Fixed Point Iteration

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Outline

1. Introduction & Theoretical Framework
2. Motivating the Algorithm: An Example
3. Fixed-Point Formulation I
4. Fixed-Point Formulation II

Functional (Fixed-Point) Iteration

Prime Objective

- In what follows, it is important not to lose sight of our prime objective:
- Given a function $f(x)$ where $a \leq x \leq b$, find values p such that

$$f(p) = 0$$

- Given such a function, $f(x)$, we now construct an auxiliary function $g(x)$ such that

$$p = g(p)$$

  whenever $f(p) = 0$ (this construction is not unique).
- The problem of finding p such that $p = g(p)$ is known as the fixed point problem.

Functional (Fixed-Point) Iteration

A Fixed Point

If g is defined on $[a, b]$ and $g(p) = p$ for some $p \in [a, b]$, then the function g is said to have the fixed point p in $[a, b]$.

Note

- The fixed-point problem turns out to be quite simple both theoretically and geometrically.
- The function $g(x)$ will have a fixed point in the interval $[a, b]$ whenever the graph of $g(x)$ intersects the line $y = x$.

Functional (Fixed-Point) Iteration

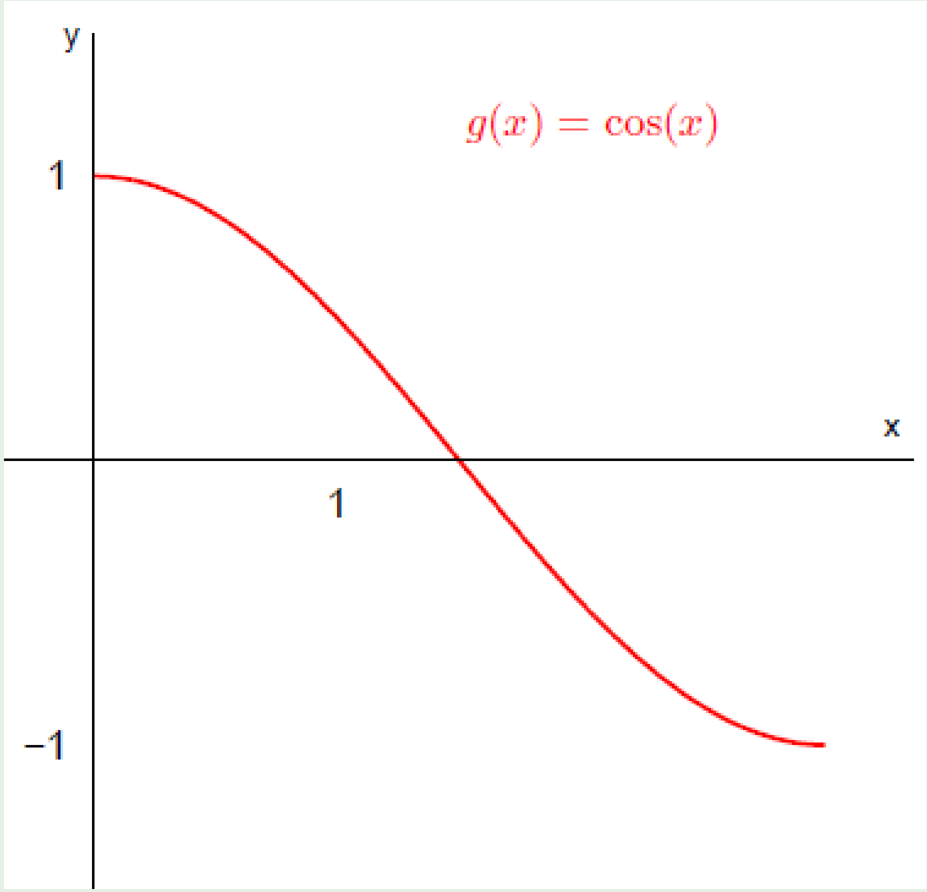
The Equation $f(x) = x - \cos(x) = 0$

If we write this equation in the form:

$$x = \cos(x)$$

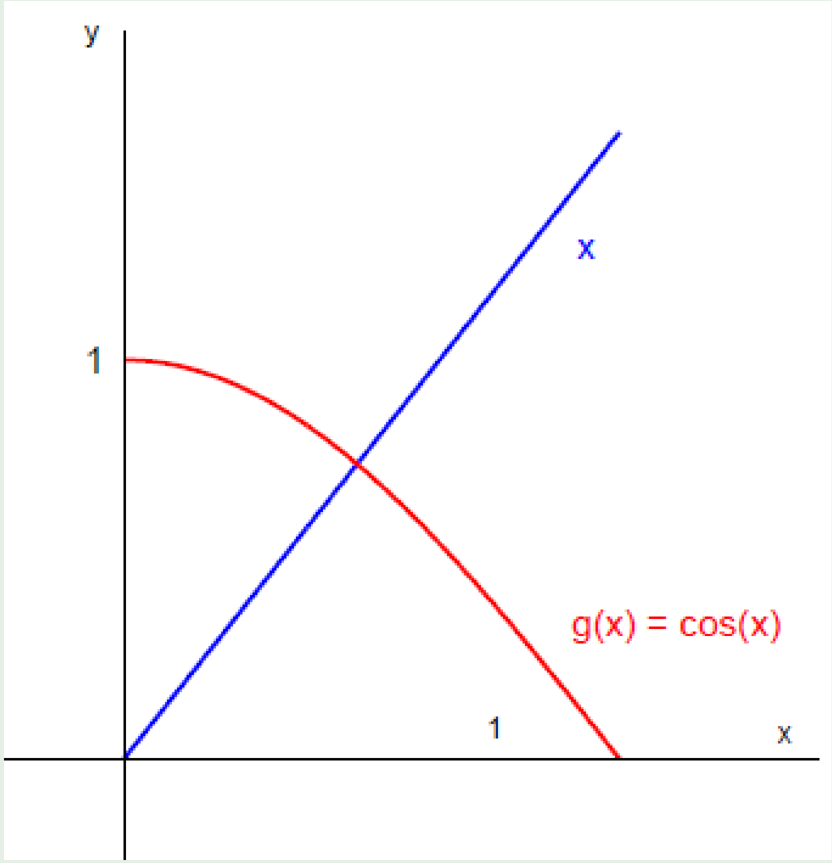
then $g(x) = \cos(x)$.

Single Nonlinear Equation $f(x) = x - \cos(x) = 0$



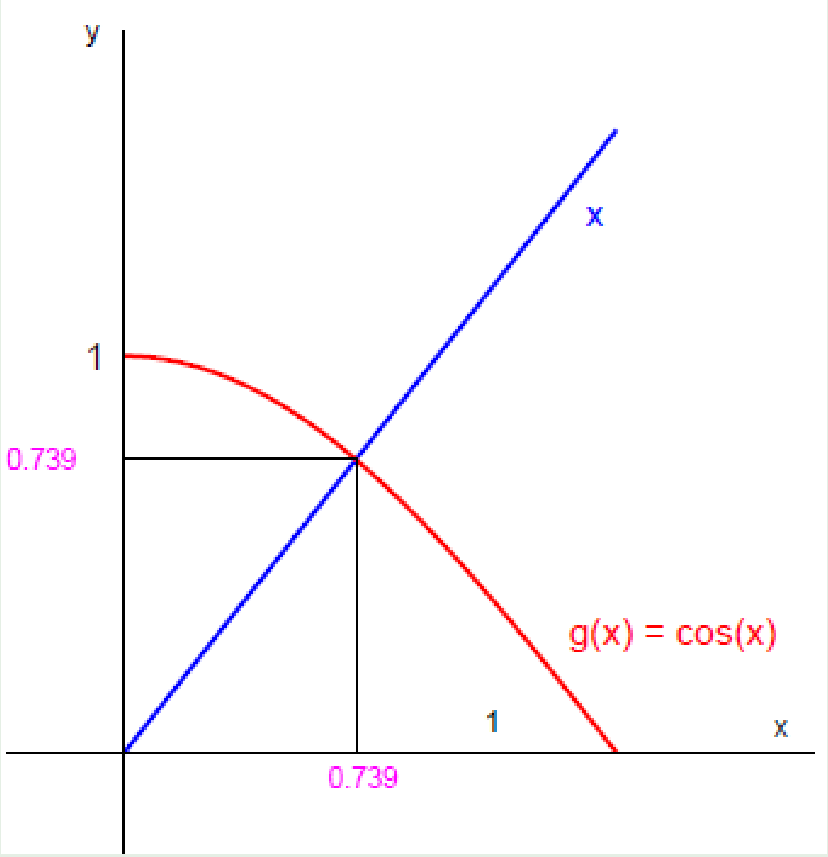
Functional (Fixed-Point) Iteration

$$x = \cos(x)$$



Functional (Fixed-Point) Iteration

$p = \cos(p)$ $\qquad\qquad p \approx 0.739$



Existence of a Fixed Point

If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$ then the function g has a fixed point in $[a, b]$.

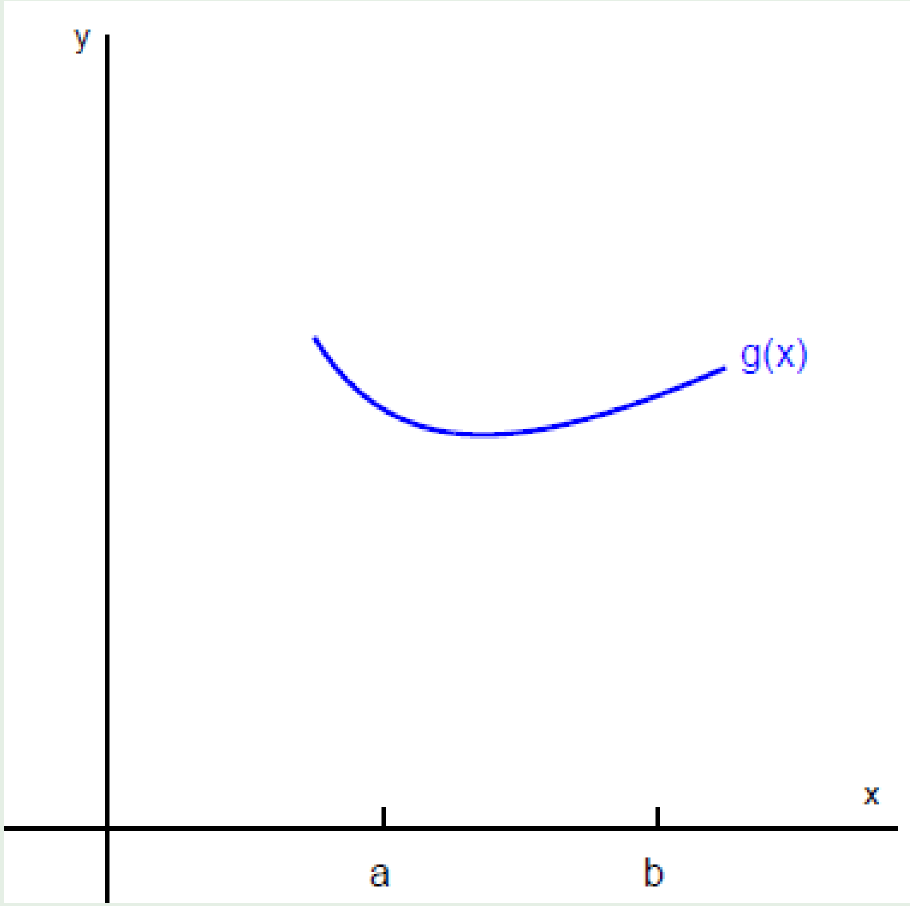
Proof

- If $g(a) = a$ or $g(b) = b$, the existence of a fixed point is obvious.
- Suppose not; then it must be true that $g(a) > a$ and $g(b) < b$.
- Define $h(x) = g(x) - x$; h is continuous on $[a, b]$ and, moreover,

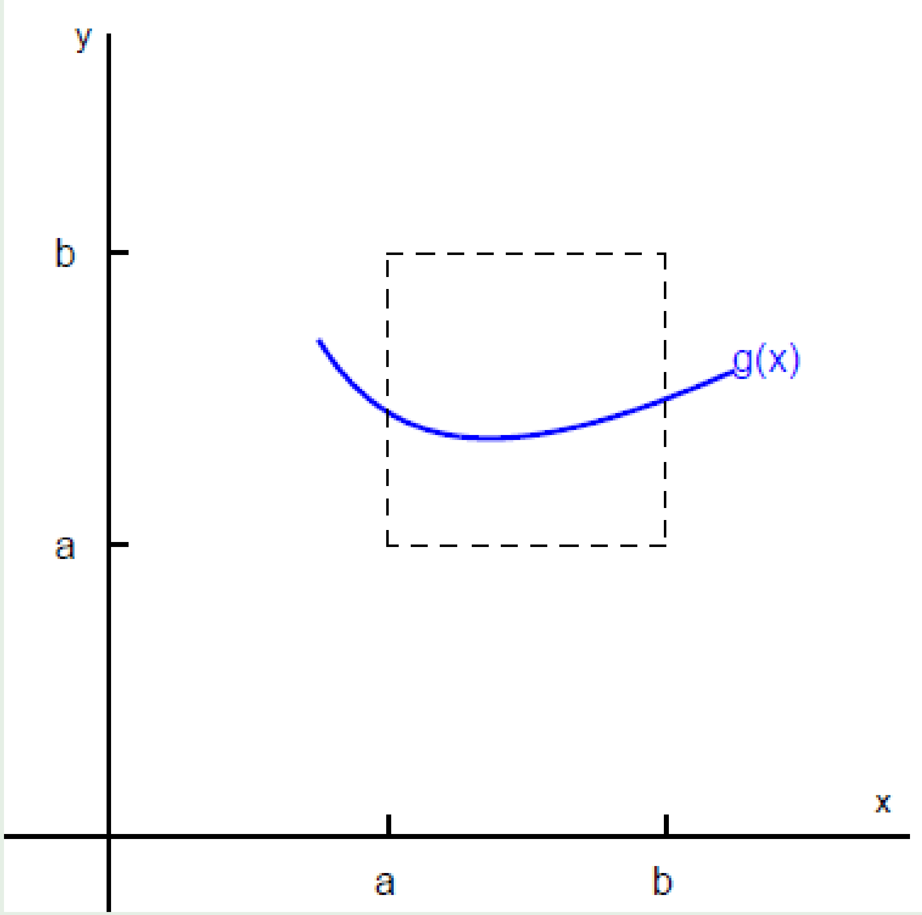
$$h(a) = g(a) - a > 0, \qquad h(b) = g(b) - b < 0.$$

- The Intermediate Value Theorem ▸ IVT implies that there exists $p \in (a, b)$ for which $h(p) = 0$.
- Thus $g(p) - p = 0$ and p is a fixed point of g.

