Mathematical Proofs
A Transition to Advanced Mathematics

Chapter 2
Logic

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Statements

In mathematics, we are constantly dealing with statements.

Definition

A **statement** is a declarative sentence or assertion that is true or false (but not both).

Statements therefore declare or assert the truth of something.

Our interest is with statements dealing with mathematics.

The following are statements:

The number π is greater than 3.14 but less than 3.15.

The integer 91 is a prime

The integer 0 is an even integer.

Definition

Every statement is true or false and has a **truth value** (T for true and F for false).

We often use P, Q, R or $P_1, P_2, \ldots, P_n$ to denote statements:

P: The number π is greater than 3.14 but less than 3.15.

Q: The integer 91 is a prime.

R: The integer 0 is an even integer.

P and R have truth value T.

Q has truth value F.

Statements

Sentences that are imperative (commands) such as

Solve the equation.

or are interrogative (questions) such as

What is the derivative of $f(x) = x^2$?

or are exclamatory such as

This problem is unfair!

are not statements since these sentences are not declarative.

A declarative sentence is a statement even if we don't know whether it's true or false.

In the decimal expansion of $\sqrt{2}$, some digit appears 100 times in a row.

Definition

An **open sentence** is a declarative sentence that contains one or more variables, each variable representing a value in some prescribed set, called the **domain** of the variable, and which becomes a statement when values from their respective domains are substituted for these variables.

An open sentence that contains a variable x is typically represented by $P(x)$, $Q(x)$ or $R(x)$.

Definition

If $P(x)$ is an open sentence, where the domain of x is S, then we say $P(x)$ is an **open sentence over the domain** S.

Suppose that the domain of the variable x in the following open sentence is the set of integers:

$$P(x): \;\; 5x^2 + 8 = 53.$$

Then $P(x)$ is a true statement if $x = 3$ or $x = -3$ and is a false statement otherwise.

Example 1

For the open sentence

$$P(x, y):\ \ |x-1|+|y-2|=3$$

where the domain of the variable x is $S=\{3,2,1\}$ and the domain of the variable y is $T=\{1,3\}$,

$P(3,3):\ |3-1|+|3-2|=3$ is a true statement.
$P(x, y)$ is a true statement if $(x, y) \in \{(3,3),(3,1)\}$
and is a false statement for all other choices $(x, y) \in S \times T$. ♦

Statements

Definition

The possible truth values of a statement are often listed in a table, called a **truth table.**

P
T
F

Q
T
F

P	Q
T	T
T	F
F	T
F	F

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Negations

Negation

The **negation** of a statement P is the statement

not P and is denoted by $\sim P$.

Q : The number 91 is a prime number.

$\sim Q$: The number 91 is not a prime number.

R: The integer 0 is an even integer.

$\sim R$: The integer 0 is not an even integer.

or $\sim R$: The integer 0 is an odd integer.

$\sim Q$ is a true statement, while $\sim R$ is a false statement.

Negations

The negation of a true statement is always false and the negation of a false statement is always true; that is, the truth value of $\sim P$ is opposite to that of P.

P	$\sim P$
T	F
F	T

The truth table for negation.

Disjunctions

Definition

The **disjunction** of the statements P and Q is the statement

$$P \textbf{ or } Q$$

and is denoted by $P \vee Q$. The disjunction $P \vee Q$ is true if at least one of P and Q is true; otherwise, $P \vee Q$ is false.

Therefore, $P \vee Q$ is true if exactly one of P and Q is true or if both P and Q are true.

Disjunctions

For two statements P and Q,

$$P : \sqrt{2} < 1.5 \text{ and } Q : \pi = \frac{22}{7},$$

P is true and Q is false and so $P \vee Q$ is true.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

The truth table for disjunction.

Conjunctions

Definition

The **conjunction** of the statements P and Q is the statement

$$P \textbf{ and } Q$$

and is denoted by $P \wedge Q$. The conjunction $P \wedge Q$ is true only when both P and Q are true; otherwise, $P \wedge Q$ is false.

Conjunctions

For the two statements P and Q,

$$P : \sqrt{2} < 1.5 \text{ and } Q : \pi = \frac{22}{7},$$

P is true and Q is false so $P \wedge Q$ is false.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

The truth table for conjunction.

Implications

Definition

For statements P and Q, the **implication** (or **conditional**) is the statement

If P, **then** Q**.**

and is denoted by $P \Rightarrow Q$. In addition to the wording "If P, then Q.", we also express $P \Rightarrow Q$ in words as

P **implies** Q**.**

Implications

$P \Rightarrow Q$ is false only when P is true and Q is false ($P \Rightarrow Q$ is true otherwise).

