Solutions of Equations in One Variable

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Outline

1. Context: The Root-Finding Problem
2. Introducing the Bisection Method
3. Applying the Bisection Method
4. A Theoretical Result for the Bisection Method

The Root-Finding Problem

A Zero of function $f(x)$

- We now consider one of the most basic problems of numerical approximation, namely the root-finding problem.
- This process involves finding a root, or solution, of an equation of the form

$$f(x) = 0$$

  for a given function f.
- A root of this equation is also called a zero of the function f.

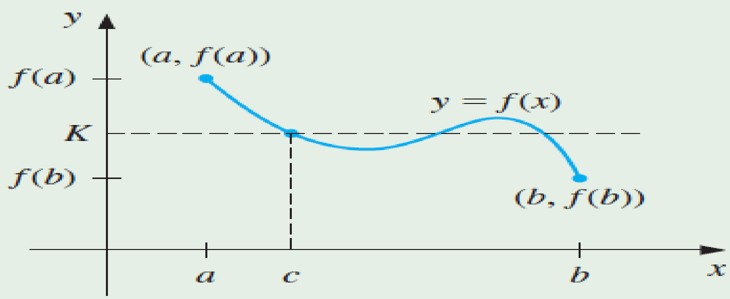
The Bisection Method

Overview

- We first consider the Bisection (Binary search) Method which is based on the Intermediate Value Theorem (IVT). ▸ IVT Illustration
- Suppose a continuous function f, defined on $[a, b]$ is given with $f(a)$ and $f(b)$ of opposite sign.
- By the IVT, there exists a point $p \in (a, b)$ for which $f(p) = 0$. In what follows, it will be assumed that the root in this interval is unique.

Intermediate Value Theorem

If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ for which $f(c) = K$.



(The diagram shows one of 3 possibilities for this function and interval.)

Bisection Technique

Main Assumptions

- Suppose f is a continuous function defined on the interval $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign.
- The Intermediate Value Theorem implies that a number p exists in (a, b) with $f(p) = 0$.
- Although the procedure will work when there is more than one root in the interval (a, b), we assume for simplicity that the root in this interval is unique.
- The method calls for a repeated halving (or bisecting) of subintervals of $[a, b]$ and, at each step, locating the half containing p.

Bisection Technique

Computational Steps

To begin, set $a_1 = a$ and $b_1 = b$, and let p_1 be the midpoint of $[a, b]$; that is,

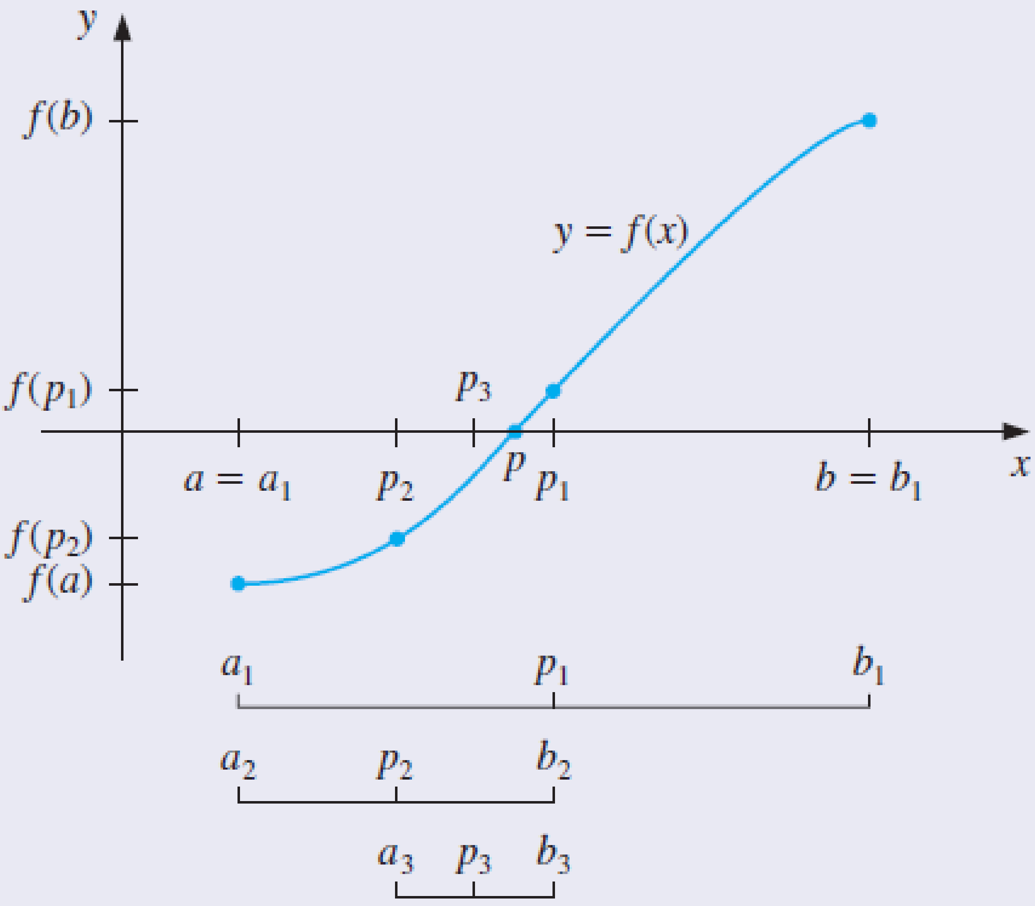
$$p_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}.$$

