# Interpolation & Polynomial Approximation
# Lagrange Interpolating Polynomials

Dr. Feras Fraige

# Weierstrass Approximation Theorem

## Algebraic Polynomials

One of the most useful and well-known classes of functions mapping the set of real numbers into itself is the algebraic polynomials, the set of functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

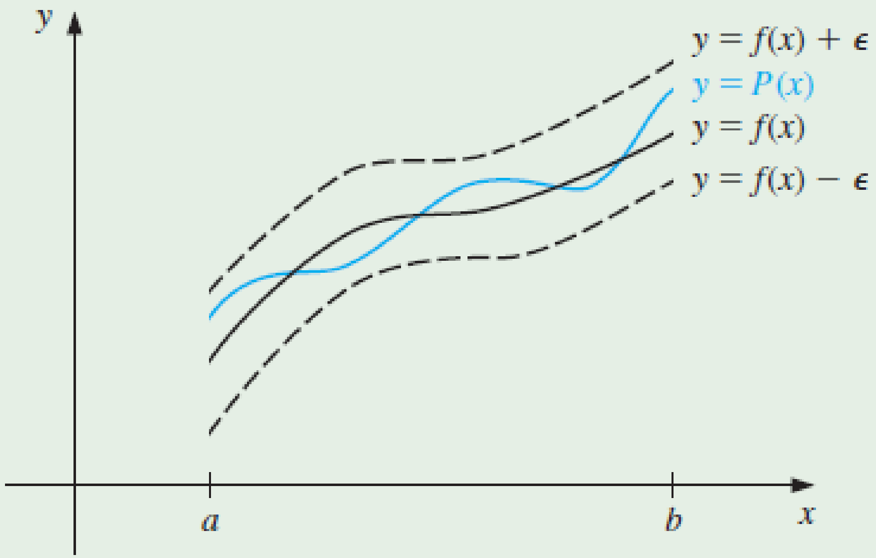
where $n$ is a nonnegative integer and $a_0, \ldots, a_n$ are real constants.

# Weierstrass Approximation Theorem

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

## Algebraic Polynomials (Cont'd)

- One reason for their importance is that they uniformly approximate continuous functions.
- By this we mean that given any function, defined and continuous on a closed and bounded interval, there exists a polynomial that is as "close" to the given function as desired.
- This result is expressed precisely in the Weierstrass Approximation Theorem.




## Weierstrass Approximation Theorem

Suppose that $f$ is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$, with the property that

$$|f(x) - P(x)| < \epsilon, \quad \text{for all } x \text{ in } [a, b].$$

## Benefits of Algebraic Polynomials

- Another important reason for considering the class of polynomials in the approximation of functions is that the derivative and indefinite integral of a polynomial are easy to determine and are also polynomials.
- For these reasons, polynomials are often used for approximating continuous functions.

# The Lagrange Polynomial: Taylor Polynomials

## Interpolating with Taylor Polynomials

- The Taylor polynomials are described as one of the fundamental building blocks of numerical analysis.
- Given this prominence, you might expect that polynomial interpolation would make heavy use of these functions.
- However this is not the case.
- The Taylor polynomials agree as closely as possible with a given function at a specific point, but they concentrate their accuracy near that point.
- A good interpolation polynomial needs to provide a relatively accurate approximation over an entire interval, and Taylor polynomials do not generally do this.

# The Lagrange Polynomial: Taylor Polynomials

**Example: $f(x) = e^x$**

We will calculate the first six Taylor polynomials about $x_0 = 0$ for $f(x) = e^x$.

**Note**

Since the derivatives of $f(x)$ are all $e^x$, which evaluated at $x_0 = 0$ gives 1.

