is to obtain approximations to the well-posed initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

- A continuous approximation to the solution $y(t)$ will not be obtained;
- Instead, approximations to $y$ will be generated at various values, called mesh points, in the interval $[a, b]$.
- Once the approximate solution is obtained at the points, the approximate solution at other points in the interval can be found by interpolation.

# Euler's Method: Derivation (Cont'd

## Set up an equally-distributed mesh

- We first make the stipulation that the mesh points are equally distributed throughout the interval $[a, b]$.
- This condition is ensured by choosing a positive integer $N$ and selecting the mesh points

$$t_i = a + ih, \quad \text{for each } i = 0, 1, 2, \ldots, N.$$

- The common distance between the points $h = (b - a)/N = t_{i+1} - t_i$ is called the step size.

# Euler's Method: Derivation (Cont'd

## Use Taylor's Theorem to derive Euler's Method

- Suppose that $y(t)$, the unique solution to

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

has two continuous derivatives on $[a, b]$, so that for each $i = 0, 1, 2, \ldots, N-1$,

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i)$$

for some number $\xi_i$ in $(t_i, t_{i+1})$.

# Euler's Method: Derivation (Cont'd

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i)$$

- Because $h = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\xi_i)$$

and, because $y(t)$ satisfies the differential equation $y' = f(t, y)$, we write

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$$

# Euler's Method: Derivation (Cont'd

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$$

## Euler's Method

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \ldots, N$, by deleting the remainder term. Thus Euler's method is

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1 \end{aligned}$$

This equation is called the difference equation associated with Euler's method.

# Euler's Method: Illustration

## Applying Euler's Method

Prior to introducing an algorithm for Euler's Method, we will illustrate the steps in the technique to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

at $t = 2$. using a step size of $h = 0.5$.

# Euler's Method: Illustration

## Solution

For this problem $f(t, y) = y - t^2 + 1$, so

$$\begin{aligned}
w_0 &= y(0) = 0.5 \\
w_1 &= w_0 + 0.5\left(w_0 - (0.0)^2 + 1\right) = 0.5 + 0.5(1.5) = 1.25 \\
w_2 &= w_1 + 0.5\left(w_1 - (0.5)^2 + 1\right) = 1.25 + 0.5(2.0) = 2.25 \\
w_3 &= w_2 + 0.5\left(w_2 - (1.0)^2 + 1\right) = 2.25 + 0.5(2.25) = 3.375
\end{aligned}$$

and

$$y(2) \approx w_4 = w_3 + 0.5\left(w_3 - (1.5)^2 + 1\right) = 3.375 + 0.5(2.125) = 4.4375$$

# Euler's Method: Algorithm (1/2)

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

at $(N + 1)$ equally spaced numbers in the interval $[a, b]$:

