

Fixed Point Iteration II

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Outline

- 1 Functional (Fixed-Point) Iteration
- 2 Convergence Criteria for the Fixed-Point Method
- 3 Sample Problem: $f(x) = x^3 + 4x^2 - 10 = 0$

Functional (Fixed-Point) Iteration

Now that we have established a condition for which $g(x)$ has a unique fixed point in I , there remains the problem of how to find it. The technique employed is known as fixed-point iteration.

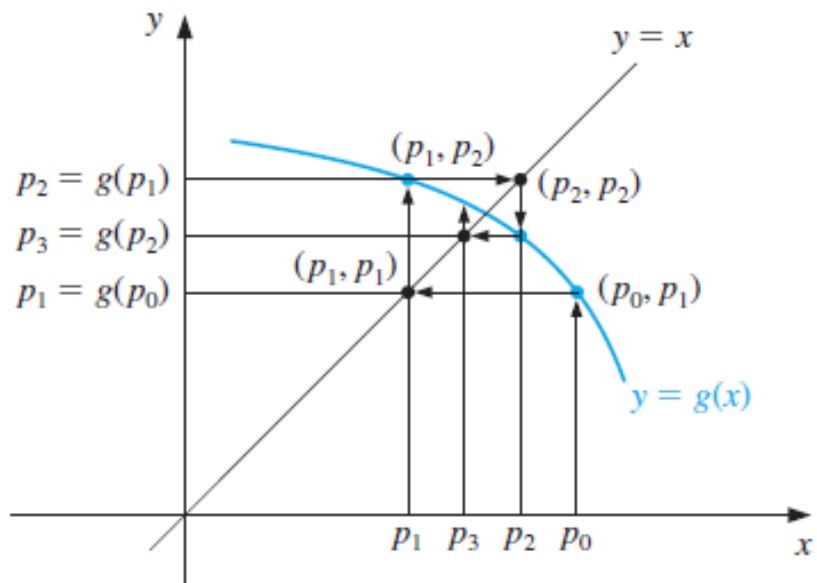
Basic Approach

- To approximate the fixed point of a function g , we choose an initial approximation p_0 and generate the sequence $\{p_n\}_{n=0}^{\infty}$ by letting $p_n = g(p_{n-1})$, for each $n \geq 1$.
- If the sequence converges to p and g is continuous, then

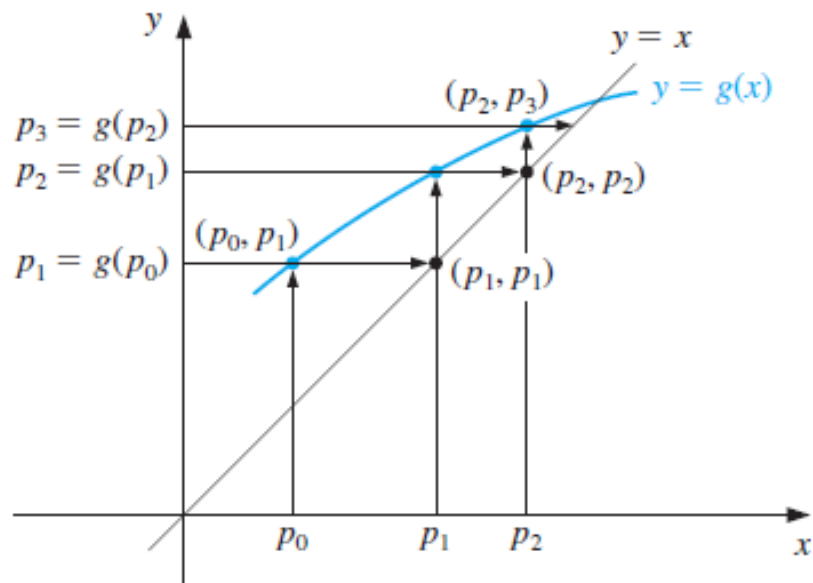
$$p = \lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} g(p_{n-1}) = g\left(\lim_{n \rightarrow \infty} p_{n-1}\right) = g(p),$$

and a solution to $x = g(x)$ is obtained.

Functional (Fixed-Point) Iteration



(a)



(b)