For $P : \sqrt{2} < 1.5$ and $Q : \pi = \frac{22}{7}$.

$P \Rightarrow Q$: If $\sqrt{2} < 1.5$, then $\pi = \frac{22}{7}$. False

$Q \Rightarrow P$: If $\pi = \frac{22}{7}$, then $\sqrt{2} < 1.5$. True

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The truth table for implication.

Implications

There are several ways of expressing $P \Rightarrow Q$ in words, namely:

If P, then Q.

Q if P.

P implies Q.

P only if Q.

P is sufficient for Q.

Q is necessary for P.

Converse

Definition

For statements (or open sentences) P and Q, the implication $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$.

The converse of

If 5 is an even integer, then 7 is an odd integer. (True)

is

If 7 is an odd integer, then 5 is an even integer. (False)

Biconditionals

Definition

For statements (or open sentences) P and Q, the conjunction

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

of the implication $P \Rightarrow Q$ and its converse is called the **biconditional** of P and Q and is denoted by $P \Leftrightarrow Q$.

Biconditionals

$P \Leftrightarrow Q$ is true whenever the statements P and Q are both true or are both false, while $P \Leftrightarrow Q$ is false otherwise.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	**T**
T	F	F	T	**F**
F	T	T	F	**F**
F	F	T	T	**T**

P	Q	$P \Leftrightarrow Q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

The truth table for a biconditional

Biconditionals

Definition

The biconditional $P \Leftrightarrow Q$ is often stated as

P **is equivalent to** Q.

or

P **if and only if** Q.

or as

P **is a necessary and sufficient condition for** Q.

For statements P and Q, it then follows that the biconditional "P if and only if Q" is true only when P and Q have the same truth values.

The biconditional

5 is an even integer if and only if 7 is an odd integer. is false.

while

5 is an odd integer if and only if 7 is an odd integer. is true.

Definition

The symbols $\sim, \vee, \wedge, \Rightarrow$ and $\Leftrightarrow$ are sometimes referred to as **logical connectives**.

From given statements, we can use these logical connectives to form more intricate statements.

For example, the statement $(P \vee Q) \wedge (P \vee R)$ is a statement formed from the given statements P, Q and R and the logical connectives $\vee$ and $\wedge$.

We call $(P \vee Q) \wedge (P \vee R)$ a compound statement.

Tautologies

Definition

More generally, a **compound statement** is a statement composed of one or more given statements (called **component statements** in this context) and at least one logical connective.

A compound statement S is called a **tautology** if it is true for all possible combinations of truth values of the component statements that comprise S.

$P \vee (\sim P)$ is a tautology.

Example 2

For statements P and Q, the compound statement

$$(\sim Q) \vee (P \Rightarrow Q)$$

is a tautology, as is verified in the truth table shown below.

P	Q	$\sim Q$	$P \Rightarrow Q$	$(\sim Q) \vee (P \Rightarrow Q)$
T	T	F	T	**T**
T	F	T	F	**T**
F	T	F	T	**T**
F	F	T	T	**T**

♦

Contradictions

Definition

A compound statement S is called a **contradiction** if it is false for all possible combinations of truth values of the component statements that are used to form S.

$P \wedge (\sim P)$ is a contradiction

Contradictions

Example 3

For statements P and Q, the compound statement

$$(P \wedge Q) \wedge (Q \Rightarrow (\sim P))$$

is a contradiction, which is verified in the truth table below.

P	Q	$\sim P$	$P \wedge Q$	$Q \Rightarrow \sim P$	$(P \wedge Q) \wedge (Q \Rightarrow \sim P)$
T	T	F	T	F	**F**
T	F	F	F	T	**F**
F	T	T	F	T	**F**
F	F	T	F	T	**F**

Indeed, if a compound statement S is a tautology,then its negation $\sim S$ is a contradiction. ♦

Logical Equivalence

Definition

Let R and S be two compound statements involving the same component statements. Then R and S are called **logically equivalent** if R and S have the same truth values for all combinations of truth values of their component statements.

If R and S are logically equivalent, then this is denoted by $R \equiv S$.

Logical Equivalence

For example, $P \Rightarrow Q$ and $(\sim P) \vee Q$ are logically equivalent and so

$$P \Rightarrow Q \equiv (\sim P) \vee Q.$$

P	Q	$\sim P$	$P \Rightarrow Q$	$(\sim P) \vee Q$
T	T	F	**T**	**T**
T	F	F	**F**	**F**
F	T	T	**T**	**T**
F	F	T	**T**	**T**

Logical Equivalence

The logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$, is especially important and we will have occasion to use this fact often.