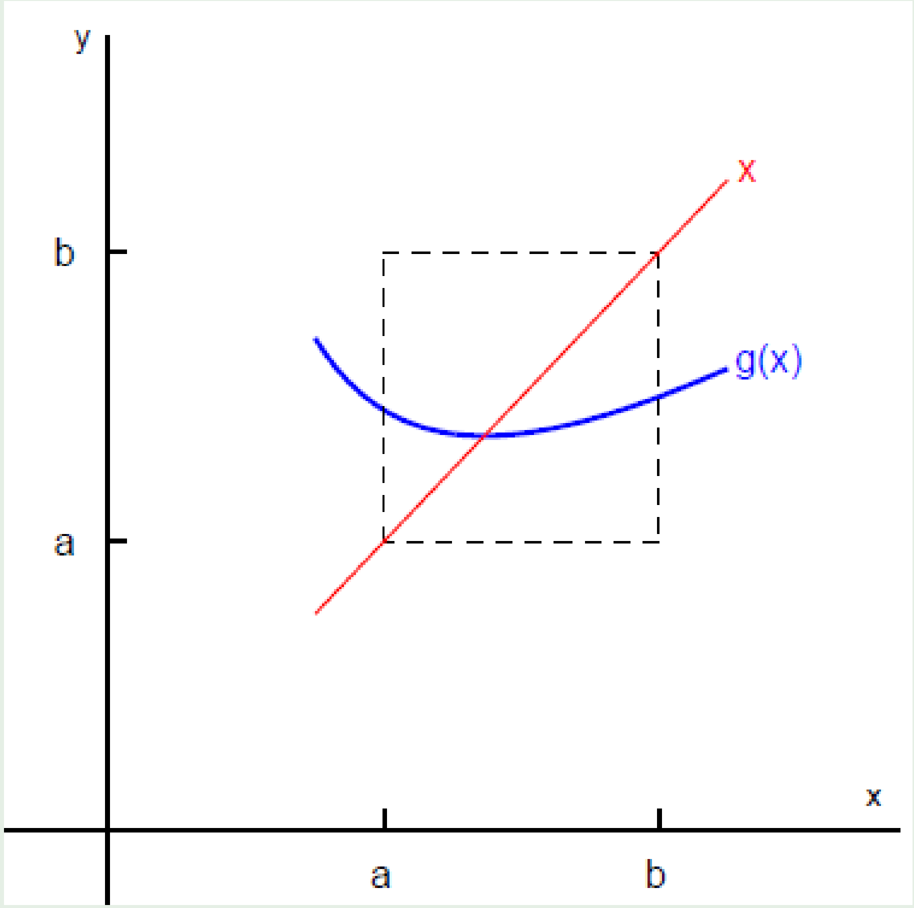
$g(x)$ is Defined on $[a, b]$



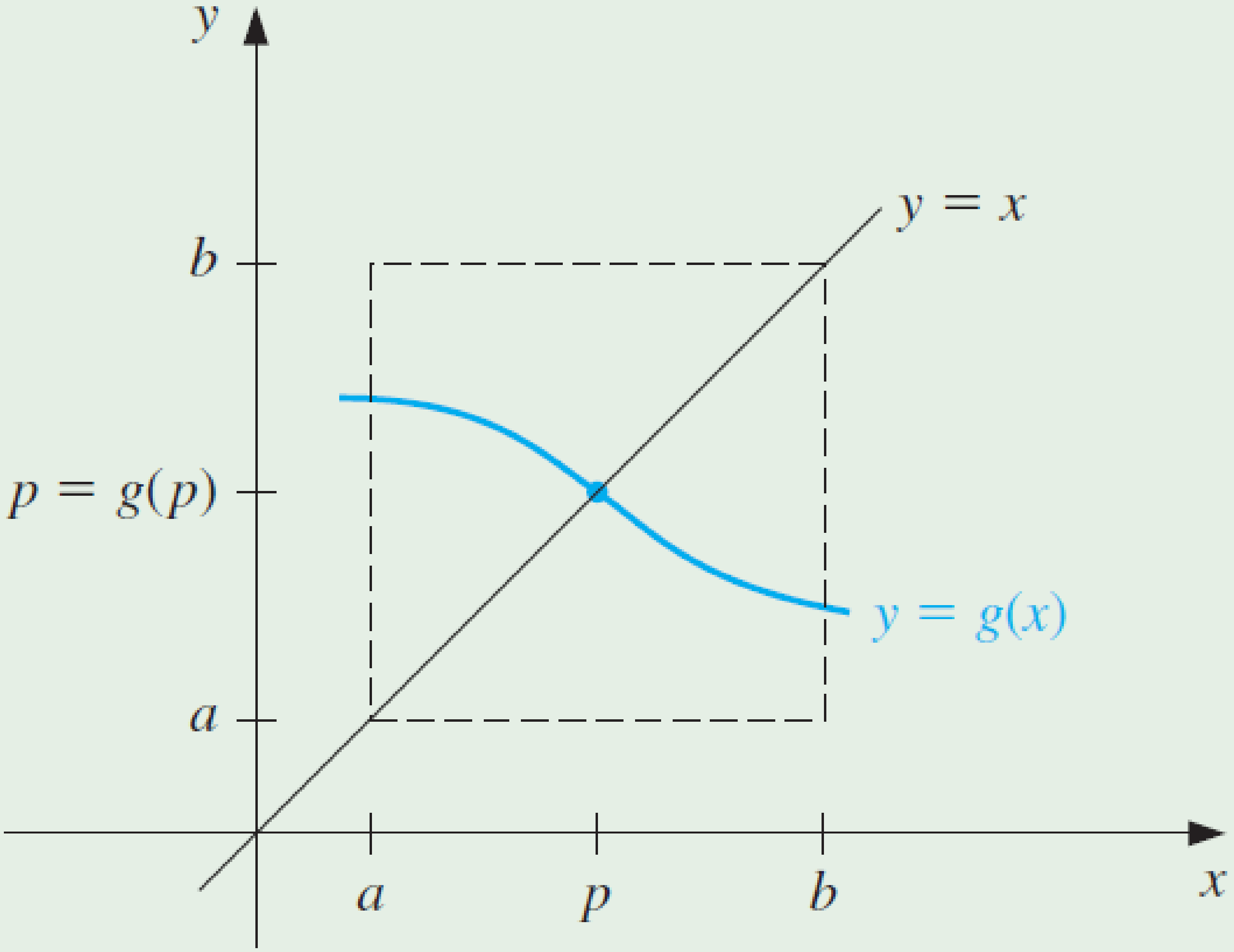
$g(x) \in [a, b]$ for all $x \in [a, b]$



$g(x)$ has a Fixed Point in $[a, b]$



$g(x)$ has a Fixed Point in $[a, b]$



Illustration

- Consider the function $g(x) = 3^{-x}$ on $0 \leq x \leq 1$. $g(x)$ is continuous and since

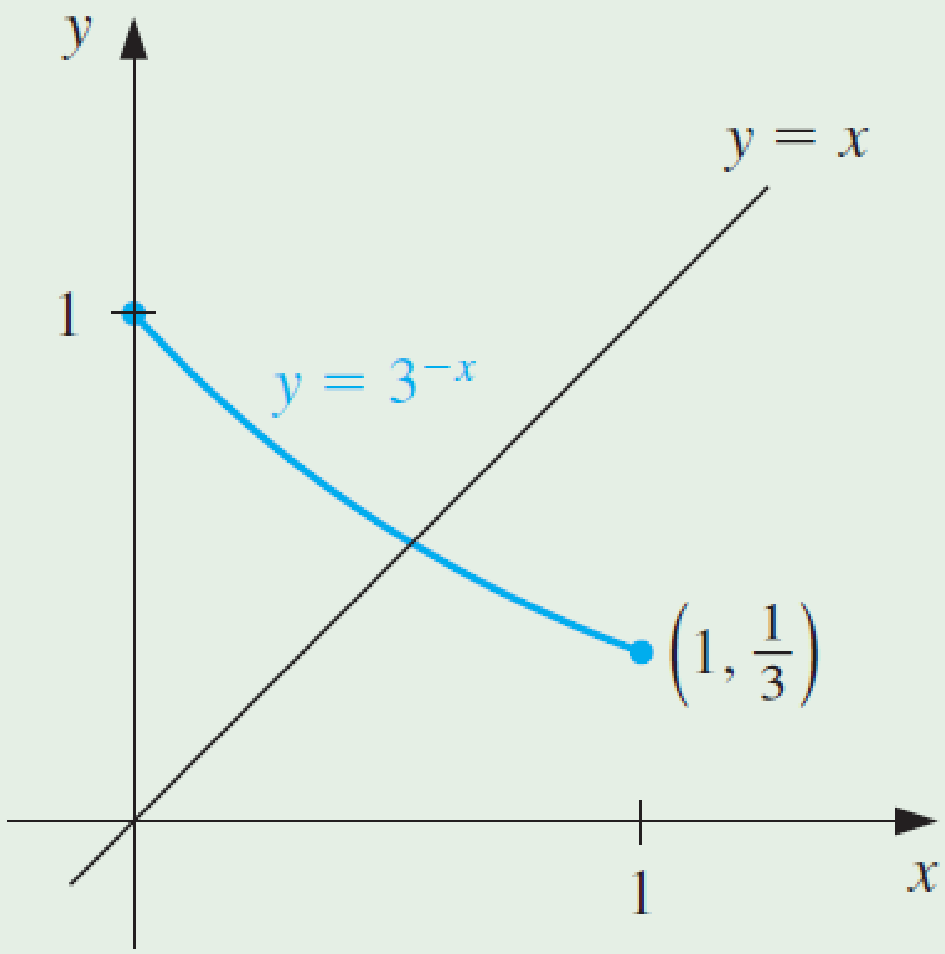
$$g'(x) = -3^{-x} \log 3 < 0 \qquad \text{on } [0, 1]$$

  $g(x)$ is decreasing on $[0, 1]$.
- Hence

$$g(1) = \frac{1}{3} \leq g(x) \leq 1 = g(0)$$

  i.e. $g(x) \in [0, 1]$ for all $x \in [0, 1]$ and therefore, by the preceding result, $g(x)$ must have a fixed point in $[0, 1]$.

$g(x) = 3^{-x}$



Functional (Fixed-Point) Iteration

An Important Observation

- It is fairly obvious that, on any given interval $I = [a, b]$, $g(x)$ may have many fixed points (or none at all).
- In order to ensure that $g(x)$ has a unique fixed point in I, we must make an additional assumption that $g(x)$ does not vary too rapidly.
- Thus we have to establish a uniqueness result.