A Single Nonlinear Equation

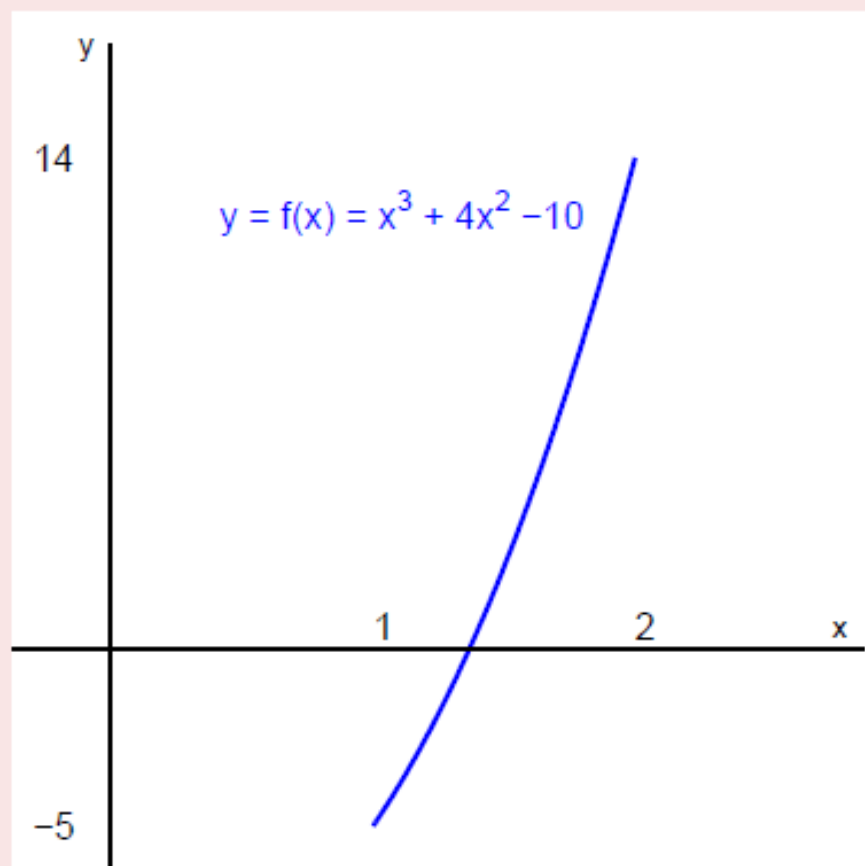
Example 1

The equation

$$x^3 + 4x^2 - 10 = 0$$

has a unique root in $[1, 2]$. Its value is approximately 1.365230013.

$$f(x) = x^3 + 4x^2 - 10 = 0 \text{ on } [1, 2]$$



$$f(x) = x^3 + 4x^2 - 10 = 0 \text{ on } [1, 2]$$

Possible Choices for $g(x)$

- There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation.
- For example, to obtain the function g described in part (c), we can manipulate the equation $x^3 + 4x^2 - 10 = 0$ as follows:

$$4x^2 = 10 - x^3, \quad \text{so} \quad x^2 = \frac{1}{4}(10 - x^3), \quad \text{and} \quad x = \pm \frac{1}{2}(10 - x^3)^{1/2}.$$

- We will consider 5 such rearrangements and, later in this section, provide a brief analysis as to why some do and some not converge to $p = 1.365230013$.

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

5 Possible Transpositions to $x = g(x)$

$$x = g_1(x) = x - x^3 - 4x^2 + 10$$

$$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$$

$$x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3}$$

$$x = g_4(x) = \sqrt{\frac{10}{4 + x}}$$

$$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

Numerical Results for $f(x) = x^3 + 4x^2 - 10 = 0$

n	g_1	g_2	g_3	g_4	g_5
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.3733333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	-469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$$x = g_1(x) = x - x^3 - 4x^2 + 10 \quad \text{Does not Converge}$$

$$x = g_2(x) = \sqrt{\frac{10}{x} - 4x} \quad \text{Does not Converge}$$

$$x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3} \quad \text{Converges after 31 Iterations}$$

$$x = g_4(x) = \sqrt{\frac{10}{4+x}} \quad \text{Converges after 12 Iterations}$$

$$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \quad \text{Converges after 5 Iterations}$$

Functional (Fixed-Point) Iteration

A Crucial Question

- How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?
- The following theorem and its corollary give us some clues concerning the paths we should pursue and, perhaps more importantly, some we should reject.

Functional (Fixed-Point) Iteration

Convergence Result

Let $g \in C[a, b]$ with $g(x) \in [a, b]$ for all $x \in [a, b]$. Let $g'(x)$ exist on (a, b) with

$$|g'(x)| \leq k < 1, \quad \forall x \in [a, b].$$

If p_0 is any point in $[a, b]$ then the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

will converge to the unique fixed point p in $[a, b]$.

Functional (Fixed-Point) Iteration

Corollary to the Convergence Result

If g satisfies the hypothesis of the Theorem, then

$$|p_n - p| \leq \frac{k^n}{1 - k} |p_1 - p_0|.$$

Functional (Fixed-Point) Iteration

Example: $g(x) = g(x) = 3^{-x}$

Consider the iteration function $g(x) = 3^{-x}$ over the interval $[\frac{1}{3}, 1]$ starting with $p_0 = \frac{1}{3}$. Determine a lower bound for the number of iterations n required so that $|p_n - p| < 10^{-5}$?

Determine the Parameters of the Problem

Note that $p_1 = g(p_0) = 3^{-\frac{1}{3}} = 0.6933612$ and, since $g'(x) = -3^{-x} \ln 3$, we obtain the bound

$$|g'(x)| \leq 3^{-\frac{1}{3}} \ln 3 \leq .7617362 \approx .762 = k.$$

Functional (Fixed-Point) Iteration

Use the Corollary

Therefore, we have

$$\begin{aligned} |p_n - p| &\leq \frac{k^n}{1 - k} |p_0 - p_1| \\ &\leq \frac{.762^n}{1 - .762} \left| \frac{1}{3} - .6933612 \right| \\ &\leq 1.513 \times 0.762^n \end{aligned}$$

We require that

$$1.513 \times 0.762^n < 10^{-5} \quad \text{or} \quad n > 43.88$$

Footnote on the Estimate Obtained

- It is important to realise that the estimate for the number of iterations required given by the theorem is an upper bound.
- In the previous example, only 21 iterations are required in practice, i.e. $p_{21} = 0.54781$ is accurate to 10^{-5} .
- The reason, in this case, is that we used

$$g'(1) = 0.762$$

whereas

$$g'(0.54781) = 0.602$$

- If one had used $k = 0.602$ (were it available) to compute the bound, one would obtain $N = 23$ which is a more accurate estimate.