Theorem

Let P and Q be two statements. Then

$$P \Rightarrow Q \text{ and } (\sim P) \vee Q$$

are logically equivalent.

Theorem

For statements P, Q and R,

(1) Commutative Laws

(a) $P \vee Q \equiv Q \vee P$.

(b) $P \wedge Q \equiv Q \wedge P$.

(2) Associative Laws

(a) $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$.

(b) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$.

(3) Distributive Laws

(a) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$.

(b) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$.

(4) De Morgan's Laws

(a) $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$.

(b) $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$.

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Example 4

Suppose that we are asked to verify

$$\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$$

for every two statements P and Q. Using the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$, we see that

$$\sim (P \Rightarrow Q) \equiv \sim ((\sim P) \vee Q) \equiv (\sim (\sim P)) \wedge (\sim Q) \equiv P \wedge (\sim Q),$$

implying that the statements $\sim (P \Rightarrow Q)$ and $P \wedge (\sim Q)$ are logically equivalent, which we alluded to earlier. ♦

Example 5

Using the second of De Morgan's Laws and

$$\sim (P \Rightarrow Q) \equiv \sim ((\sim P) \vee Q) \equiv (\sim (\sim P)) \wedge (\sim Q) \equiv P \wedge (\sim Q),$$

we can establish a useful logically equivalent form of the negation of $P \Leftrightarrow Q$ by the following string of logical equivalences:

$$\begin{aligned} \sim (P \Leftrightarrow Q) &\equiv \sim ((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \\ &\equiv (\sim (P \Rightarrow Q)) \vee (\sim (Q \Rightarrow P)) \\ &\equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P)). \end{aligned}$$

♦

Logical Equivalence

What we have observed about the negation of an implication and a biconditional is repeated in the following theorem.

Theorem

For statements P and Q,

(a) $\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$.

(b) $\sim (P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$.

Quantified Statements

Definition

There are ways that an open sentence can be converted into a statement by a method called **quantification**.

Let $P(x)$ be an open sentence over a domain S. Adding the phrase "For every $x \in S$" to $P(x)$ produces a statement called a **quantified statement**.

The phrase "for every" is referred to as the **universal quantifier** and is denoted by the symbol $\forall$.

Quantified Statements

Other ways to express the universal quantifier are "for each" and "for all." This quantified statement is expressed in symbols by

$$\forall x \in S, P(x) \tag{1}$$

and is expressed in words by

$$\text{For every } x \in S,\ P(x). \tag{2}$$

The quantified statement (1) (or (2)) is true if $P(x)$ is true for every $x \in S$; while the quantified statement (1) is false if $P(x)$ is false for at least one element $x \in S$.

Another way to convert an open sentence $P(x)$ over a domain S into a statement through quantification is by the introduction of a quantifier called an existential quantifier.

Definition

Each of the phrases "there exists," "there is," "for some," and "for at least one" is referred to as an **existential quantifier** and is denoted by the symbol $\exists$.

Quantified Statements

The quantified statement

$$\exists x \in S, P(x) \tag{3}$$

can be expressed in words by

$$\text{There exists } x \in S \text{ such that } P(x). \tag{4}$$

The quantified statement (3) (or (4)) is true if $P(x)$ is true for at least one element $x \in S$; while the quantified statement (3) is false if $P(x)$ is false for all $x \in S$.

Example 6

For the open sentence

$$P(n): \ \frac{2n^2 + 5 + (-1)^n}{2} \text{ is prime.}$$

over the domain $S = \{1, 2, \cdots, 7\}$, the quantified statement

$$\forall n \in S, P(n): \text{ For every } n \in S, \ \frac{2n^2 + 5 + (-1)^n}{2} \text{ is prime.}$$

is false since $P(5)$ is false, for example; while the quantified statement

$$\exists n \in S, P(n): \text{ There exists } n \in S \text{ such that } \frac{2n^2 + 5 + (-1)^n}{2} \text{ is prime.}$$

is true since $P(1)$ is true, for example. ♦

Suppose now that we were to consider the open sentence

$$Q(x) : x^2 \leq 0.$$

The statement $\forall x \in \mathbf{R}, Q(x)$ (that is, for every real number x, we have $x^2 \leq 0$) is false since, for example, $Q(1)$ is false. Of course, this means that its negation is true.

If it were not the case that for every real number x, we have $x^2 \leq 0$, then there must exist some real number x such that $x^2 > 0$.