Functional (Fixed-Point) Iteration

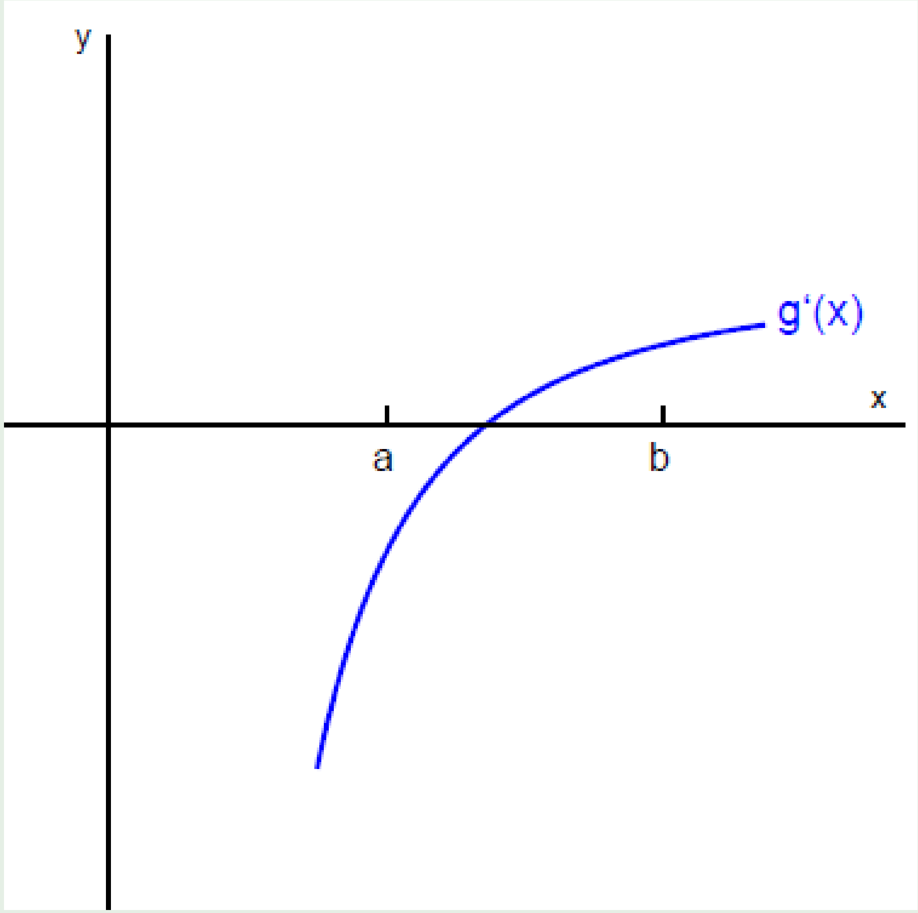
Uniqueness Result

Let $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$. Further if $g'(x)$ exists on (a, b) and

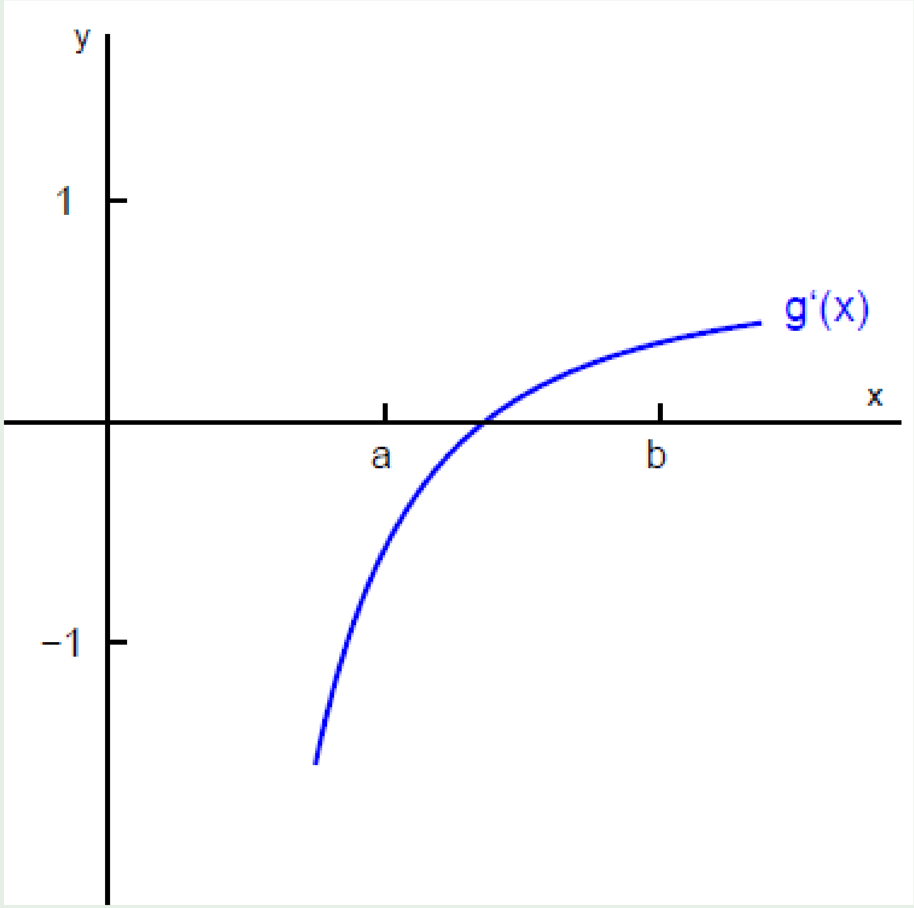
$$|g'(x)| \leq k < 1, \quad \forall\, x \in [a, b],$$

then the function g has a unique fixed point p in $[a, b]$.

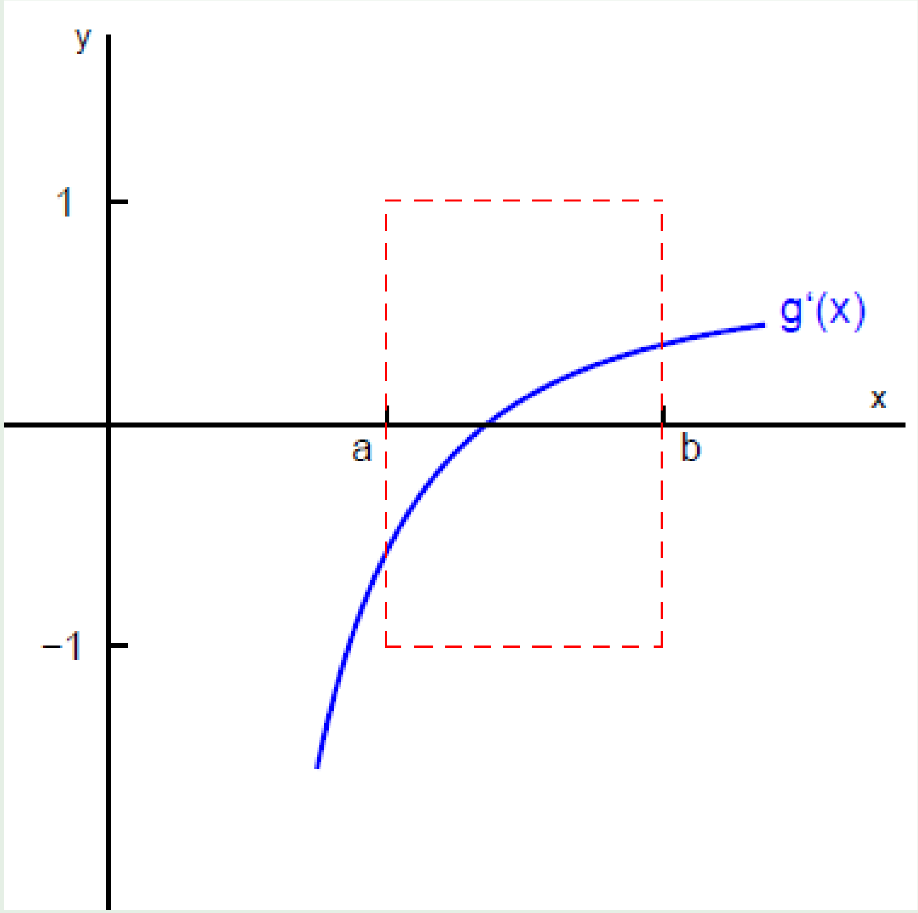
$g'(x)$ is Defined on $[a, b]$



$-1 \leq g'(x) \leq 1$ for all $x \in [a, b]$



Unique Fixed Point: $|g'(x)| \leq 1$ for all $x \in [a, b]$



A Single Nonlinear Equaton

Model Problem

Consider the quadratic equation:

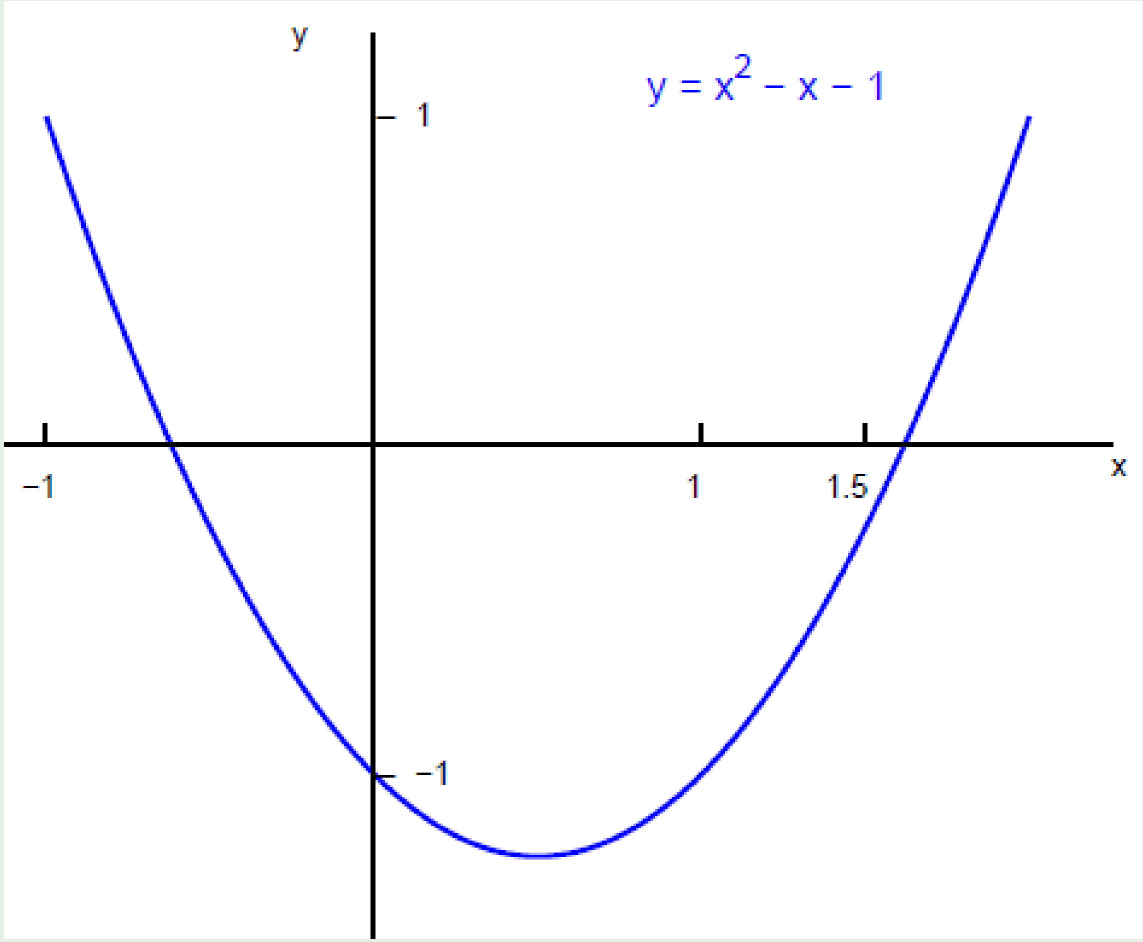
$$x^2 - x - 1 = 0$$

Positive Root

The positive root of this equations is:

$$x = \frac{1 + \sqrt{5}}{2} \approx 1.618034$$

Single Nonlinear Equation $f(x) = x^2 - x - 1 = 0$



We can convert this equation into a fixed –point problem

Single Nonlinear Equation $f(x) = x^2 - x - 1 = 0$

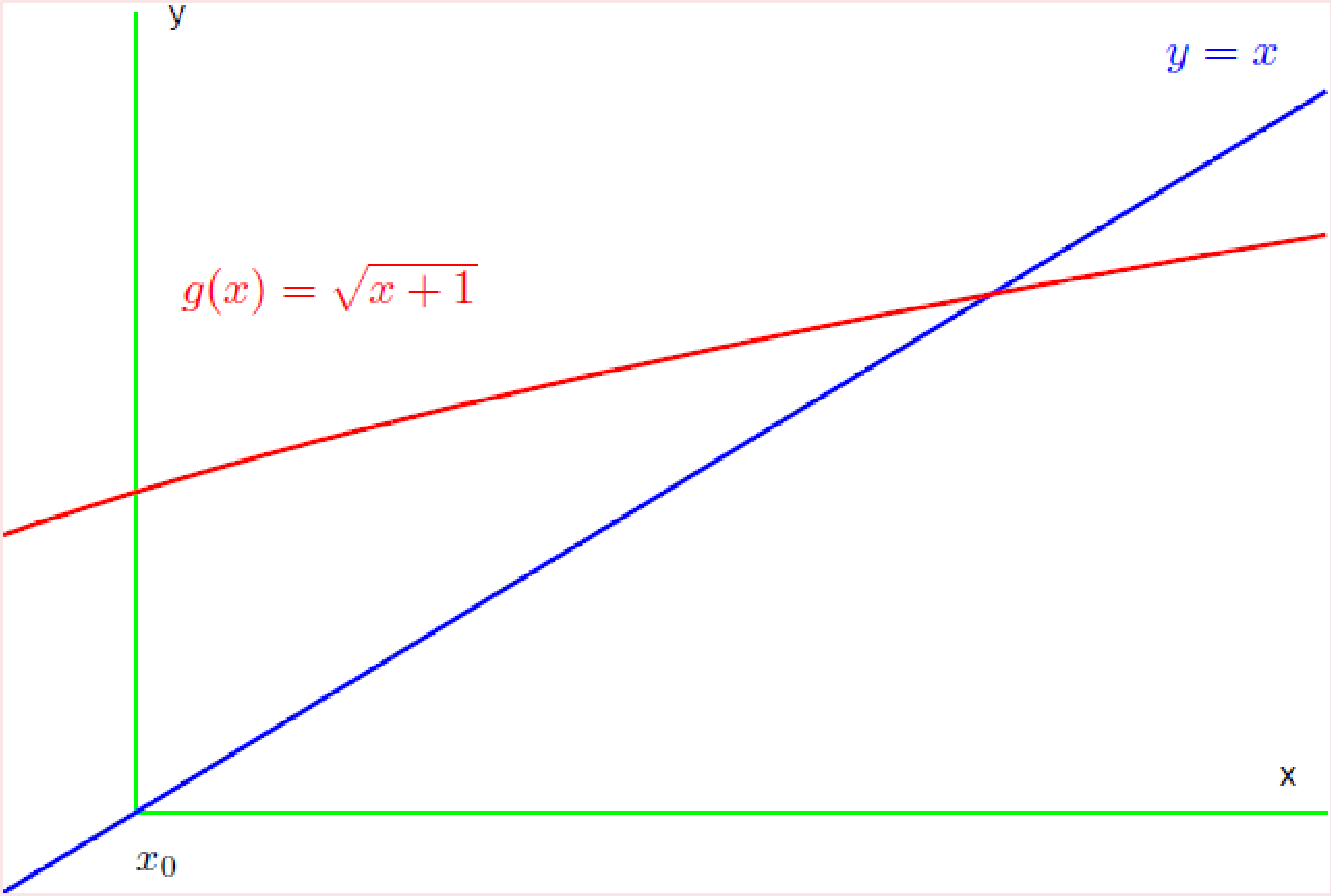
One Possible Formulation for $g(x)$

Transpose the equation $f(x) = 0$ for variable x:

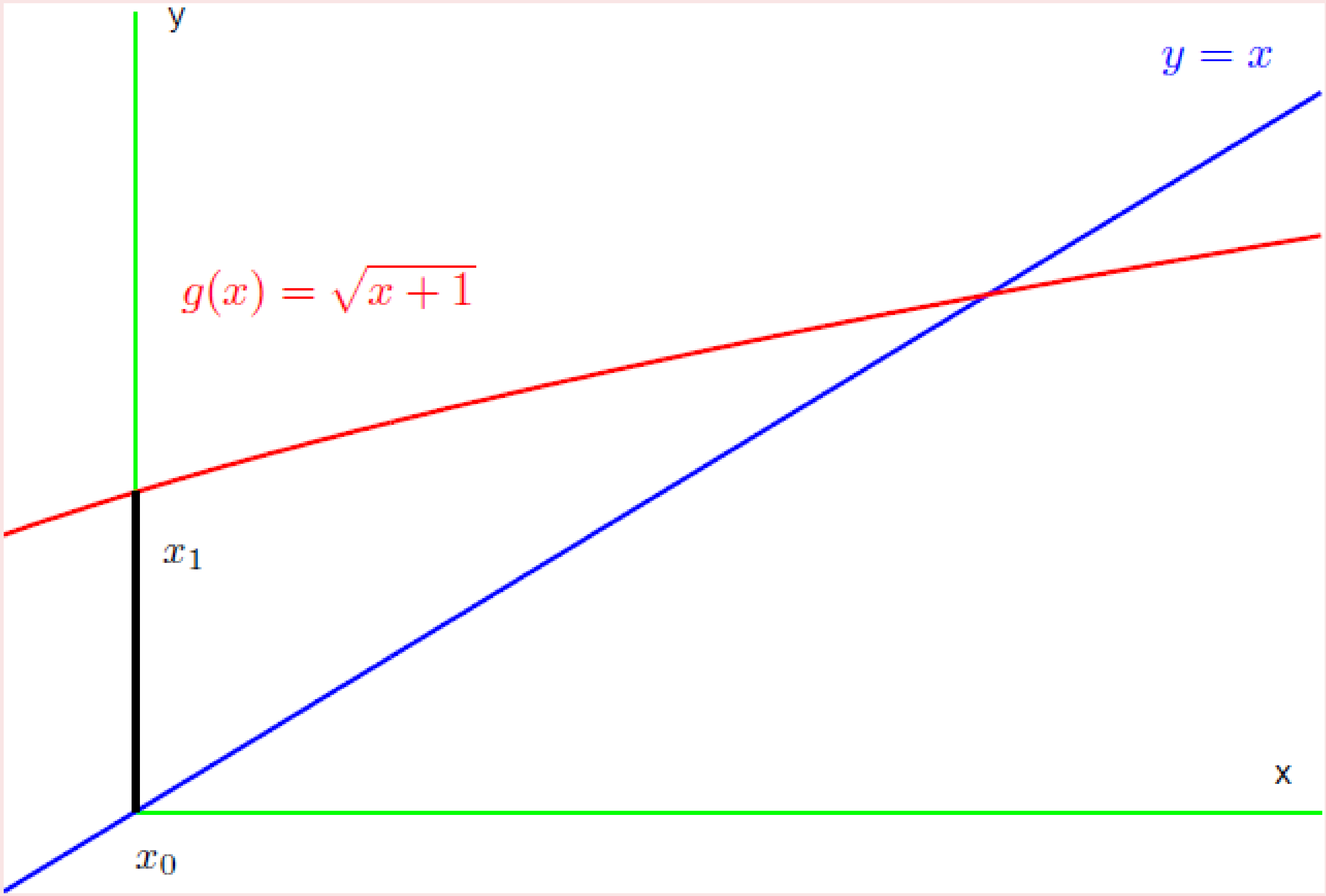
$$\begin{aligned} x^2 - x - 1 &= 0 \\ \Rightarrow \qquad x^2 &= x + 1 \\ \Rightarrow \qquad x &= \pm\sqrt{x+1} \end{aligned}$$

$$g(x) = \sqrt{x+1}$$

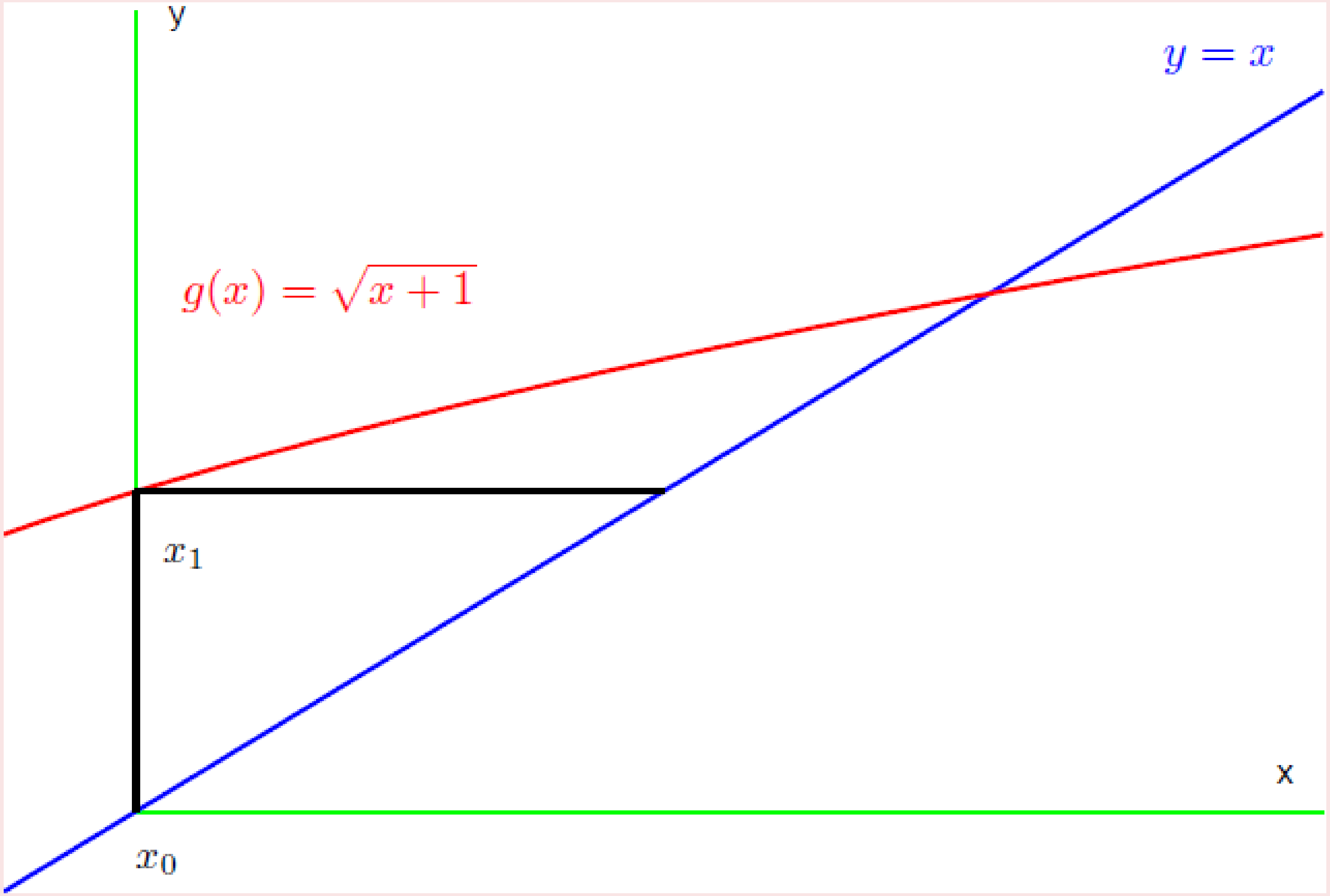
$x_{n+1} = g(x_n) = \sqrt{x_n + 1}$ with $x_0 = 0$



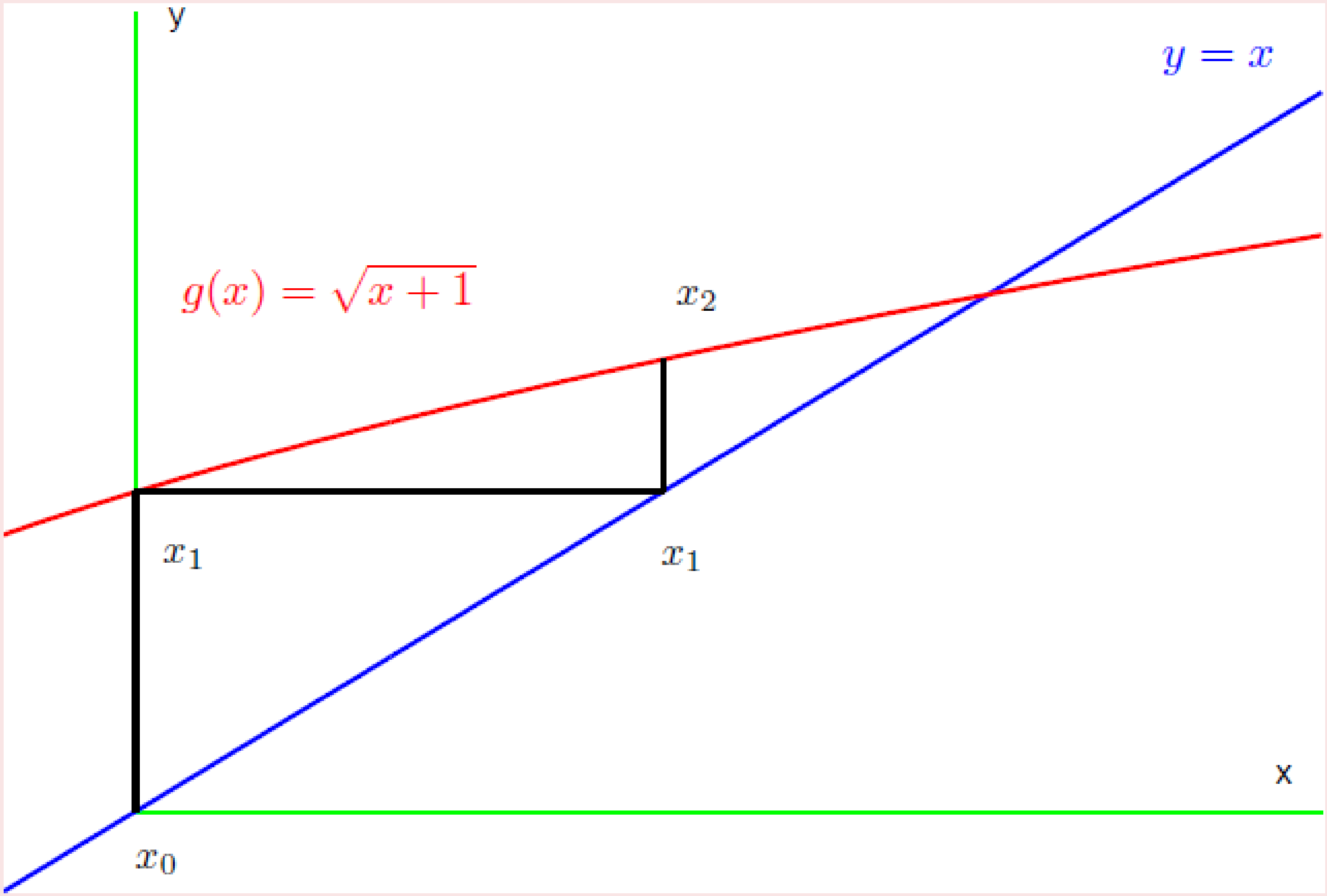
Fixed Point: $g(x) = \sqrt{x+1}$ $x_0 = 0$			
n	p_n	p_{n+1}	$\|p_{n+1} - p_n\|$
1	0.000000000	1.000000000	1.000000000
2	1.000000000	1.414213562	0.414213562
3	1.414213562	1.553773974	0.139560412
4	1.553773974	1.598053182	0.044279208
5	1.598053182	1.611847754	0.013794572