A Single Nonlinear Equation

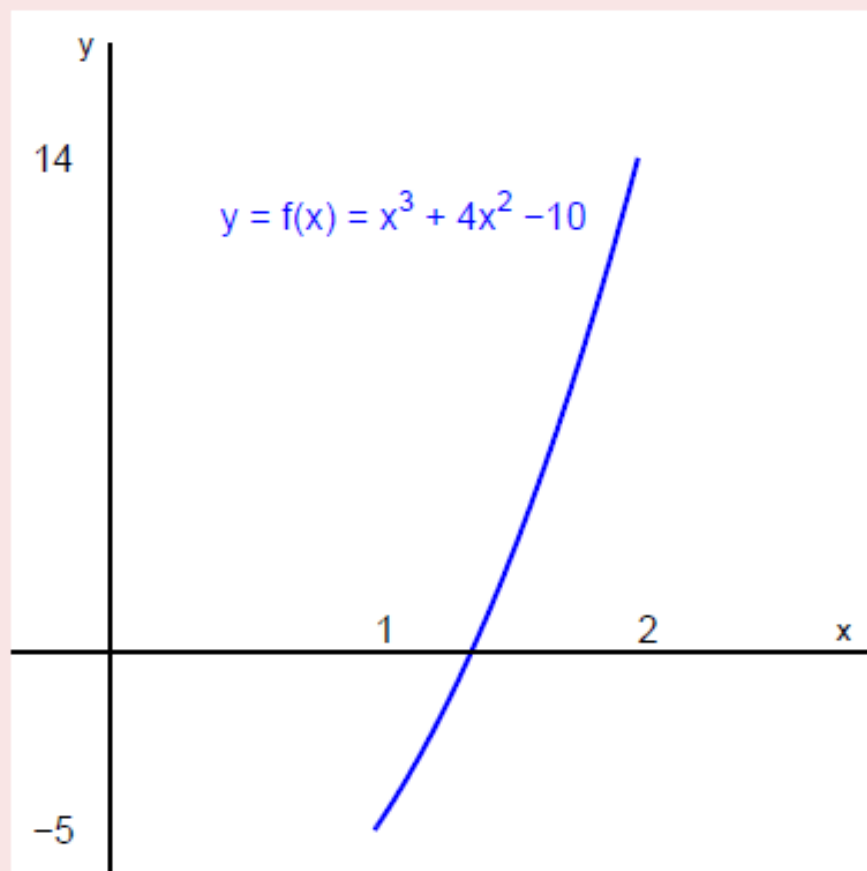
Example 2

We return to Example 1 and the equation

$$x^3 + 4x^2 - 10 = 0$$

which has a unique root in $[1, 2]$. Its value is approximately 1.365230013.

$$f(x) = x^3 + 4x^2 - 10 = 0 \text{ on } [1, 2]$$



Solving $f(x) = x^3 + 4x^2 - 10 = 0$

Earlier, we listed 5 possible transpositions to $x = g(x)$

$$x = g_1(x) = x - x^3 - 4x^2 + 10$$

$$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$$

$$x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3}$$

$$x = g_4(x) = \sqrt{\frac{10}{4+x}}$$

$$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

Results Observed for $x = g(x)$ with $x_0 = 1.5$

$$x = g_1(x) = x - x^3 - 4x^2 + 10 \quad \text{Does not Converge}$$

$$x = g_2(x) = \sqrt{\frac{10}{x} - 4x} \quad \text{Does not Converge}$$

$$x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3} \quad \text{Converges after 31 Iterations}$$

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Solving $f(x) = x^3 + 4x^2 - 10 = 0$

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Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_1(x) = x - x^3 - 4x^2 + 10$$

Iteration for $x = g_1(x)$ Does Not Converge

Since

$$g_1'(x) = 1 - 3x^2 - 8x$$

$$g_1'(1) = -10$$

$$g_1'(2) = -27$$

there is no interval $[a, b]$ containing p for which $|g_1'(x)| < 1$. Also, note that

$$g_1(1) = 6$$

and

$$g_1(2) = -12$$

so that $g(x) \notin [1, 2]$ for $x \in [1, 2]$.