The Taylor polynomials are as follows:

## Taylor Polynomials for $f(x) = e^x$ about $x_0 = 0$

$$P_0(x) = 1$$

$$P_1(x) = 1 + x$$

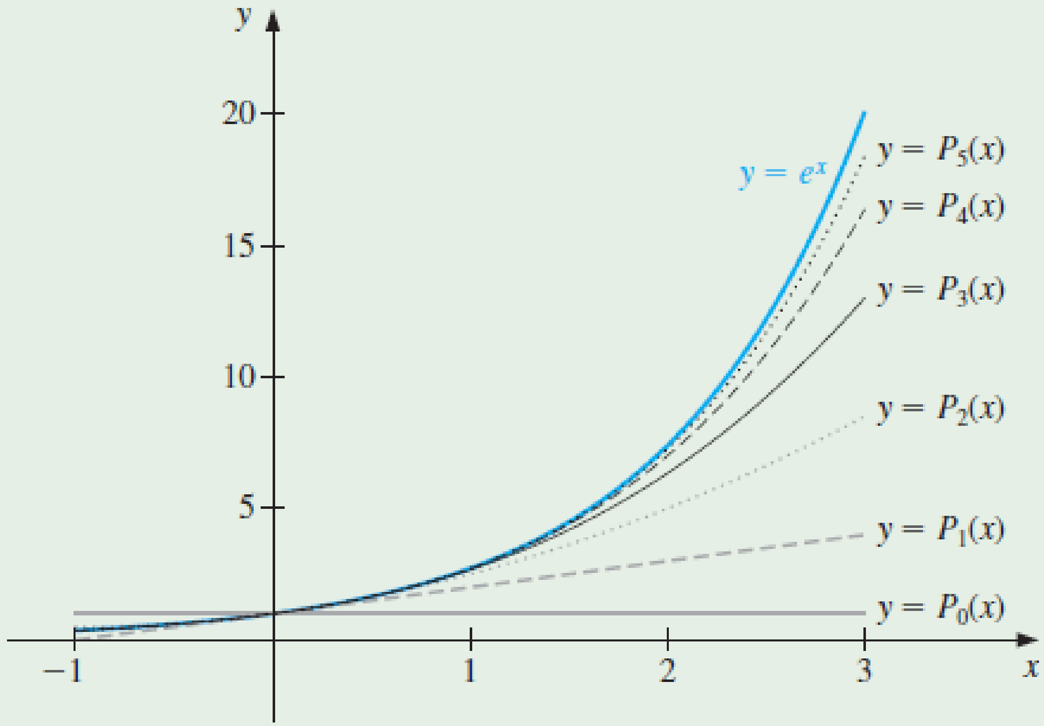
$$P_2(x) = 1 + x + \frac{x^2}{2}$$

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

# Taylor Polynomials for $f(x) = e^x$ about $x_0 = 0$




Notice that even for the higher-degree polynomials, the error becomes progressively worse as we move away from zero.

# Taylor Polynomials for $f(x) = \frac{1}{x}$ about $x_0 = 1$

**Example: A more extreme case**

- Although better approximations are obtained for $f(x) = e^x$ if higher-degree Taylor polynomials are used, this is not true for all functions.
- Consider, as an extreme example, using Taylor polynomials of various degrees for $f(x) = \frac{1}{x}$ expanded about $x_0 = 1$ to approximate $f(3) = \frac{1}{3}$.

# Taylor Polynomials for $f(x) = \frac{1}{x}$ about $x_0 = 1$

## Calculations

Since

$$f(x) = x^{-1},\ f'(x) = -x^{-2},\ f''(x) = (-1)^2 2 \cdot x^{-3},$$

and, in general,

$$f^{(k)}(x) = (-1)^k k! x^{-k-1},$$

the Taylor polynomials are

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{n} (-1)^k (x-1)^k.$$

# Taylor Polynomials for $f(x) = \frac{1}{x}$ about $x_0 = 1$

## To Approximate $f(3) = \frac{1}{3}$ by $P_n(3)$

- To approximate $f(3) = \frac{1}{3}$ by $P_n(3)$ for increasing values of $n$, we obtain the values shown below — rather a dramatic failure!
- When we approximate $f(3) = \frac{1}{3}$ by $P_n(3)$ for larger values of $n$, the approximations become increasingly inaccurate.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $P_n(3)$ | 1 | $-1$ | 3 | $-5$ | 11 | $-21$ | 43 | $-85$ |