This negation

There exists a real number x such that $x^2 > 0$.

can be written in symbols as

$$\exists x \in \mathbf{R}, x^2 > 0 \quad \text{or} \quad \exists x \in \mathbf{R}, \sim Q(x).$$

More generally, if we are considering an open sentence $P(x)$ over a domain S, then

$$\sim (\forall x \in S, P(x)) \equiv \exists x \in S, \sim P(x).$$

Example 7

The following statement contains the existential quantifier.

$$\text{There exists a real number } x \text{ such that } x^2 = 3. \qquad (5)$$

If we let $P(x) : x^2 = 3$, then (5) can be rewritten as $\exists x \in \mathbf{R}, P(x)$. The statement (5) is true since $P(x)$ is true when $x = \sqrt{3}$ (or when $x = -\sqrt{3}$). The negation of (5) is:

$$\text{For every real number } x,\ x^2 \neq 3. \qquad (6)$$

The statement (6) is therefore false. ♦

Example 8

The negation of

For all integers a and b,
if ab is even, then a is even and b is even.

is

There exist integers a and b
such that ab is even and a or b is odd. ♦

Example 9

Let $A = \{\sqrt{2}, \pi\}$ and $B = \{-\sqrt{2}, \sqrt{3}, e\}$. The negation of

There exists a rational number r such that $r \in A$ or $r \in B$.

is

For every rational number r, both $r \notin A$ and $r \notin B$. ♦

Example 10

Consider the open sentence

$$P(x, y):\ \ x + y \text{ is prime},$$

where the domain of x is $S = \{2, 3\}$ and the domain of y is $T = \{3, 4\}$. The quantified statement

$$\forall x \in S, \exists y \in T, P(x, y),$$

expressed in words, is

For every $x \in S$, there exists $y \in T$ such that $x + y$ is prime.

This statement is true. For $x = 2$, $P(2, 3)$ is true and for $x = 3$, $P(3, 4)$ is true. ♦

Example 11

Once again, consider the open sentence

$$P(x, y): \quad x + y \text{ is prime},$$

where the domain of x is $S = \{2, 3\}$ and the domain of y is $T = \{3, 4\}$. Here, we consider the quantified statement

$$\exists y \in T, \forall x \in S, P(x, y), \tag{7}$$

which, expressed in words, is

There exists some $y \in T$ such that $x + y$ is prime for every $x \in S$.

The statement is false, however, for if $y = 3$, then for $x = 3$, $x + y = 6$ is not prime. Also, for $y = 4$, it follows for $x = 2$ that $x + y = 6$ is not prime. Hence, the quantified statement (7) is false. ♦

Quantified Statements

A review of the symbols introduced in this chapter:

$\sim$	negation (not)
$\vee$	disjunction (or)
$\wedge$	conjunction (and)
$\Rightarrow$	implication
$\Leftrightarrow$	biconditional
$\forall$	universal quantifier (for every)
$\exists$	existential quantifier (there exists)

Characterizations

Definition

Suppose that some concept (or object) is expressed in an open sentence $P(x)$ over a domain S and $Q(x)$ is another open sentence over the domain S concerning this concept.

We say that this concept is **characterized** by $Q(x)$ if $\forall x \in S, P(x) \Leftrightarrow Q(x)$ is a true statement.

The statement $\forall x \in S, P(x) \Leftrightarrow Q(x)$ is then called a **characterization** of this concept.

Characterizations

Irrational numbers are defined as real numbers that are not rational and are characterized as real numbers whose decimal expansions are nonrepeating. This provides a characterization of irrational numbers:

A real number r is irrational if and only if
r has a nonrepeating decimal expansion.

Equilateral triangles are defined as triangles whose sides are equal. They are characterized however as triangles whose angles are equal. Therefore, we have the characterization:

A triangle T is equilateral if and only if
T has three equal angles.

Characterizations

You might think that equilateral triangles are also characterized as those triangles having three equal sides but the associated biconditional:

A triangle T is equilateral if and only if T has three equal sides.

is not a characterization of equilateral triangles. Indeed, this is the definition we gave of equilateral triangles.

A characterization of a concept then gives an alternative, but equivalent, way of looking at this concept.

Characterizations are often valuable in studying concepts or in proving other results.

We will see examples of this in future chapters.

We mentioned that the following biconditional, though true, is not a characterization:

A triangle T is equilateral if and only if T has three equal sides.

Although this is the definition of equilateral triangles, mathematicians rarely use the phrase "if and only if" in a definition since this is what is meant in a definition. That is, a triangle is defined to be equilateral if it has three equal sides. Consequently, a triangle with three equal sides is equilateral but a triangle that does not have three equal sides is not equilateral.