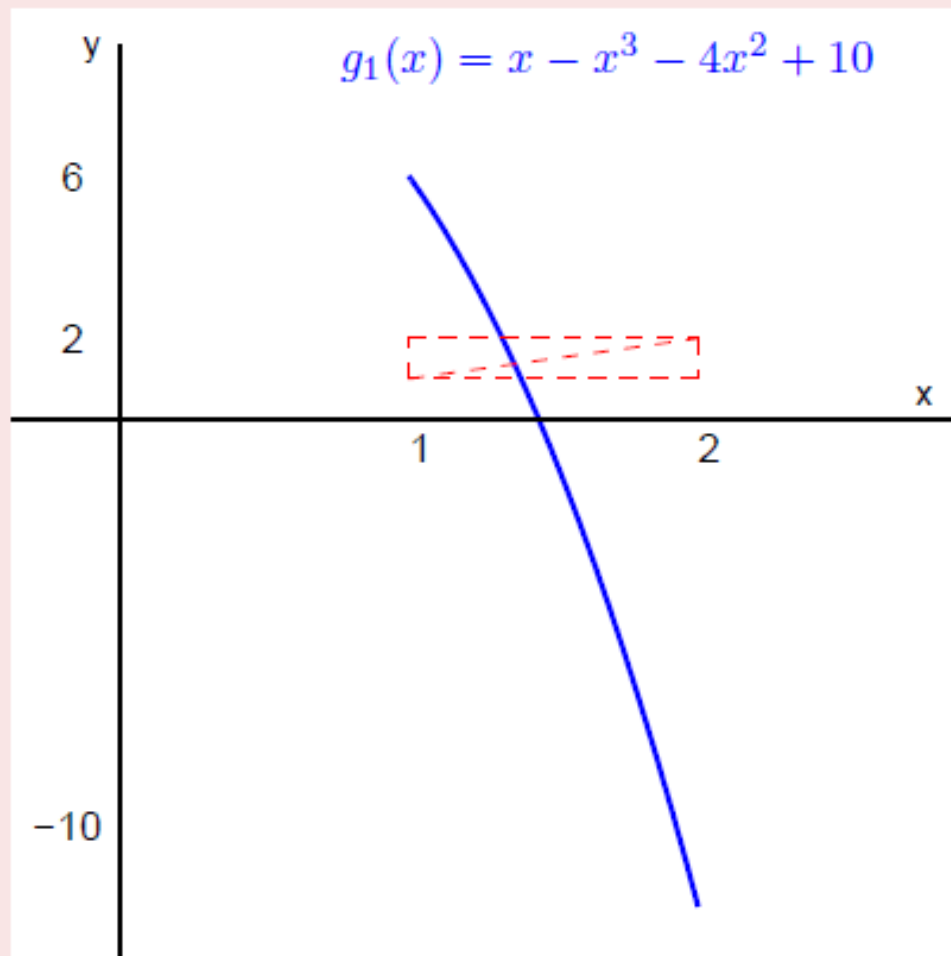
Iteration Function: $x = g_1(x) = x - x^3 - 4x^2 + 10$

Iterations starting with $p_0 = 1.5$

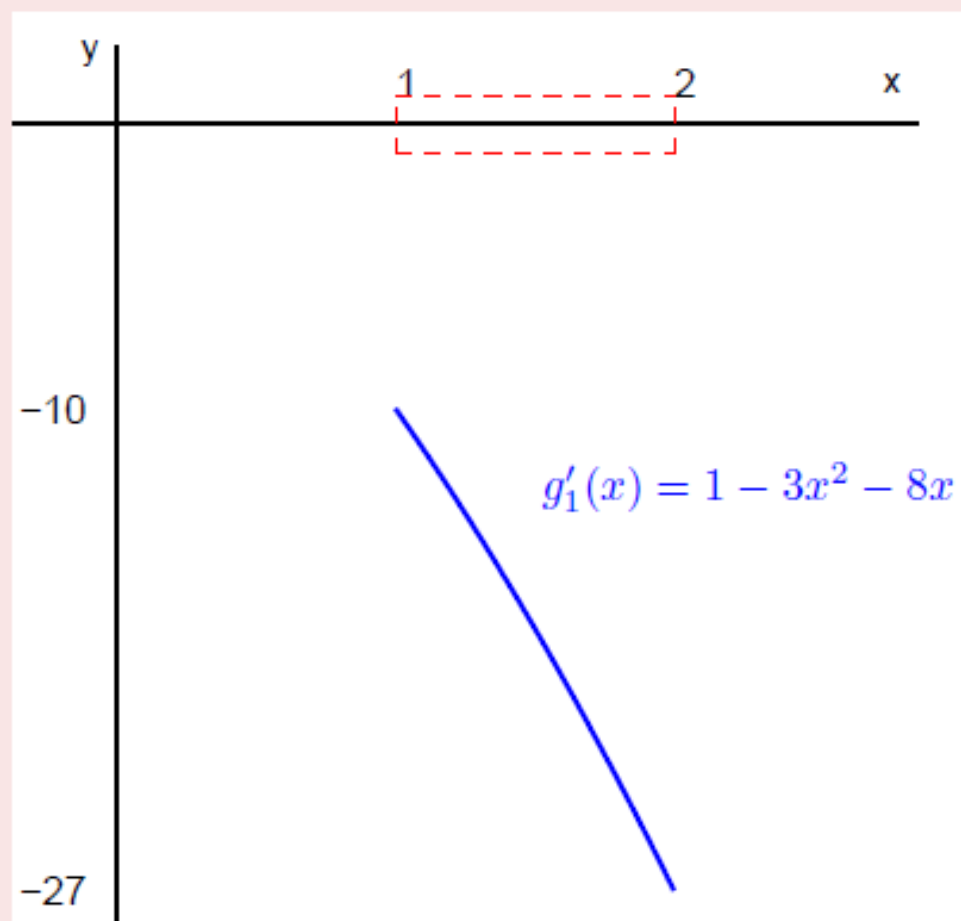
n	p_{n-1}	p_n	$ p_n - p_{n-1} $
1	1.5000000	-0.8750000	2.3750000
2	-0.8750000	6.7324219	7.6074219
3	6.7324219	-469.7200120	476.4524339

$$p_4 \approx 1.03 \times 10^8$$

g_1 Does Not Map $[1, 2]$ into $[1, 2]$



$|g'_1(x)| > 1$ on $[1, 2]$



Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$$x = g_1(x) = x - x^3 - 4x^2 + 10 \quad \text{Does not Converge}$$