# The Lagrange Polynomial: Taylor Polynomials

## Footnotes

- For the Taylor polynomials, all the information used in the approximation is concentrated at the single number $x_0$, so these polynomials will generally give inaccurate approximations as we move away from $x_0$.
- This limits Taylor polynomial approximation to the situation in which approximations are needed only at numbers close to $x_0$.
- For ordinary computational purposes, it is more efficient to use methods that include information at various points.
- The primary use of Taylor polynomials in numerical analysis is not for approximation purposes, but for the derivation of numerical techniques and error estimation.

# The Lagrange Polynomial: The Linear Case

## Polynomial Interpolation

- The problem of determining a polynomial of degree one that passes through the distinct points

$$(x_0, y_0) \quad \text{and} \quad (x_1, y_1)$$

  is the same as approximating a function $f$ for which

$$f(x_0) = y_0 \quad \text{and} \quad f(x_1) = y_1$$

  by means of a first-degree polynomial interpolating, or agreeing with, the values of $f$ at the given points.
- Using this polynomial for approximation within the interval given by the endpoints is called polynomial interpolation.

# The Lagrange Polynomial: The Linear Case

Define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

**Definition**

The linear Lagrange interpolating polynomial though $(x_0, y_0)$ and $(x_1, y_1)$ is

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

# The Lagrange Polynomial: The Linear Case

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

Note that

$$L_0(x_0) = 1, \quad L_0(x_1) = 0, \quad L_1(x_0) = 0, \quad \text{and} \quad L_1(x_1) = 1,$$

which implies that

$$P(x_0) = 1 \cdot f(x_0) + 0 \cdot f(x_1) = f(x_0) = y_0$$

and

$$P(x_1) = 0 \cdot f(x_0) + 1 \cdot f(x_1) = f(x_1) = y_1.$$

So $P$ is the unique polynomial of degree at most 1 that passes through $(x_0, y_0)$ and $(x_1, y_1)$.

# The Lagrange Polynomial: The Linear Case

## Example: Linear Interpolation

Determine the linear Lagrange interpolating polynomial that passes through the points $(2,4)$ and $(5,1)$.