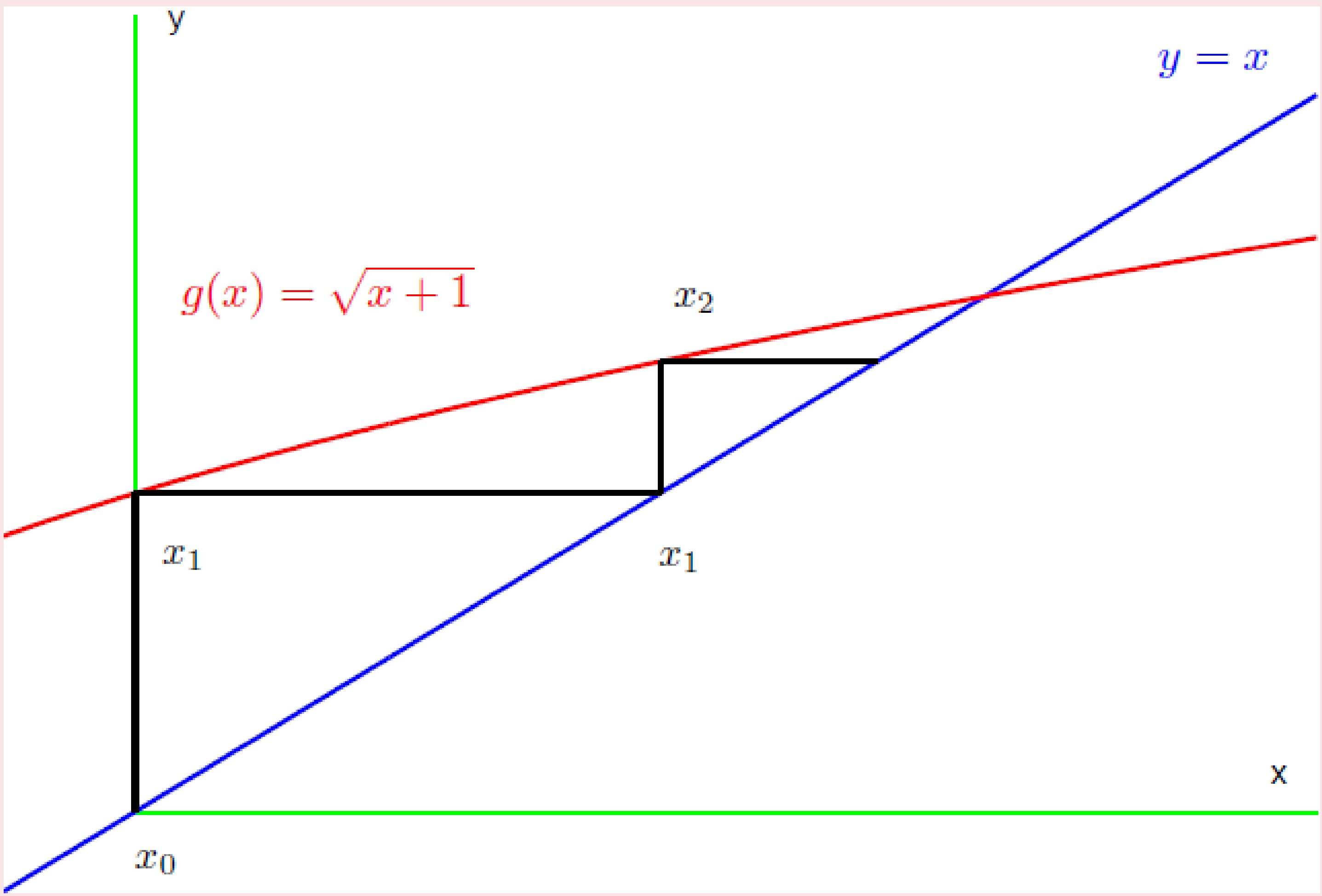
y
$y = x$
$g(x) = \sqrt{x+1}$
x_1
x
x_0



y
y = x
g(x) = √x + 1
x1
x
x0



y
$y = x$
$g(x) = \sqrt{x+1}$
x_2
x_1
x_1
x
x_0


y
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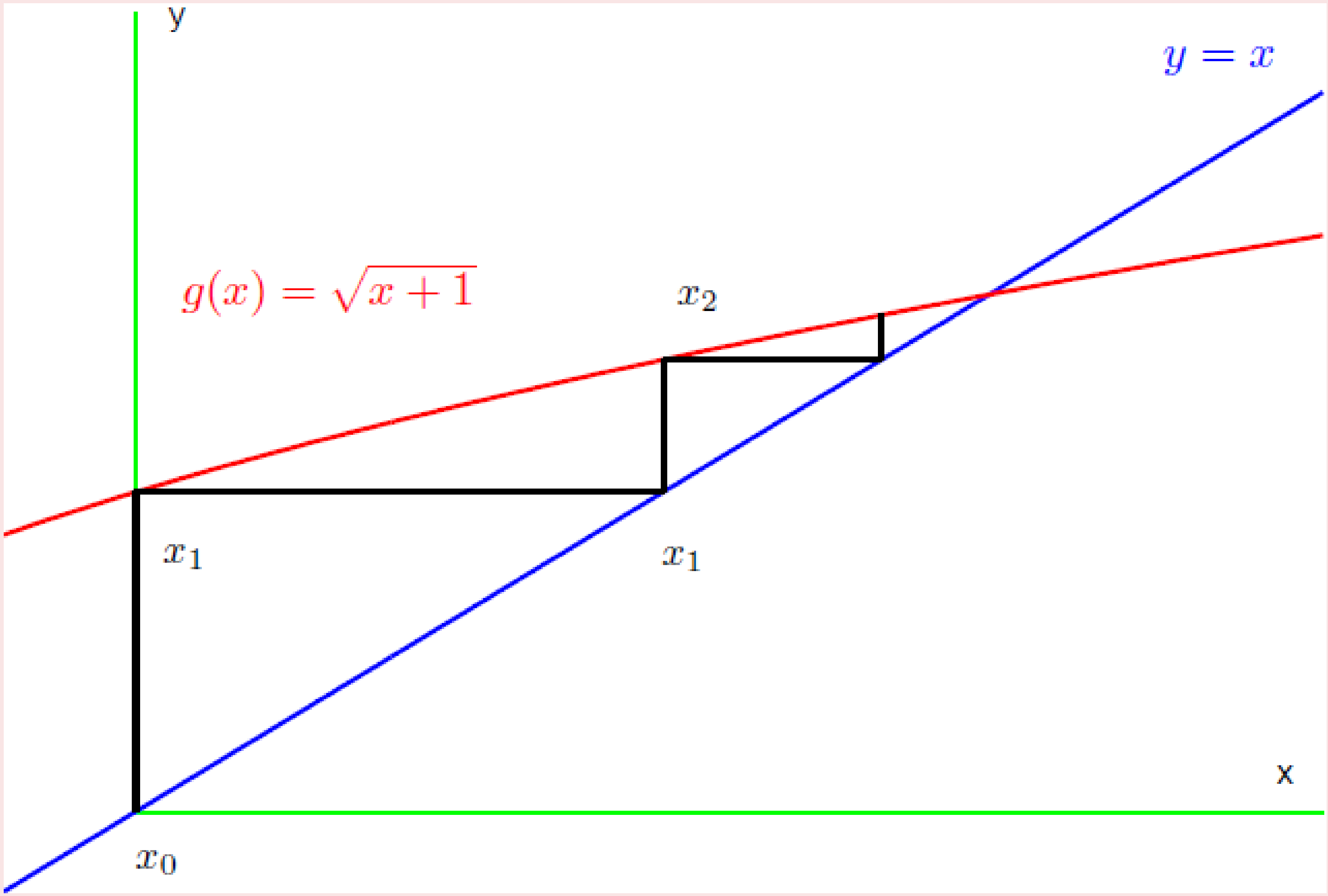
x_2
x_1
x_1
x
x_0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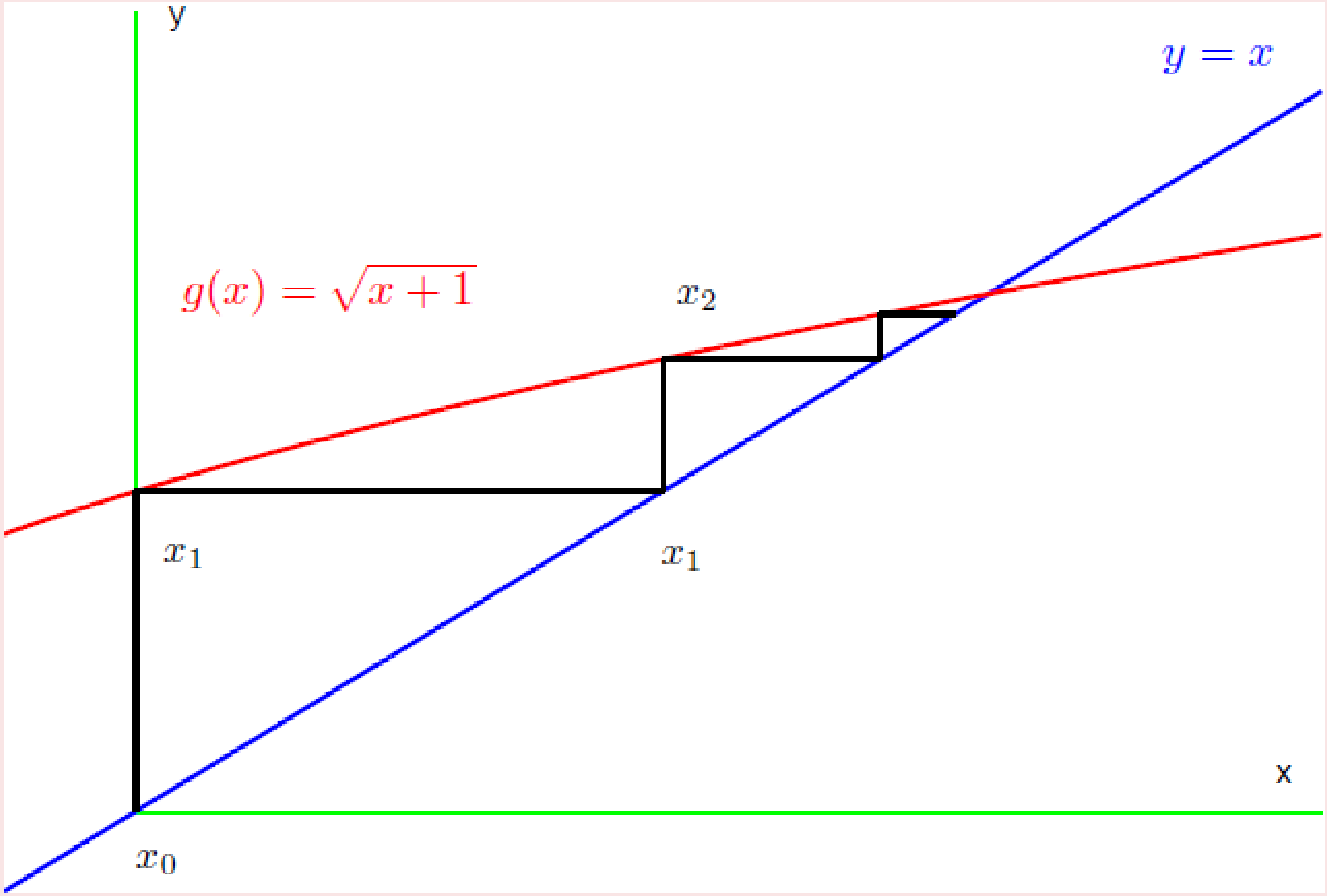
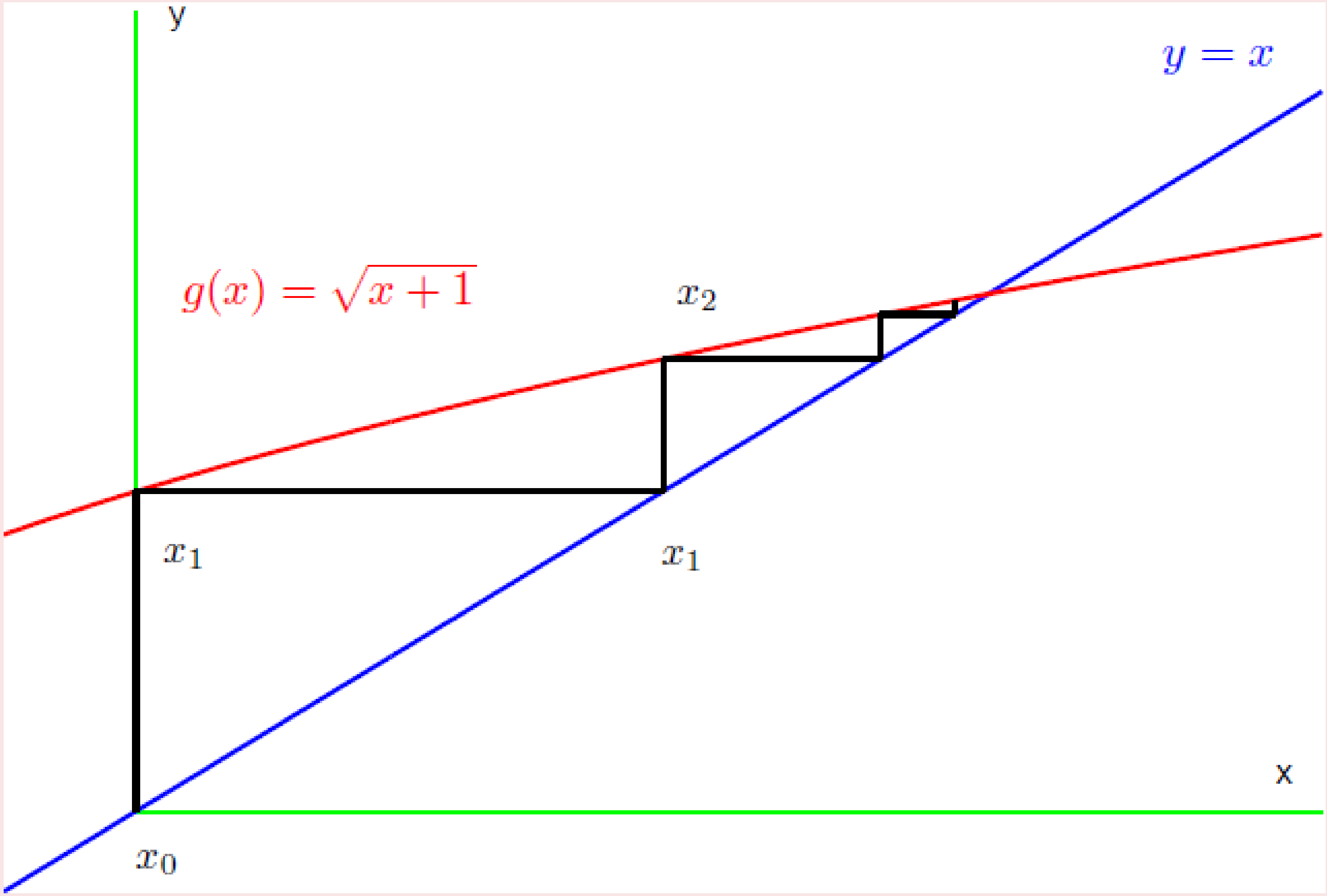
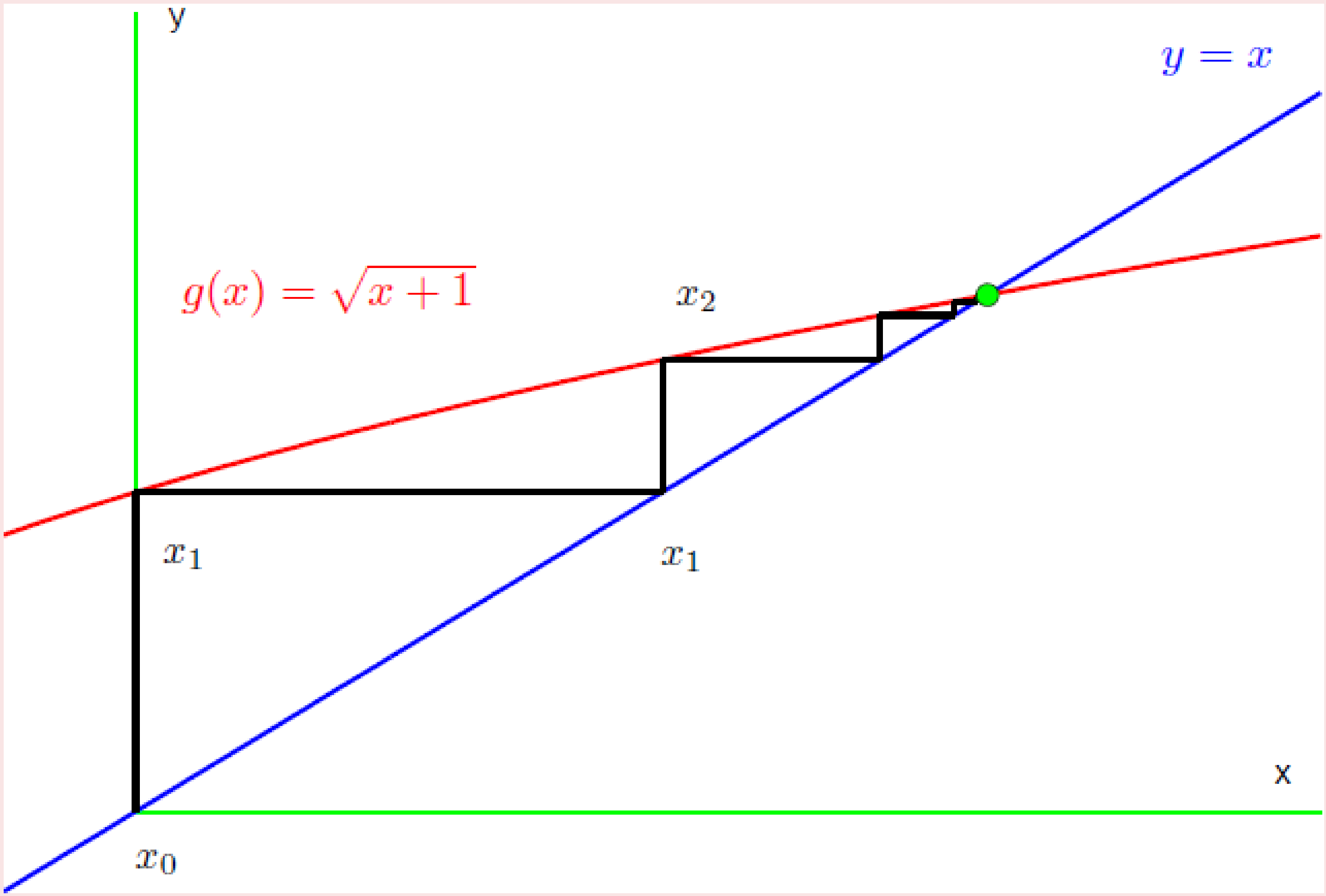
$g(x) = \sqrt{x+1}$
x_2
x_1
x_1
x
x_0