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Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$$

Iteration for $x = g_2(x)$ is Not Defined

It is clear that $g_2(x)$ does not map $[1, 2]$ onto $[1, 2]$ and the sequence $\{p_n\}_{n=0}^{\infty}$ is not defined for $p_0 = 1.5$. Also, there is no interval containing p such that

$$|g_2'(x)| < 1$$

since

$$g_2'(1) \approx -2.86$$

$$g_2'(p) \approx -3.43$$

and $g_2'(x)$ is not defined for $x > 1.58$.

Iteration Function: $x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$

Iterations starting with $p_0 = 1.5$

n	p_{n-1}	p_n	$ p_n - p_{n-1} $
1	1.5000000	0.8164966	0.6835034
2	0.8164966	2.9969088	2.1804122
3	2.9969088	$\sqrt{-8.6509}$	—

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

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Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3}$$

Iteration for $x = g_3(x)$ Converges (Slowly)

By differentiation,

$$g'_3(x) = -\frac{3x^2}{4\sqrt{10 - x^3}} < 0 \quad \text{for } x \in [1, 2]$$

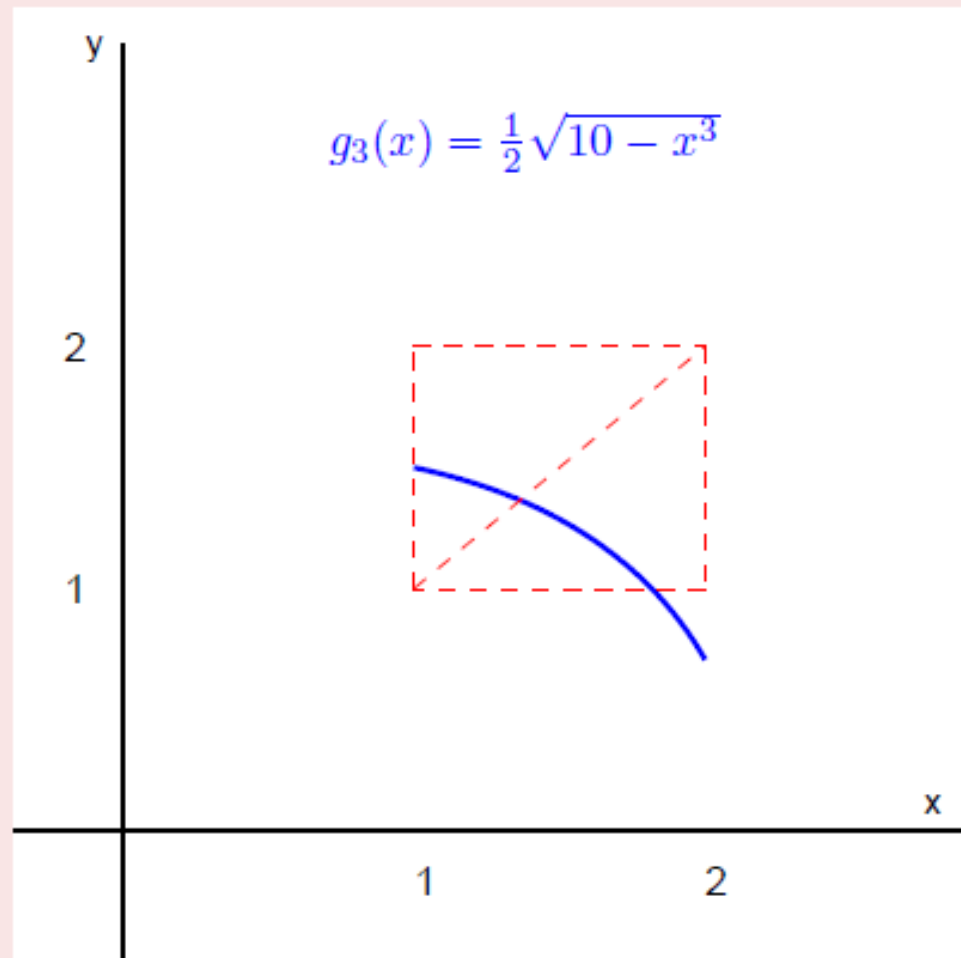
and so $g = g_3$ is strictly decreasing on $[1, 2]$. However, $|g'_3(x)| > 1$ for $x > 1.71$ and $|g'_3(2)| \approx -2.12$. A closer examination of $\{p_n\}_{n=0}^{\infty}$ will show that it suffices to consider the interval $[1, 1.7]$ where $|g'_3(x)| < 1$ and $g(x) \in [1, 1.7]$ for $x \in [1, 1.7]$.