## Solution

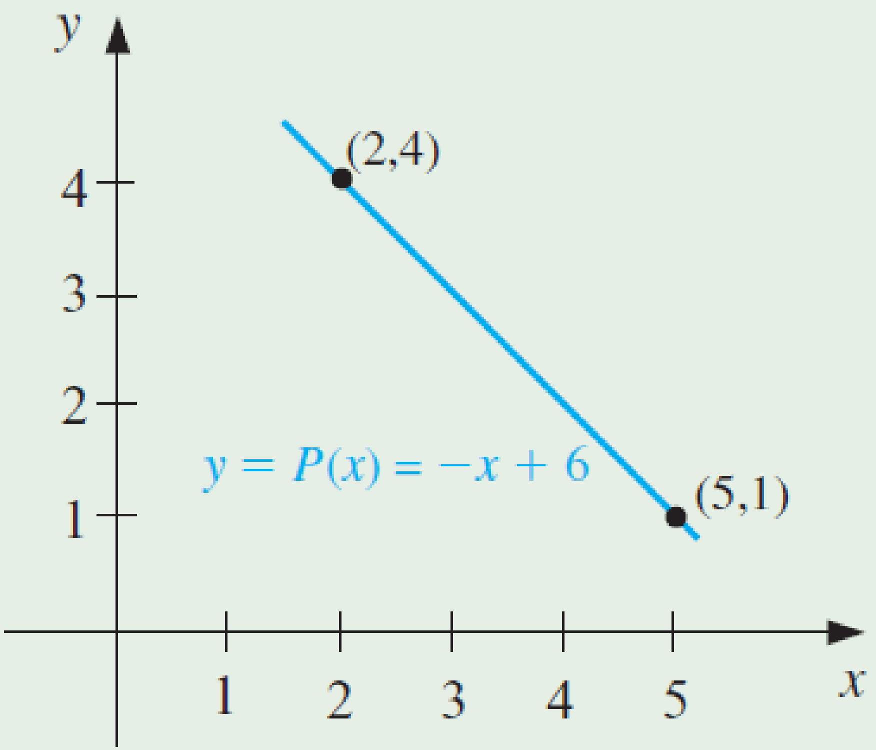
In this case we have

$$L_0(x) = \frac{x-5}{2-5} = -\frac{1}{3}(x-5) \quad \text{and} \quad L_1(x) = \frac{x-2}{5-2} = \frac{1}{3}(x-2),$$

so

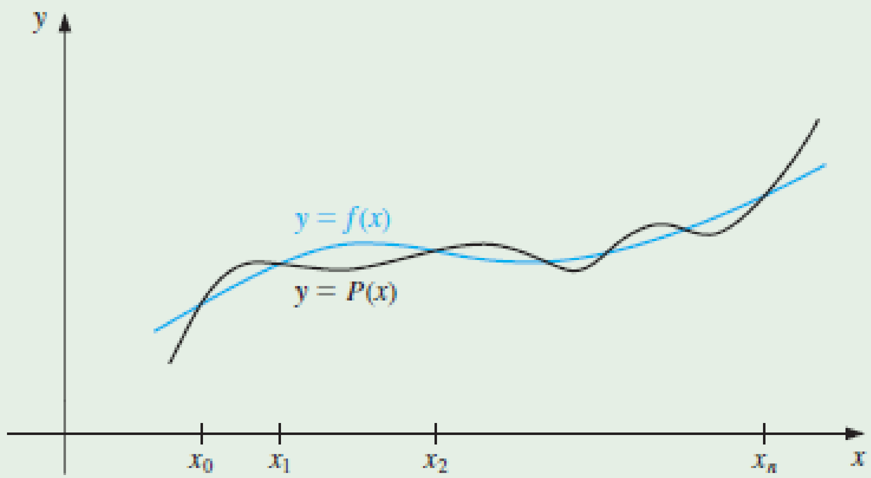
$$P(x) = -\frac{1}{3}(x-5)\cdot 4 + \frac{1}{3}(x-2)\cdot 1 = -\frac{4}{3}x + \frac{20}{3} + \frac{1}{3}x - \frac{2}{3} = -x + 6.$$

# The Lagrange Polynomial: The Linear Case




The linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

# The Lagrange Polynomial: Degree $n$ Construction




To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most $n$ that passes through the $n+1$ points

$$(x_0, f(x_0)),\ (x_1, f(x_1)), \ldots, (x_n, f(x_n)).$$

# The Lagrange Polynomial: The General Case

## Constructing the Degree $n$ Polynomial

- We first construct, for each $k = 0, 1, \ldots, n$, a function $L_{n,k}(x)$ with the property that $L_{n,k}(x_i) = 0$ when $i \neq k$ and $L_{n,k}(x_k) = 1$.
- To satisfy $L_{n,k}(x_i) = 0$ for each $i \neq k$ requires that the numerator of $L_{n,k}(x)$ contain the term