# Euler's Method: Algorithm (2/2)

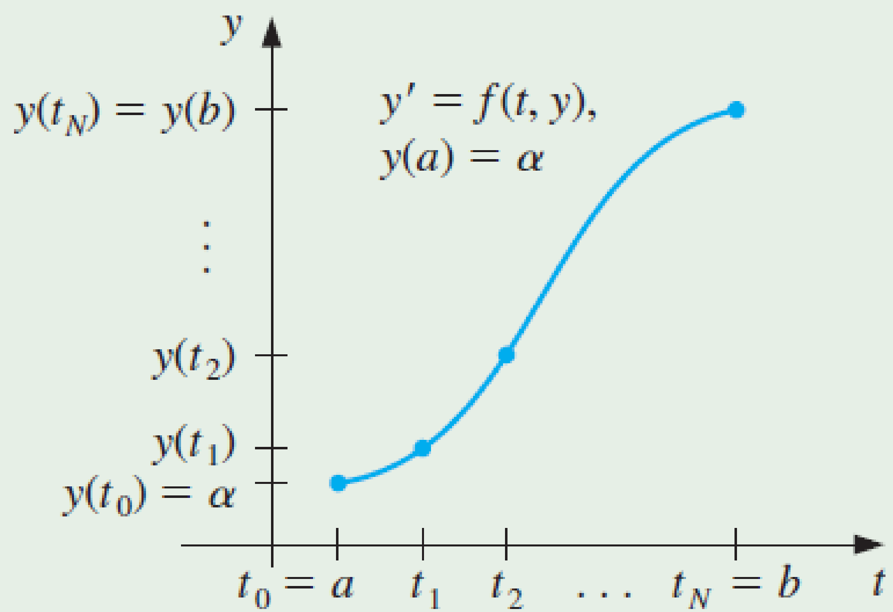
| | |
|---|---|
| INPUT | endpoints $a, b$; integer $N$; initial condition $\alpha$. |
| OUTPUT | approximation $w$ to $y$ at the $(N+1)$ values of $t$. |
| Step 1 | Set $h = (b-a)/N$ |
| | $t = a$ |
| | $w = \alpha$ |
| | OUTPUT $(t, w)$ |
| Step 2 | For $i = 1, 2, \ldots, N$ do Steps 3 & 4 |
| | Step 3 Set $w = w + hf(t, w)$; (*Compute* $w_i$) |
| | $t = a + ih$. (*Compute* $t_i$) |
| | Step 4 OUTPUT $(t, w)$ |
| Step 5 | STOP |

# Euler's Method: Geometric Interpretation

To interpret Euler's method geometrically, note that when $w_i$ is a close approximation to $y(t_i)$, the assumption that the problem is well-posed implies that

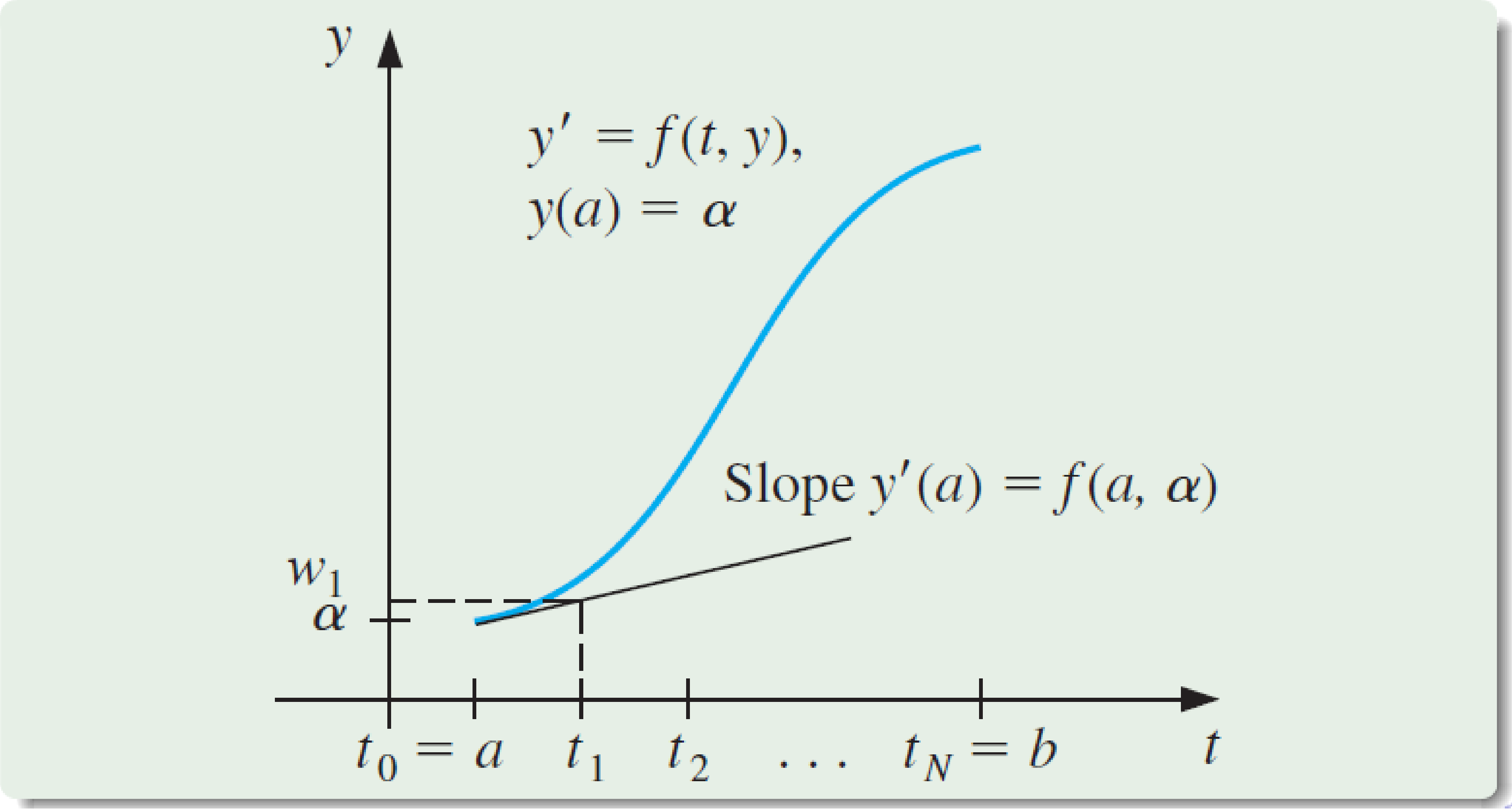
$$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$$