Iteration Function: $x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$

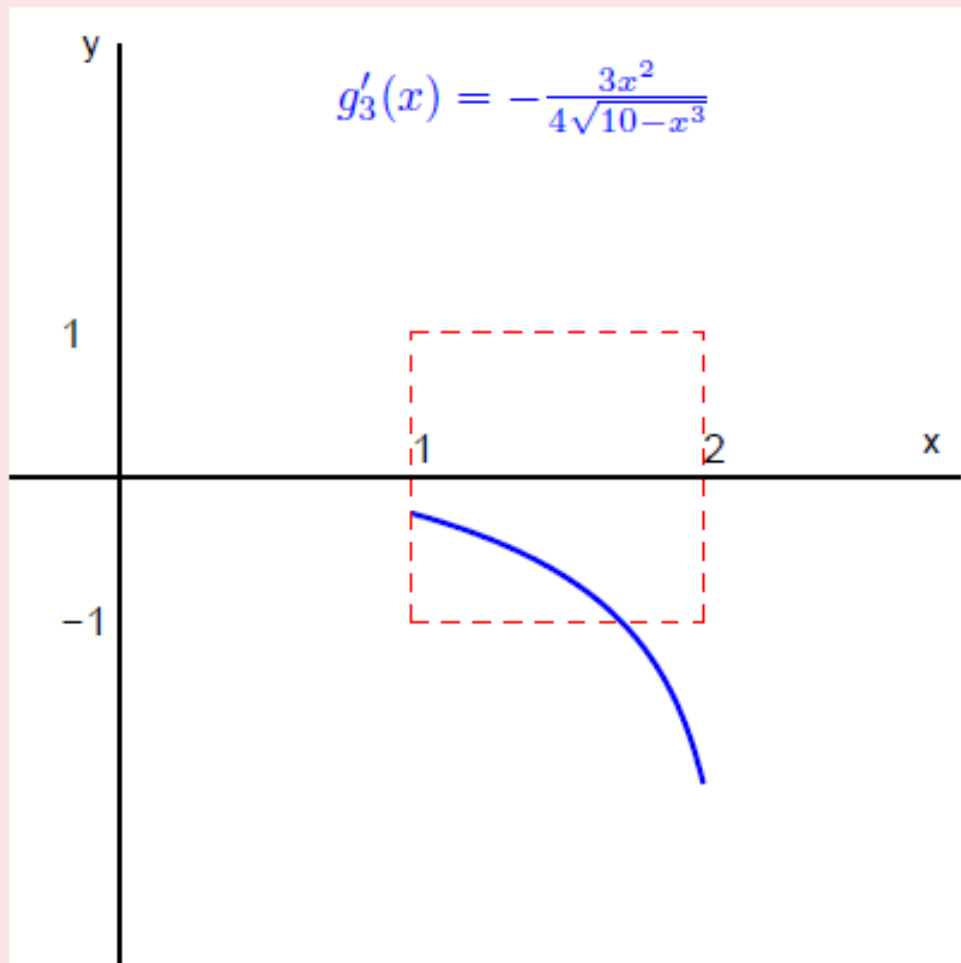
Iterations starting with $p_0 = 1.5$

n	p_{n-1}	p_n	$ p_n - p_{n-1} $
1	1.5000000000	1.286953768	0.213046232
2	1.286953768	1.402540804	0.115587036
3	1.402540804	1.345458374	0.057082430
4	1.345458374	1.375170253	0.029711879
5	1.375170253	1.360094193	0.015076060
6	1.360094193	1.367846968	0.007752775
30	1.365230013	1.365230014	0.000000001
31	1.365230014	1.365230013	0.000000000

g_3 Maps $[1, 1.7]$ into $[1, 1.7]$



$$|g'_3(x)| < 1 \text{ on } [1, 1.7]$$



Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$$x = g_1(x) = x - x^3 - 4x^2 + 10 \quad \text{Does not Converge}$$

$$x = g_2(x) = \sqrt{\frac{10}{x} - 4x} \quad \text{Does not Converge}$$

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$$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \quad \text{Converges after 5 Iterations}$$

$$\text{Solving } f(x) = x^3 + 4x^2 - 10 = 0$$

$$x = g_4(x) = \sqrt{\frac{10}{4+x}}$$

Iteration for $x = g_4(x)$ Converges (Moderately)

By differentiation,

$$g'_4(x) = -\sqrt{\frac{10}{4(4+x)^3}} < 0$$

and it is easy to show that

$$0.10 < |g'_4(x)| < 0.15 \quad \forall x \in [1, 2]$$

The bound on the magnitude of $|g'_4(x)|$ is much smaller than that for $|g'_3(x)|$ and this explains the reason for the much faster convergence.