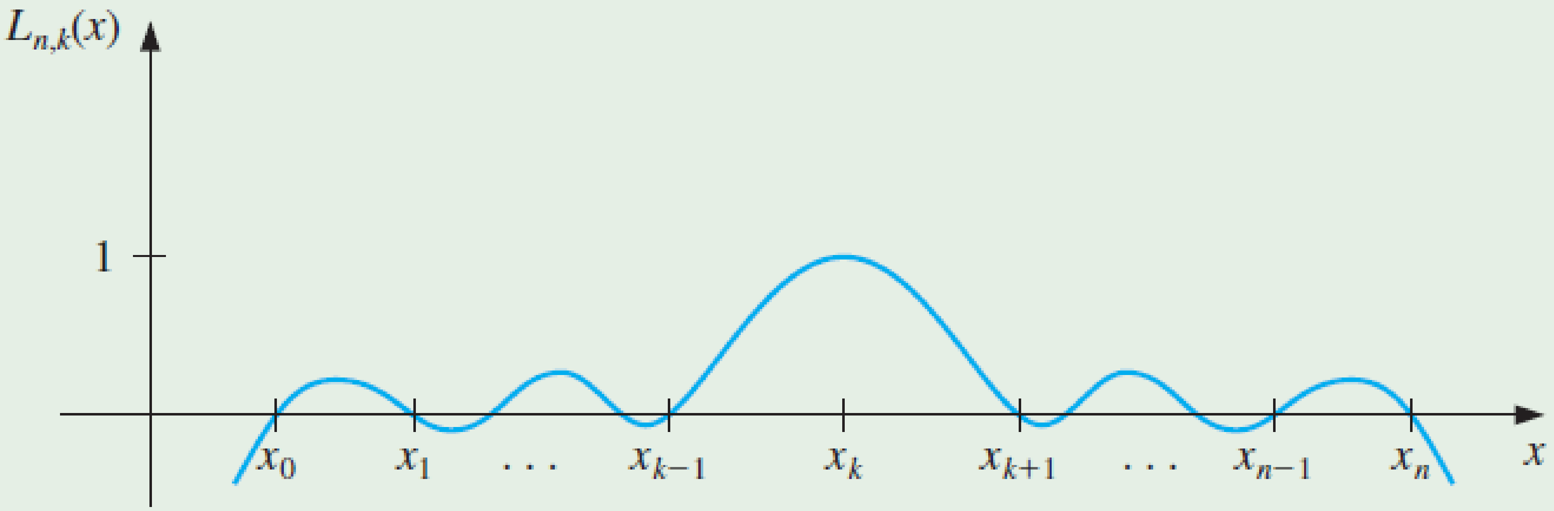
$$(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n).$$

- To satisfy $L_{n,k}(x_k) = 1$, the denominator of $L_{n,k}(x)$ must be this same term but evaluated at $x = x_k$.
- Thus

$$L_{n,k}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}.$$

## The Lagrange Polynomial: The General Case

$$L_{n,k}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}.$$



# The Lagrange Polynomial: The General Case

**Theorem: *n*-th Lagrange interpolating polynomial**

If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct numbers and $f$ is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most $n$ exists with

$$f(x_k) = P(x_k), \quad \text{for each } k = 0, 1, \ldots, n.$$

This polynomial is given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

where, for each $k = 0, 1, \ldots, n$, $L_{n,k}(x)$ is defined as follows:

# The Lagrange Polynomial: The General Case

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

**Definition of $L_{n,k}(x)$**

$$\begin{aligned} L_{n,k}(x) &= \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} \\ &= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)} \end{aligned}$$

We will write $L_{n,k}(x)$ simply as $L_k(x)$ when there is no confusion as to its degree.

# The Lagrange Polynomial: 2nd Degree Polynomial

**Example: $f(x) = \frac{1}{x}$**

(a) Use the numbers (called nodes) $x_0 = 2$, $x_1 = 2.75$ and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$.

(b) Use this polynomial to approximate $f(3) = \frac{1}{3}$.