- If $f(p_1) = 0$, then $p = p_1$, and we are done.
- If $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.
  - If $f(p_1)$ and $f(a_1)$ have the same sign, $p \in (p_1, b_1)$. Set $a_2 = p_1$ and $b_2 = b_1$.
  - If $f(p_1)$ and $f(a_1)$ have opposite signs, $p \in (a_1, p_1)$. Set $a_2 = a_1$ and $b_2 = p_1$.

Then re-apply the process to the interval $[a_2, b_2]$, etc.

The Bisection Method to solve $f(x) = 0$

Interval Halving to Bracket the Root



The Bisection Method to solve $f(x) = 0$

Given the function f defined on [a,b] satisfying $f(a)f(b) < 0$.

1. $a_1 = a$, $b_1 = b$, $p_0 = a$;
2. $i = 1$;
3. $p_i = \frac{1}{2}(a_i + b_i)$;
4. If $|p_i - p_{i-1}| < \epsilon$ or $|f(p_i)| < \epsilon$ then 10;
5. If $f(p_i)f(a_i) > 0$, then 6;
   If $f(p_i)f(a_i) < 0$, then 8;
6. $a_{i+1} = p_i$, $b_{i+1} = b_i$;
7. $i = i + 1$; go to 3;
8. $a_{i+1} = a_i$; $b_{i+1} = p_i$;
9. $i = i + 1$; go to 3;
10. End of Procedure.

The Bisection Method

Comment on Stopping Criteria for the Algorithm

- Other stopping procedures can be applied in Step 4.
- For example, we can select a tolerance $\epsilon > 0$ and generate $p_1, \ldots, p_N$ until one of the following conditions is met:

$$|p_N - p_{N-1}| < \epsilon \quad (1)$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \epsilon, \quad p_N \neq 0, \quad \text{or} \quad (2)$$

$$|f(p_N)| < \epsilon \quad (3)$$

- Without additional knowledge about f or p, Inequality (2) is the best stopping criterion to apply because it comes closest to testing relative error.

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

Example: The Bisction Method

Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$ and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4}.

Relative Error Test

Note that, for this example, the iteration will be terminated when a bound for the relative error is less than 10^{-4}, implemented in the form:

$$\frac{|p_n - p_{n-1}|}{|p_n|} < 10^{-4}.$$

Bisection Method applied to $f(x) = x^3 + 4x^2 - 10$

Solution

- Because $f(1) = -5$ and $f(2) = 14$ the Intermediate Value Theorem ensures that this continuous function has a root in $[1, 2]$.
  IVT
- For the first iteration of the Bisection method we use the fact that at the midpoint of $[1, 2]$ we have $f(1.5) = 2.375 > 0$.
- This indicates that we should select the interval $[1, 1.5]$ for our second iteration.
- Then we find that $f(1.25) = -1.796875$ so our new interval becomes $[1.25, 1.5]$, whose midpoint is 1.375.
- Continuing in this manner gives the values shown in the following table.

Bisection Method applied to $f(x) = x^3 + 4x^2 - 10$

Iter	a_n	b_n	p_n	$f(a_n)$	$f(p_n)$	RelErr
1	1.000000	2.000000	1.500000	-5.000	2.375	0.33333
2	1.000000	1.500000	1.250000	-5.000	-1.797	0.20000
3	1.250000	1.500000	1.375000	-1.797	0.162	0.09091
4	1.250000	1.375000	1.312500	-1.797	-0.848	0.04762
5	1.312500	1.375000	1.343750	-0.848	-0.351	0.02326
6	1.343750	1.375000	1.359375	-0.351	-0.096	0.01149
7	1.359375	1.375000	1.367188	-0.096	0.032	0.00571
8	1.359375	1.367188	1.363281	-0.096	-0.032	0.00287
9	1.363281	1.367188	1.365234	-0.032	0.000	0.00143
10	1.363281	1.365234	1.364258	-0.032	-0.016	0.00072
11	1.364258	1.365234	1.364746	-0.016	-0.008	0.00036
12	1.364746	1.365234	1.364990	-0.008	-0.004	0.00018
13	1.364990	1.365234	1.365112	-0.004	-0.002	0.00009