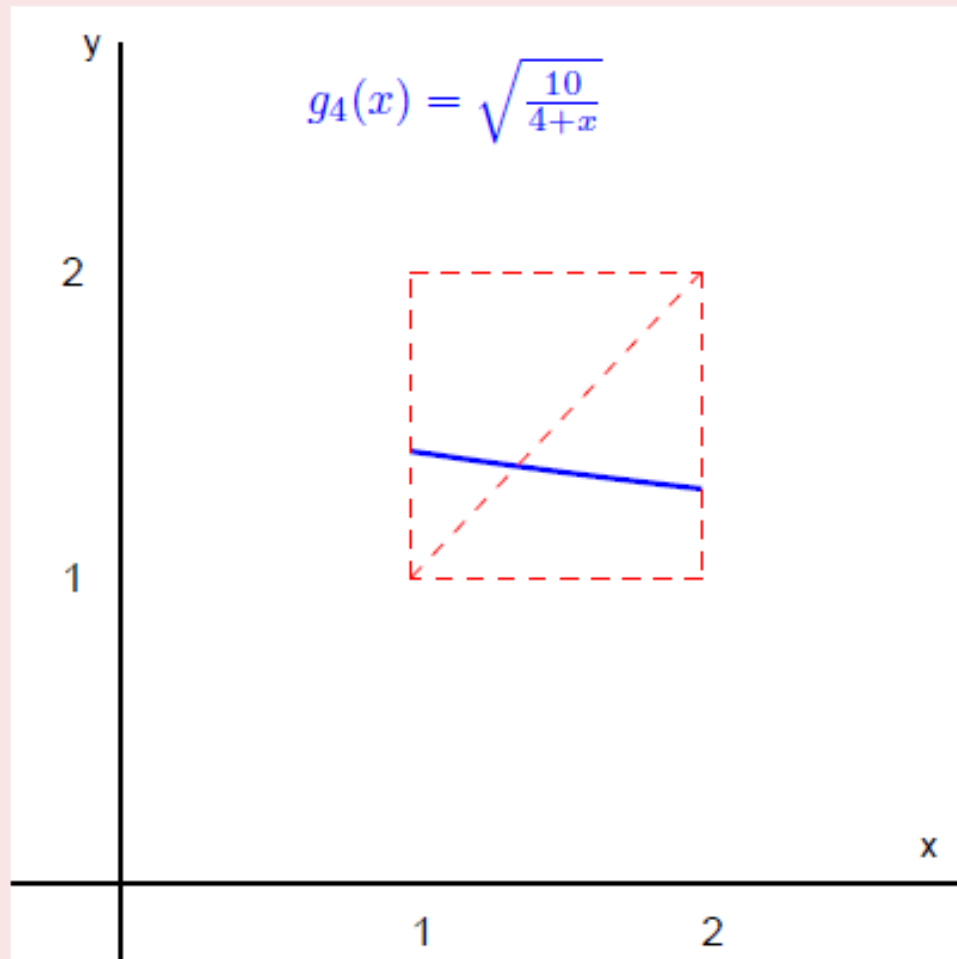
Iteration Function: $x = g_4(x) = \sqrt{\frac{10}{4+x}}$

Iterations starting with $p_0 = 1.5$

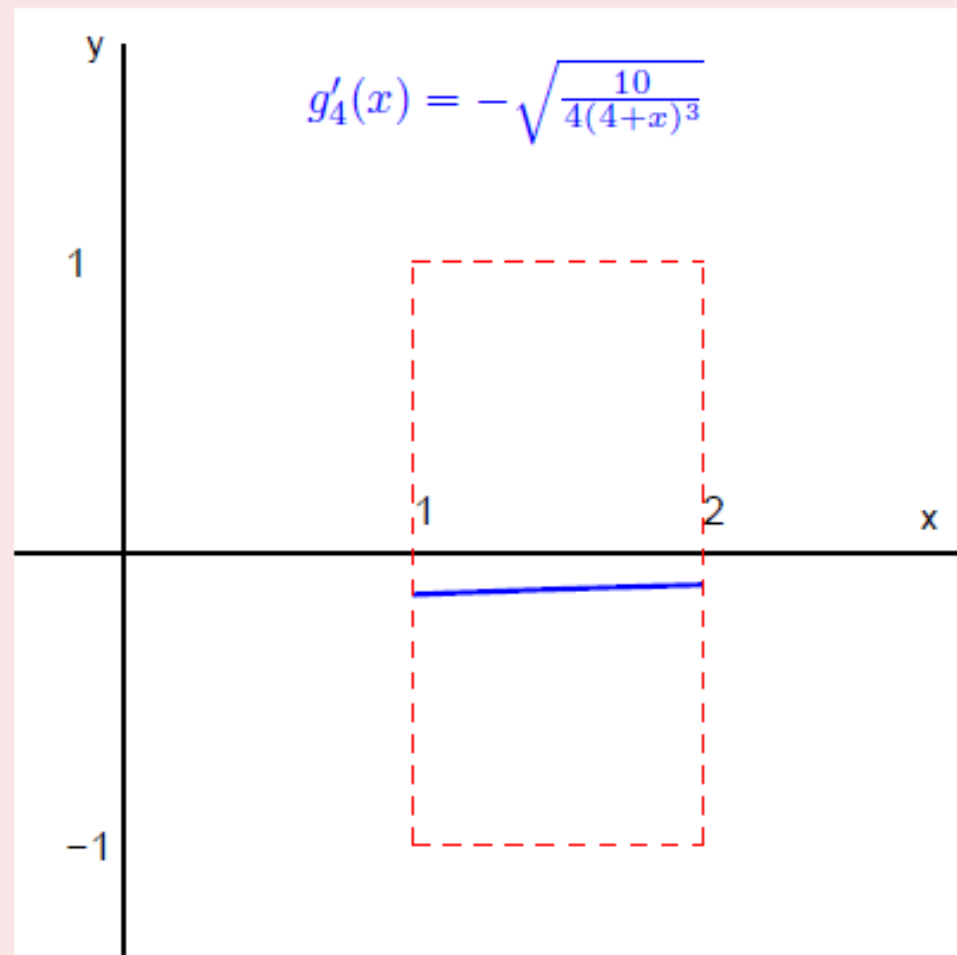
n	p_{n-1}	p_n	$ p_n - p_{n-1} $
1	1.500000000	1.348399725	0.151600275
2	1.348399725	1.367376372	0.018976647
3	1.367376372	1.364957015	0.002419357
4	1.364957015	1.365264748	0.000307733
5	1.365264748	1.365225594	0.000039154
6	1.365225594	1.365230576	0.000004982