# The Lagrange Polynomial: 2nd Degree Polynomial

## Part (a): Solution

We first determine the coefficient polynomials $L_0(x)$, $L_1(x)$, and $L_2(x)$:

$$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.5)(2-4)} = \frac{2}{3}(x-2.75)(x-4)$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4)$$

$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.5)} = \frac{2}{5}(x-2)(x-2.75)$$

Also, since $f(x) = \frac{1}{x}$:

$$f(x_0) = f(2) = 1/2, \qquad f(x_1) = f(2.75) = 4/11, \qquad f(x_2) = f(4) = 1/4$$

# The Lagrange Polynomial: 2nd Degree Polynomial

## Part (a): Solution (Cont'd)

Therefore, we obtain

$$\begin{aligned} P(x) &= \sum_{k=0}^{2} f(x_k) L_k(x) \\ &= \frac{1}{3}(x - 2.75)(x - 4) - \frac{64}{165}(x - 2)(x - 4) + \frac{1}{10}(x - 2)(x - 2.75) \\ &= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}. \end{aligned}$$

# The Lagrange Polynomial: 2nd Degree Polynomial

$$P(x) = \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$$

(b) Use this polynomial to approximate $f(3) = \frac{1}{3}$.

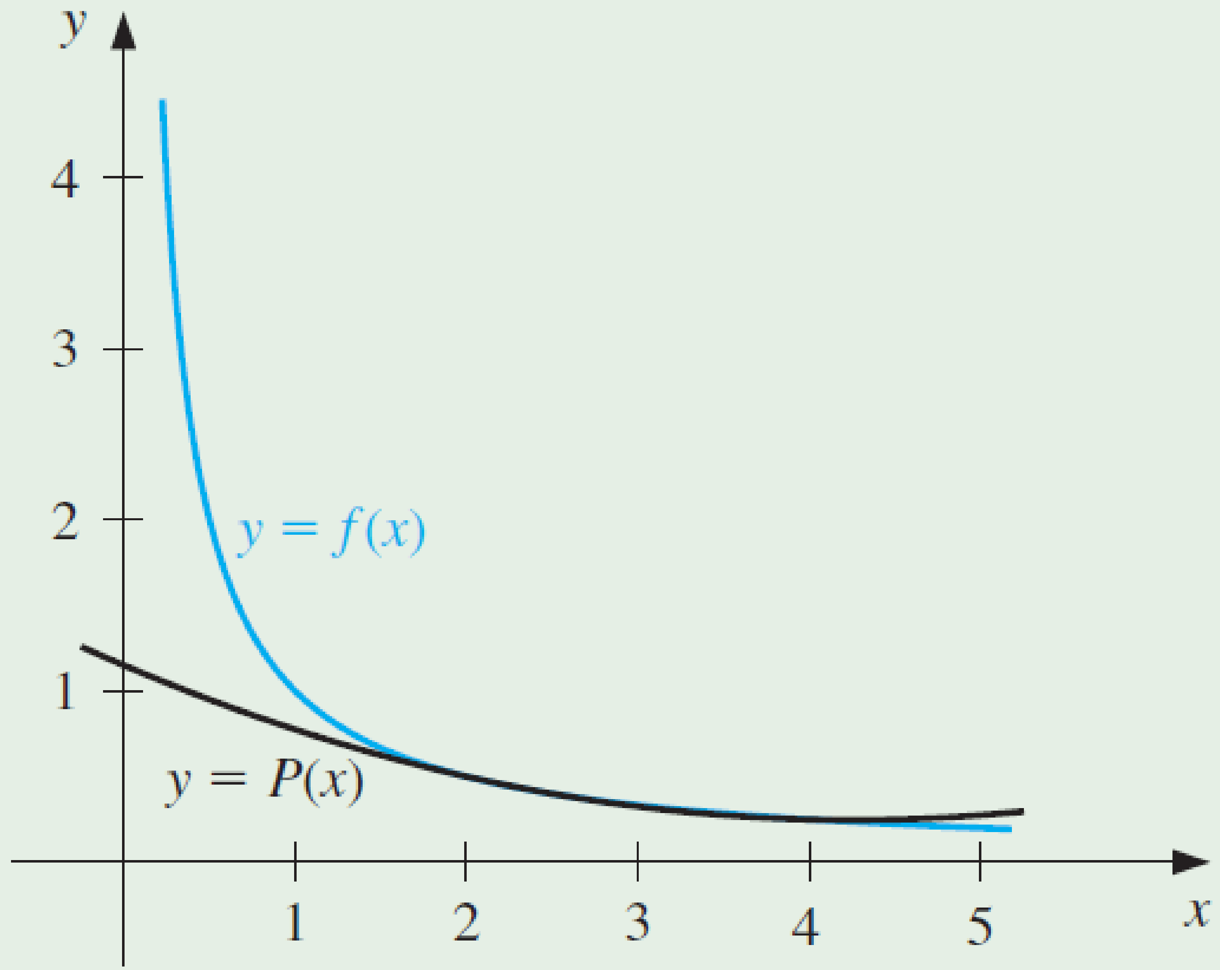
## Part (b): Solution

An approximation to $f(3) = \frac{1}{3}$ is

$$f(3) \approx P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{29}{88} \approx 0.32955.$$

Earlier, we we found that no Taylor polynomial expanded about $x_0 = 1$ could be used to reasonably approximate $f(x) = 1/x$ at $x = 3$.

## Second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$



# Error Approximation

# The Lagrange Polynomial: Theoretical Error Bound

**Theorem**

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each $x$ in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \ldots, x_n$, and hence in $(a, b)$, exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where $P(x)$ is the interpolating polynomial given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