y
$y = x$
$g(x) = \sqrt{x+1}$
x_2
x_1
x_1
x
x_0


y
$y = x$
$g(x) = \sqrt{x+1}$
x_2
x_1
x_1
x
x_0



y
$y = x$
$g(x) = \sqrt{x+1}$
x_2
x_1
x_1
x
x_0

$x_{n+1} = g(x_n) = \sqrt{x_n + 1}$ with $x_0 = 0$

Rate of Convergence

We require that $|g'(x)| \leq k < 1$. Since

$$g(x) = \sqrt{x+1} \quad \text{and} \quad g'(x) = \frac{1}{2\sqrt{x+1}} > 0 \quad \text{for} \quad x \geq 0$$

we find that

$$g'(x) = \frac{1}{2\sqrt{x+1}} < 1 \quad \text{for all} \quad x > -\frac{3}{4}$$

Note

$$g'(p) \approx 0.30902$$

Fixed Point: $g(x) = \sqrt{x+1}$ $p_0 = 0$

n	p_{n-1}	p_n	$\|p_n - p_{n-1}\|$	e_n/e_{n-1}
1	0.0000000	1.0000000	1.0000000	—
2	1.0000000	1.4142136	0.4142136	0.41421
3	1.4142136	1.5537740	0.1395604	0.33693
4	1.5537740	1.5980532	0.0442792	0.31728
5	1.5980532	1.6118478	0.0137946	0.31154
⋮	⋮	⋮	⋮	⋮
12	1.6180286	1.6180323	0.0000037	0.30902
13	1.6180323	1.6180335	0.0000012	0.30902
14	1.6180335	1.6180338	0.0000004	0.30902
15	1.6180338	1.6180339	0.0000001	0.30902

Single Nonlinear Equation $f(x) = x^2 - x - 1 = 0$

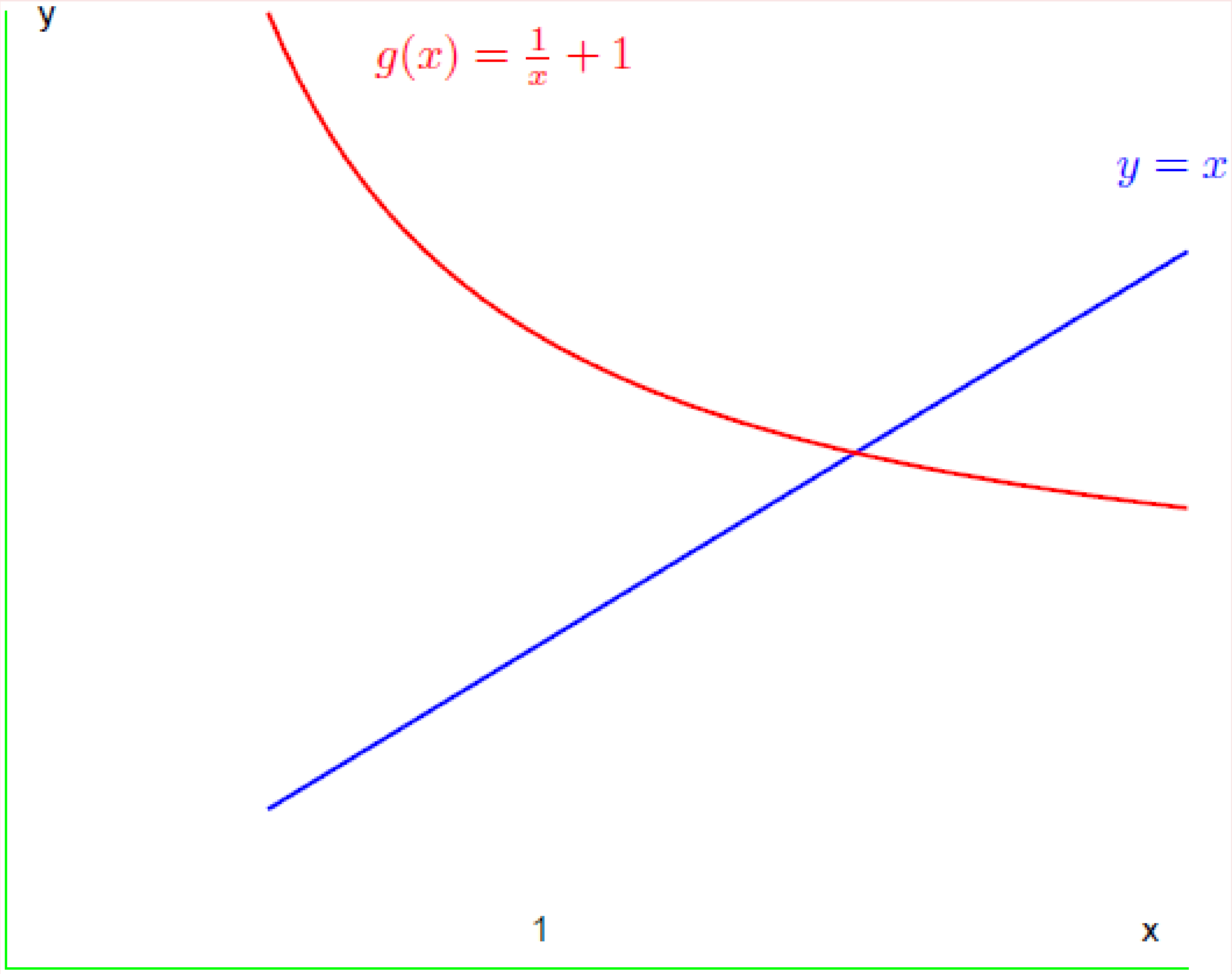
A Second Formulation for $g(x)$

Transpose the equation $f(x) = 0$ for variable x:

$$\begin{aligned} x^2 - x - 1 &= 0 \\ \Rightarrow \qquad x^2 &= x + 1 \\ \Rightarrow \qquad x &= 1 + \frac{1}{x} \end{aligned}$$