The graph of the function highlighting $y(t_i)$ is shown below.



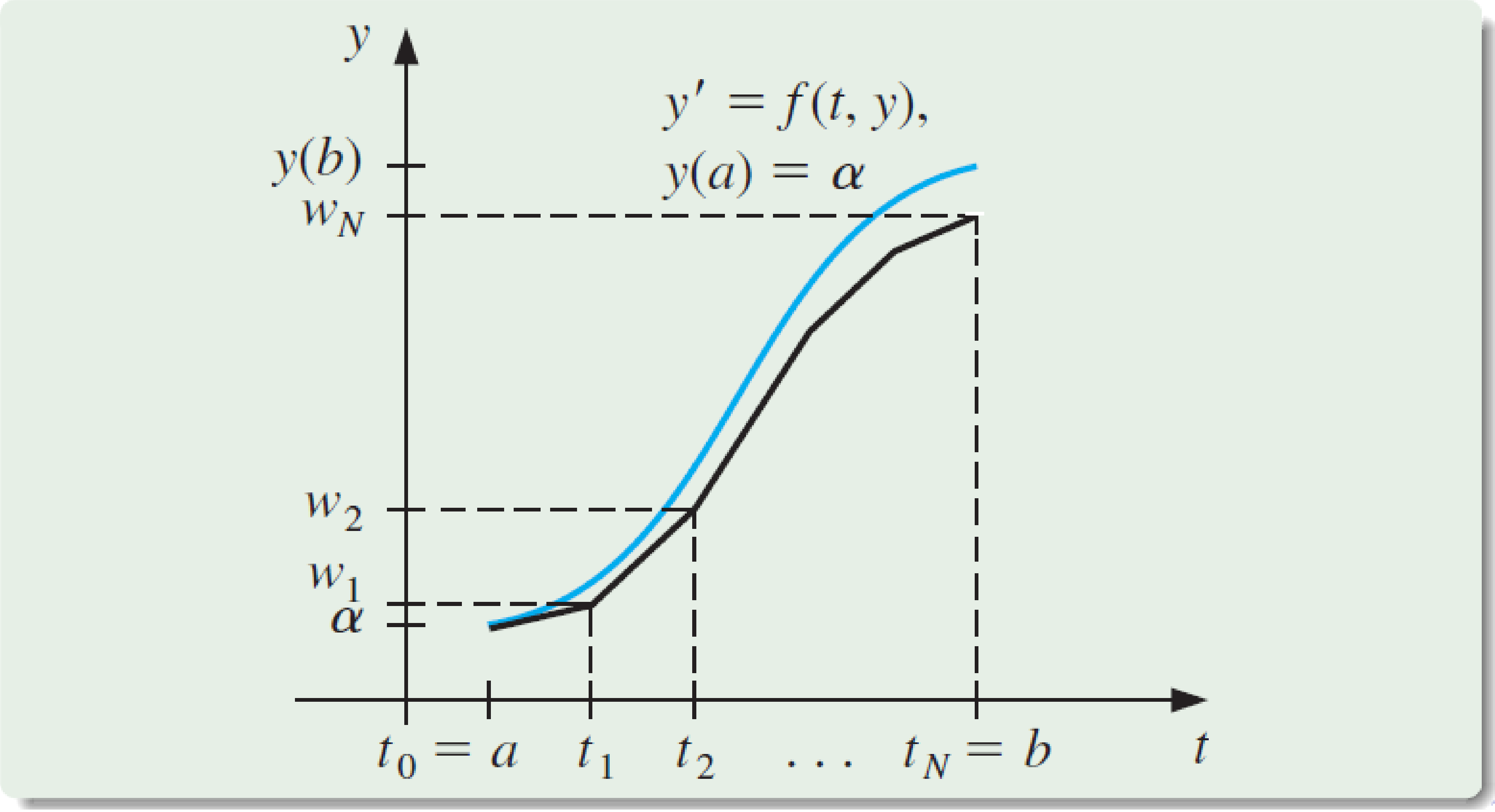
# Euler's Method: Geometric Interpretation