11	1.365230014	1.365230013	0.000000000
12	1.365230013	1.365230013	0.000000000

g_4 Maps $[1, 2]$ into $[1, 2]$



$$|g'_4(x)| < 1 \text{ on } [1, 2]$$



Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$$x = g_1(x) = x - x^3 - 4x^2 + 10 \quad \text{Does not Converge}$$

$$x = g_2(x) = \sqrt{\frac{10}{x} - 4x} \quad \text{Does not Converge}$$

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Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

Iteration for $x = g_5(x)$ Converges (Rapidly)

For the iteration function $g_5(x)$, we obtain:

$$g_5(x) = x - \frac{f(x)}{f'(x)} \Rightarrow g_5'(x) = \frac{f(x)f''(x)}{[f'(x)]^2} \Rightarrow g_5'(p) = 0$$

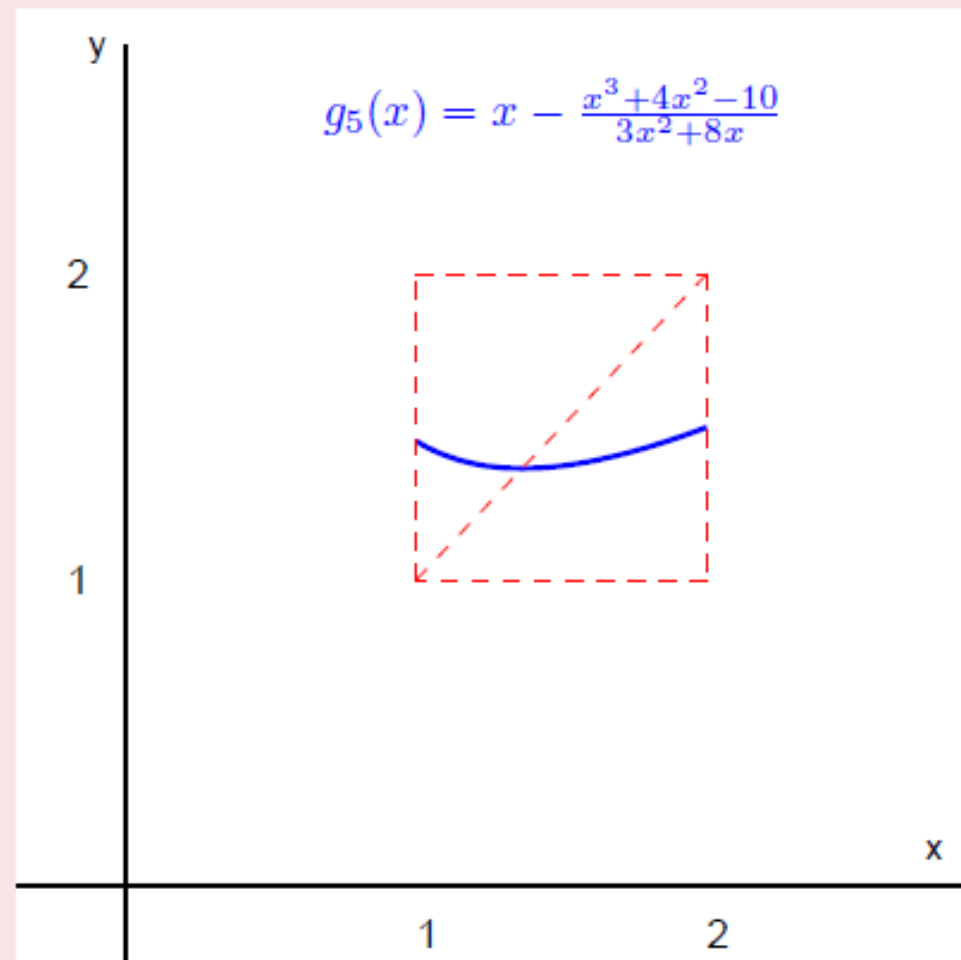
It is straightforward to show that $0 \leq |g_5'(x)| < 0.28 \quad \forall x \in [1, 2]$ and the order of convergence is quadratic since $g_5'(p) = 0$.

Iteration Function: $x = g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$

Iterations starting with $p_0 = 1.5$

n	p_{n-1}	p_n	$ p_n - p_{n-1} $
1	1.500000000	1.373333333	0.126666667
2	1.373333333	1.365262015	0.008071318
3	1.365262015	1.365230014	0.000032001
4	1.365230014	1.365230013	0.000000001
5	1.365230013	1.365230013	0.000000000

g_5 Maps $[1, 2]$ into $[1, 2]$



$|g'_5(x)| < 1$ on $[1, 2]$