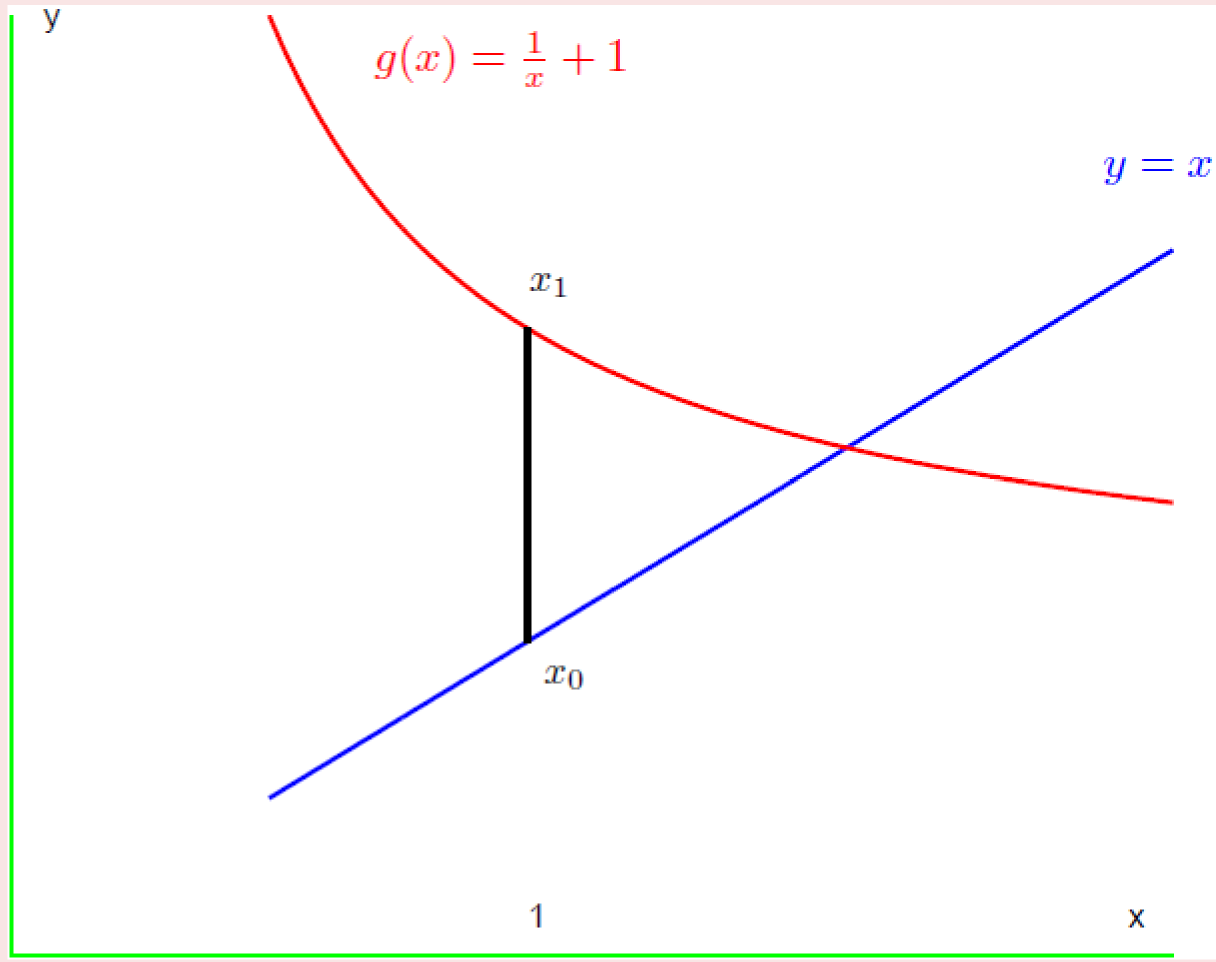
$$g(x) = 1 + \frac{1}{x}$$

$x_{n+1} = g(x_n) = \frac{1}{x_n} + 1$ with $x_0 = 1$

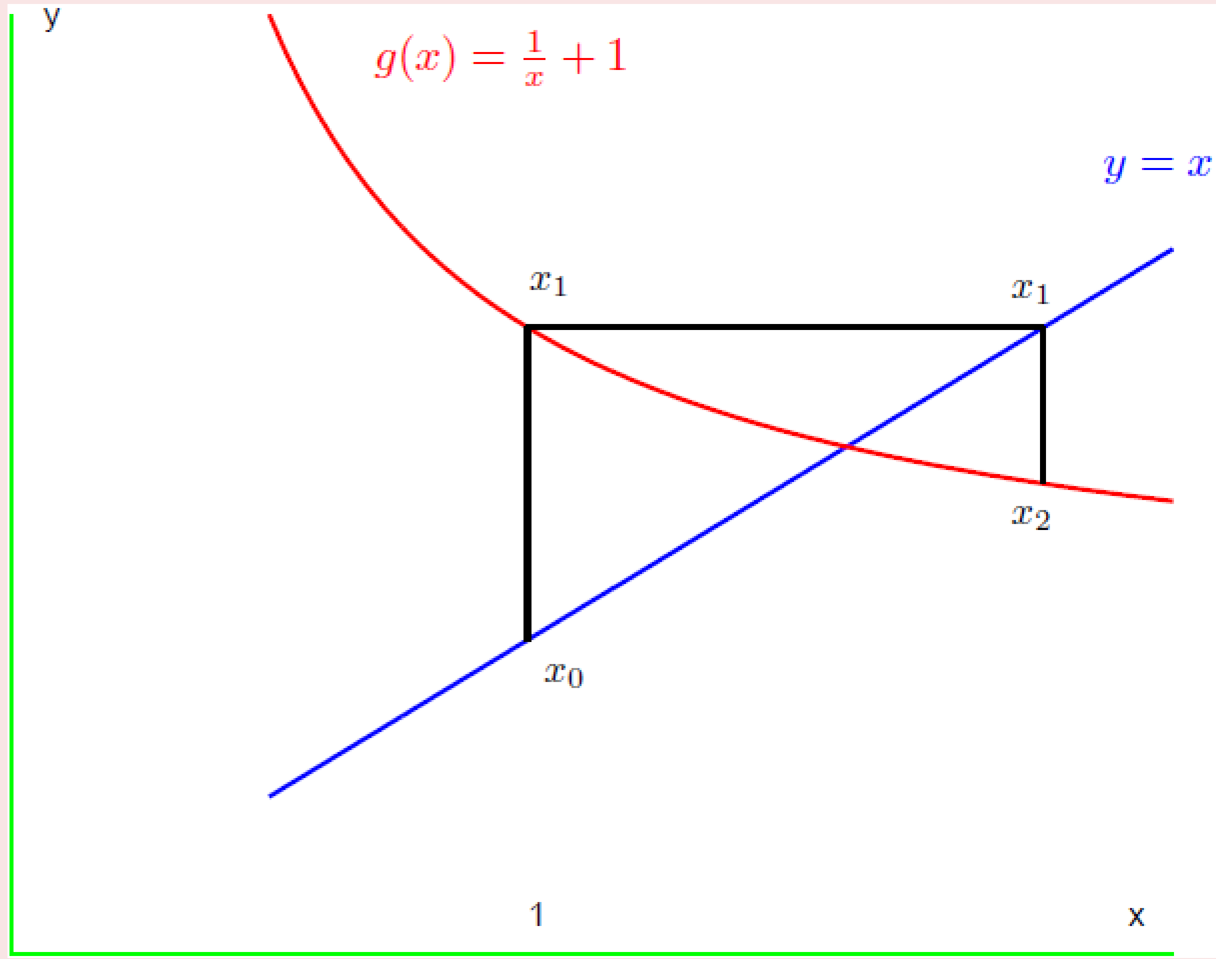


Fixed Point: $g(x) = \frac{1}{x} + 1$ $x_0 = 1$

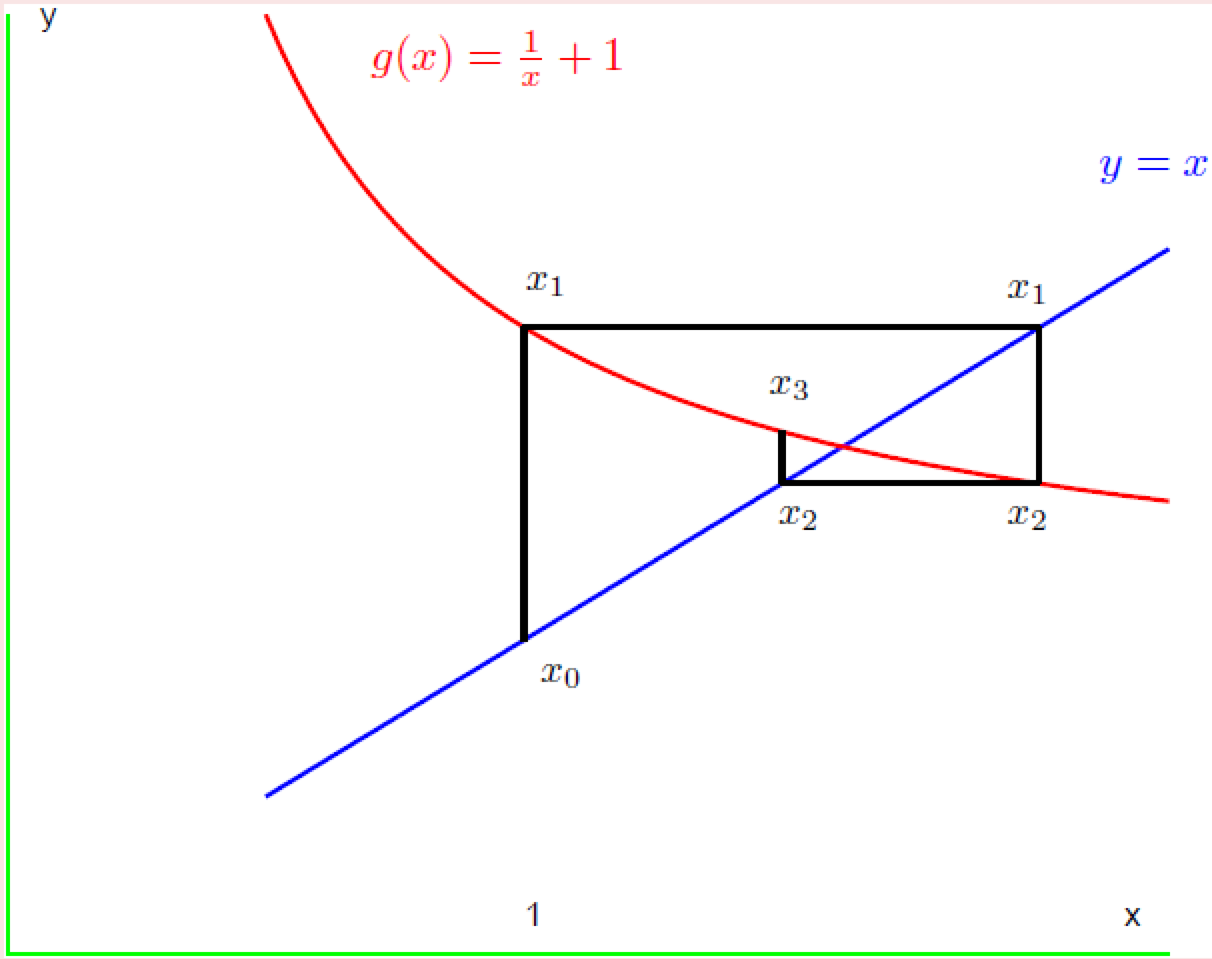
n	p_n	p_{n+1}	$\|p_{n+1} - p_n\|$
1	1.000000000	2.000000000	1.000000000
2	2.000000000	1.500000000	0.500000000
3	1.500000000	1.666666667	0.166666667
4	1.666666667	1.600000000	0.066666667
5	1.600000000	1.625000000	0.025000000



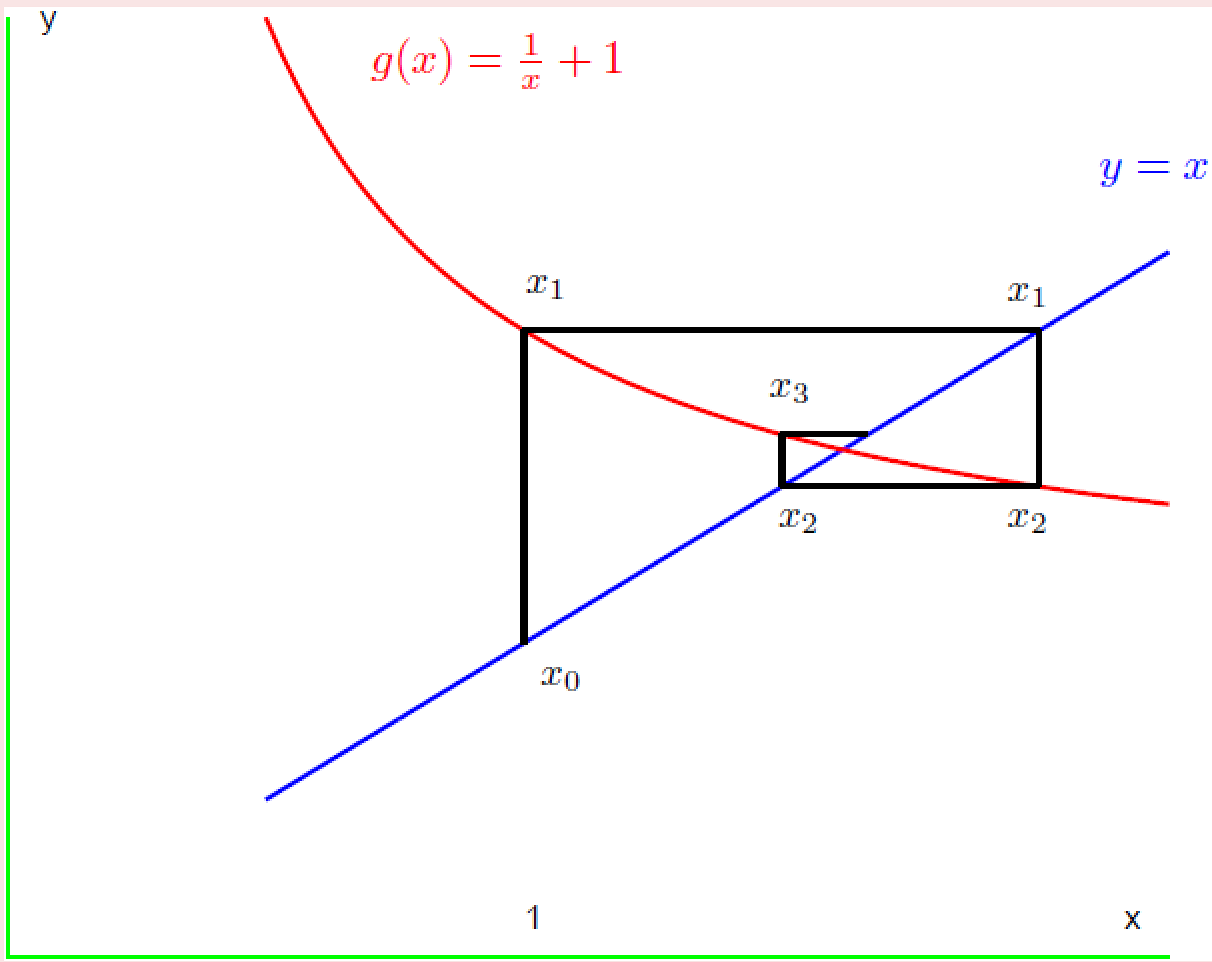
y
$g(x) = \frac{1}{x} + 1$
$y = x$
x_1
x_0
1
x