One step in Euler's method:



# Euler's Method: Geometric Interpretation

A series of steps in Euler's method:



# Euler's Method: Numerical Example (1/4)

## Application of Euler's Method

Use the algorithm for Euler's method with $N = 10$ to determine approximations to the solution to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

and compare these with the exact values given by

$$y(t) = (t+1)^2 - 0.5e^t$$

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \ldots, N$:

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1 \end{aligned}$$

# Euler's Method: Numerical Example (2/4)

## Solution

With $N = 10$, we have $h = 0.2$, $t_i = 0.2i$, $w_0 = 0.5$, so that:

$$\begin{aligned} w_{i+1} &= w_i + h(w_i - t_i^2 + 1) \\ &= w_i + 0.2[w_i - 0.04i^2 + 1] \\ &= 1.2w_i - 0.008i^2 + 0.2 \end{aligned}$$

for $i = 0, 1, \ldots, 9$. So

$$\begin{aligned} w_1 &= 1.2(0.5) - 0.008(0)^2 + 0.2 = 0.8 \\ w_2 &= 1.2(0.8) - 0.008(1)^2 + 0.2 = 1.152 \end{aligned}$$

and so on.

The following table shows the comparison between the approximate values at $t_i$ and the actual values.

# Euler's Method: Numerical Example (3/4)

## Results for $y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$

| $t_i$ | $w_i$ | $y_i = y(t_i)$ | $\lvert y_i - w_i \rvert$ |
|---|---|---|---|
| 0.0 | 0.5000000 | 0.5000000 | 0.0000000 |
| 0.2 | 0.8000000 | 0.8292986 | 0.0292986 |
| 0.4 | 1.1520000 | 1.2140877 | 0.0620877 |
| 0.6 | 1.5504000 | 1.6489406 | 0.0985406 |
| 0.8 | 1.9884800 | 2.1272295 | 0.1387495 |
| 1.0 | 2.4581760 | 2.6408591 | 0.1826831 |
| 1.2 | 2.9498112 | 3.1799415 | 0.2301303 |
| 1.4 | 3.4517734 | 3.7324000 | 0.2806266 |
| 1.6 | 3.9501281 | 4.2834838 | 0.3333557 |
| 1.8 | 4.4281538 | 4.8151763 | 0.3870225 |
| 2.0 | 4.8657845 | 5.3054720 | 0.4396874 |

# Euler's Method: Numerical Example (4/4)

## Comments

- Note that the error grows slightly as the value of $t$ increases.
- This controlled error growth is a consequence of the stability of Euler's method, which implies that the error is expected to grow in no worse than a linear manner.