# Lagrange Interpolating Polynomial Error Bound

## Example: Second Lagrange Polynomial for $f(x) = \frac{1}{x}$

In an earlier example, ▸ Original Example we found the second Lagrange polynomial for $f(x) = \frac{1}{x}$ on $[2, 4]$ using the nodes $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$. Determine the error form for this polynomial, and the maximum error when the polynomial is used to approximate $f(x)$ for $x \in [2, 4]$.

## Note

We will make use of the theoretical result ▸ Theorem written in the form

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| \cdot \max_{[2,4]} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

with $n = 2$

# The Lagrange Polynomial: 2nd Degree Error Bound

## Solution (1/3)

Because $f(x) = x^{-1}$, we have

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad \text{and} \quad f'''(x) = -\frac{6}{x^4}.$$

As a consequence, the second Lagrange polynomial has the error form

$$\frac{f'''(\xi(x))}{3!}(x - x_0)(x - x_1)(x - x_2) = -\frac{1}{\xi(x)^4}(x - 2)(x - 2.75)(x - 4)$$

for $\xi(x)$ in $(2, 4)$. The maximum value of $\frac{1}{\xi(x)^4}$ on the interval is $\frac{1}{2^4} = 1/16$.

# The Lagrange Polynomial: 2nd Degree Error Bound

## Solution (2/3)

We now need to determine the maximum value on $[2, 4]$ of the absolute value of the polynomial

$$g(x) = (x - 2)(x - 2.75)(x - 4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22$$

Because

$$g'(x) = 3x^2 - \frac{35}{2}x + \frac{49}{2} = \frac{1}{2}(3x - 7)(2x - 7),$$

the critical points occur at

$$x = \frac{7}{3} \text{ with } g\left(\frac{7}{3}\right) = \frac{25}{108} \qquad \text{and} \qquad x = \frac{7}{2} \text{ with } g\left(\frac{7}{2}\right) = -\frac{9}{16}$$

# The Lagrange Polynomial: 2nd Degree Error Bound

## Solution (3/3)

Hence, the maximum error is

$$\begin{aligned}
& \max_{[2,4]} \left| \frac{f'''(\xi(x))}{3!} \right| \cdot \max_{[2,4]} |(x - x_0)(x - x_1)(x - x_2)| \\
\leq \; & \frac{1}{3!} \cdot \frac{1}{16} \cdot \frac{9}{16} \\
= \; & \frac{3}{512} \\
\approx \; & 0.00586
\end{aligned}$$