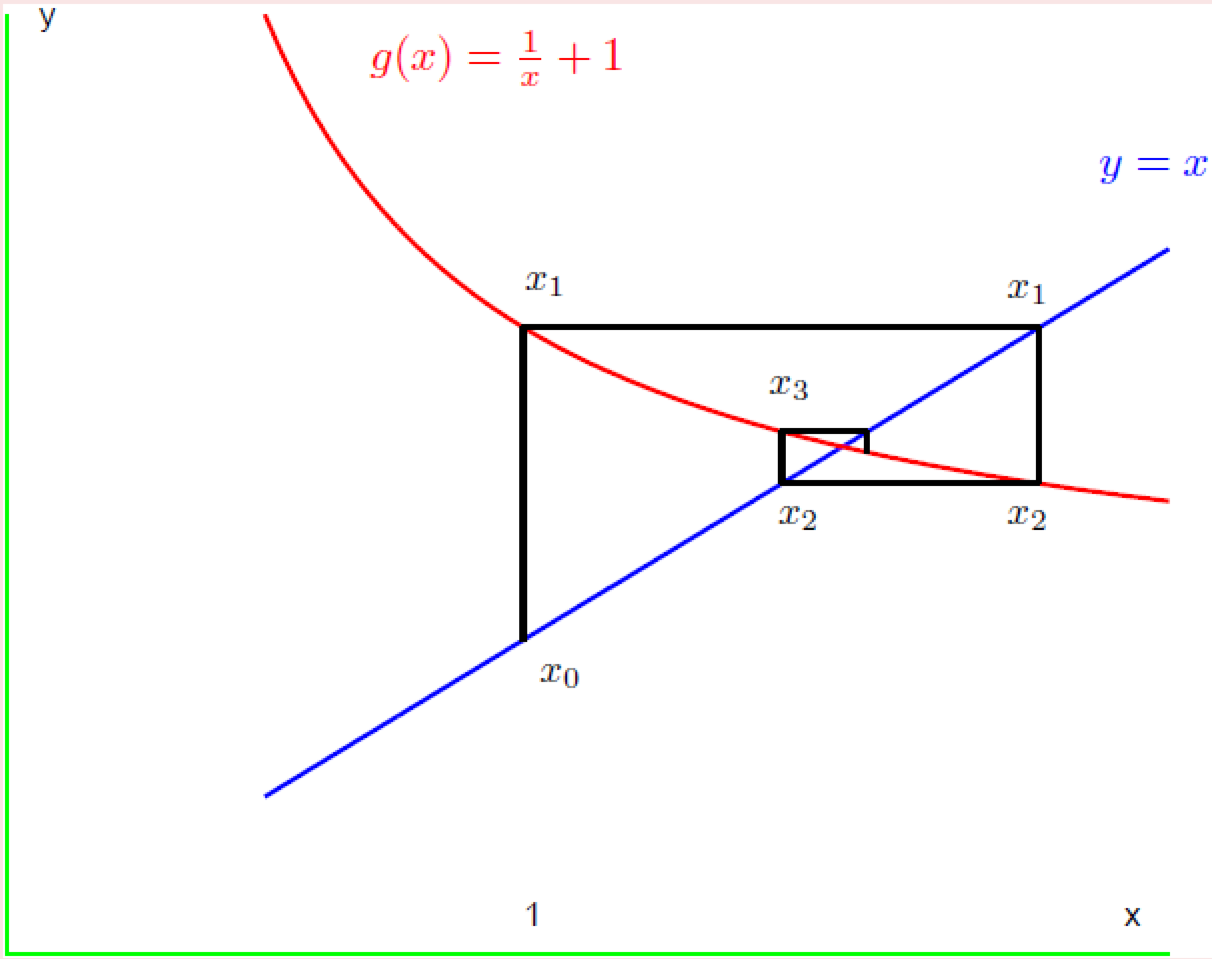
y
$g(x) = \frac{1}{x} + 1$
$y = x$
x_1
x_1
x_2
x_0
1
x

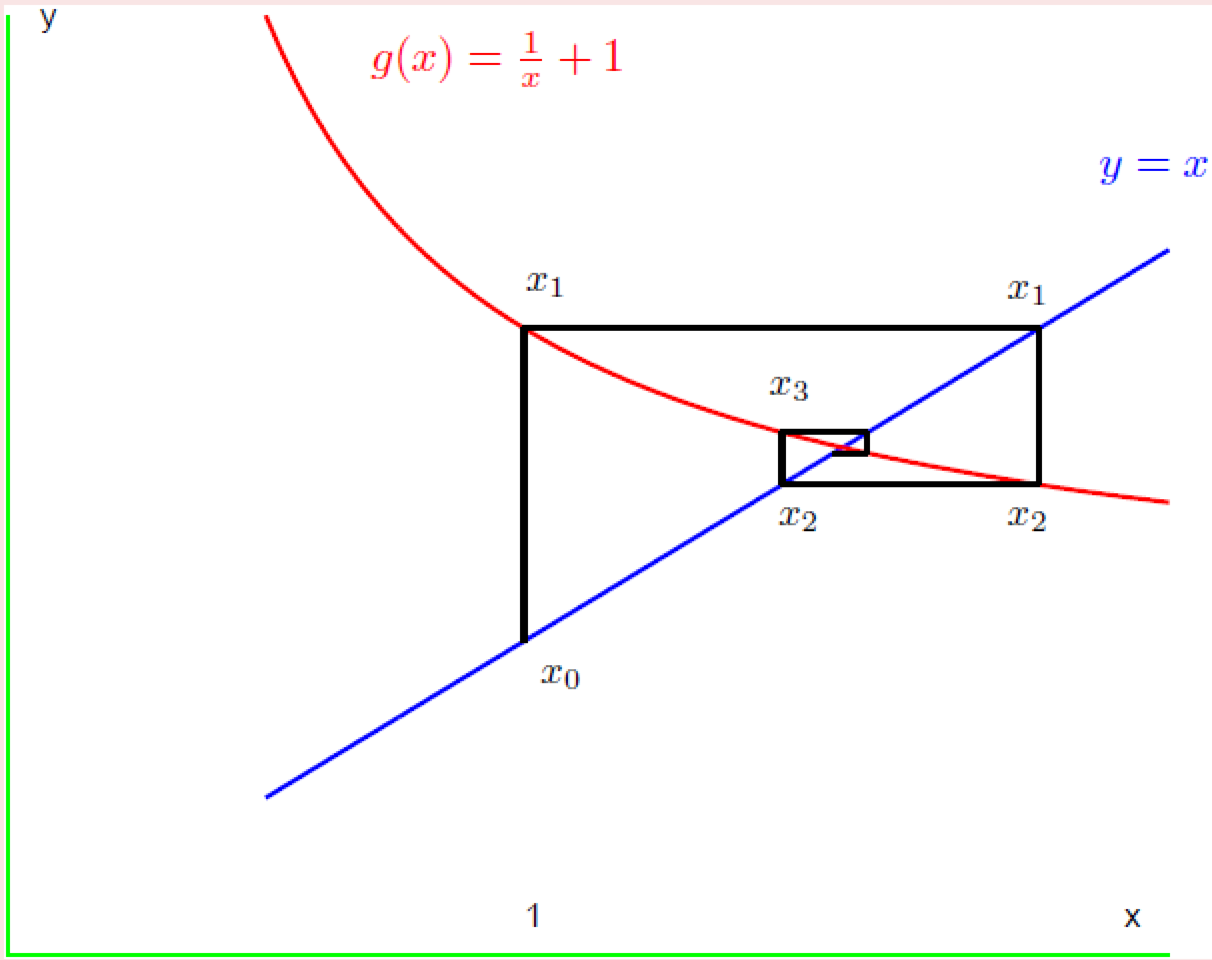


$g(x) = \frac{1}{x} + 1$
$y = x$
y
x
1
x_0
x_1
x_1
x_2
x_2
x_3

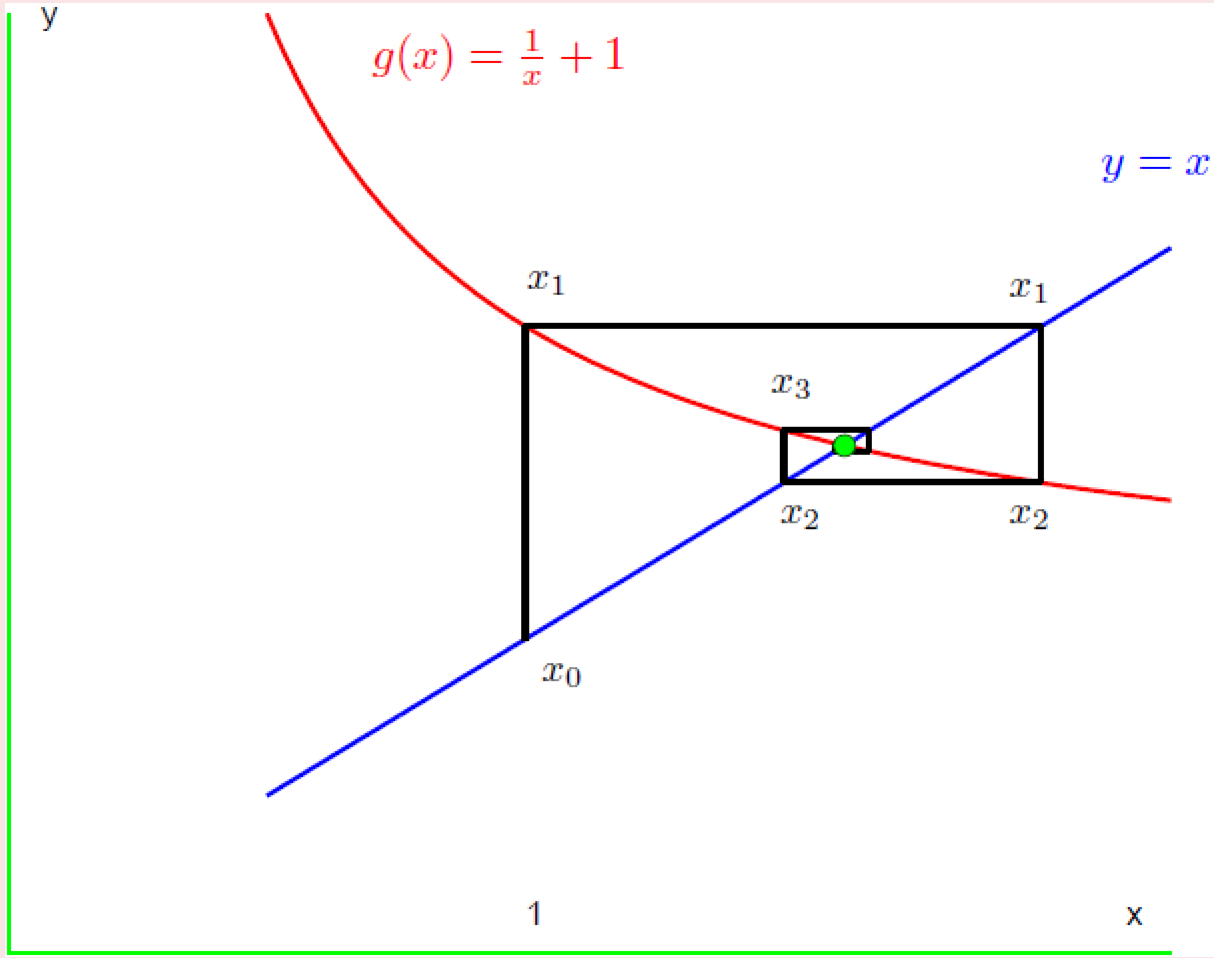


y
$g(x) = \frac{1}{x} + 1$
$y = x$
x_1
x_1
x_3
x_2
x_2
x_0
1
x


y
$g(x) = \frac{1}{x} + 1$
$y = x$
x_1
x_1
x_3
x_2
x_2
x_0
1
x



y
$g(x) = \frac{1}{x} + 1$
$y = x$
x_1
x_1
x_3
x_2
x_2
x_0
1
x



y
$g(x) = \frac{1}{x} + 1$
$y = x$
x_1
x_1
x_3
x_2
x_2
x_0
1
x

$x_{n+1} = g(x_n) = \frac{1}{x_n} + 1$ with $x_0 = 1$

Rate of Convergence

We require that $|g'(x)| \leq k < 1$. Since

$$g(x) = \frac{1}{x} + 1 \quad \text{and} \quad g'(x) = -\frac{1}{x^2} < 0 \quad \text{for} \quad x \neq 0$$

we find that

$$g'(x) = \frac{1}{2\sqrt{x+1}} > -1 \quad \text{for all} \quad x > 1$$

Note

$$g'(p) \approx -0.38197$$

Fixed Point: $g(x) = \frac{1}{x} + 1 \qquad p_0 = 1$

n	p_{n-1}	p_n	$\|p_n - p_{n-1}\|$	e_n/e_{n-1}
1	1.0000000	2.0000000	1.0000000	—
2	2.0000000	1.5000000	0.5000000	0.50000
3	1.5000000	1.6666667	0.1666667	0.33333
4	1.6666667	1.6000000	0.0666667	0.40000
5	1.6000000	1.6250000	0.0250000	0.37500
⋮	⋮	⋮	⋮	⋮
12	1.6180556	1.6180258	0.0000298	0.38197
13	1.6180258	1.6180371	0.0000114	0.38196
14	1.6180371	1.6180328	0.0000043	0.38197
15	1.6180328	1.6180344	0.0000017	0.38197