Bisection Method applied to $f(x) = x^3 + 4x^2 - 10$

Solution (Cont'd)

- After 13 iterations, $p_{13} = 1.365112305$ approximates the root p with an error

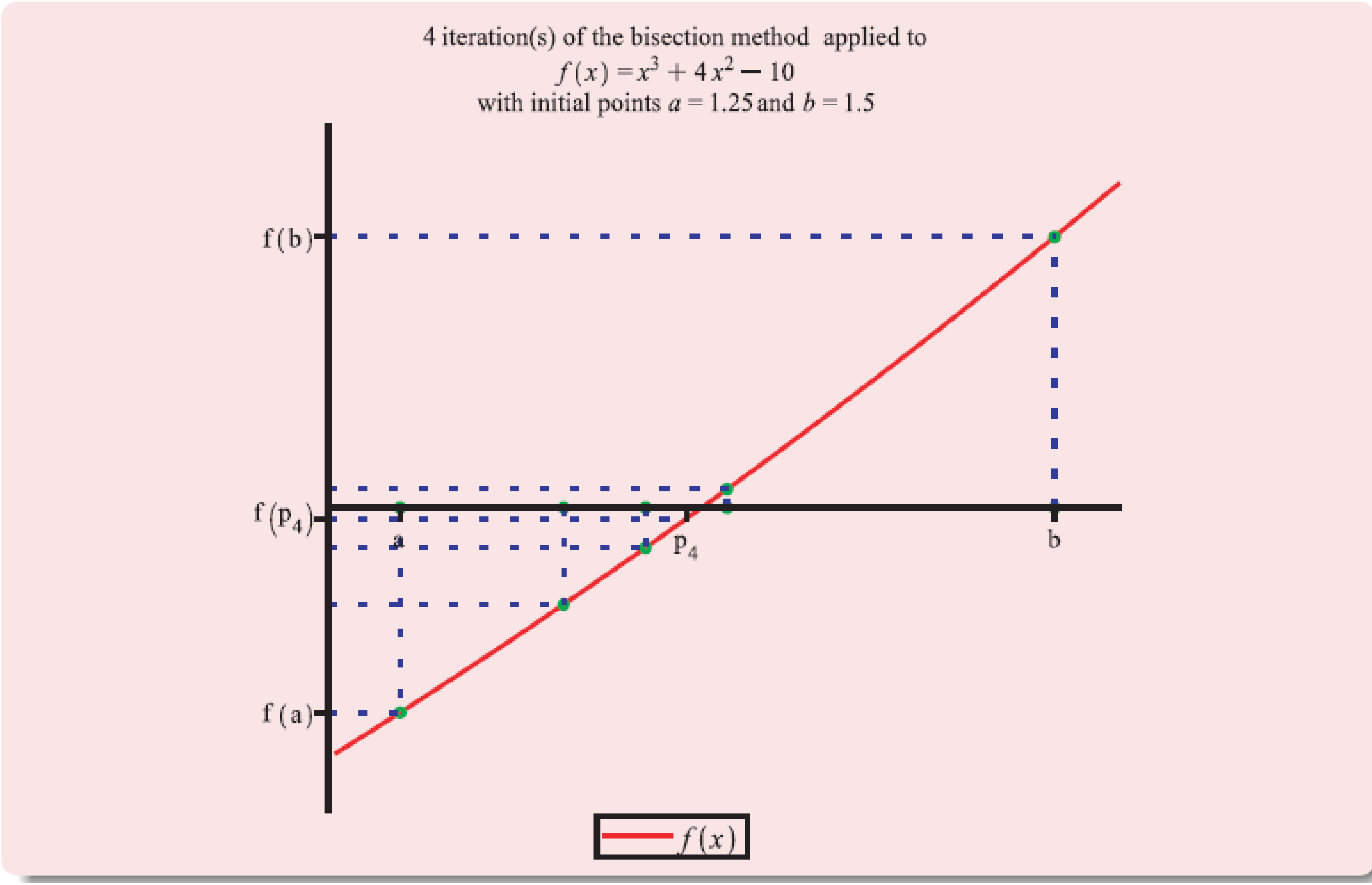
$$|p - p_{13}| < |b_{14} - a_{14}| = |1.3652344 - 1.3651123| = 0.0001221$$

- Since $|a_{14}| < |p|$, we have

$$\frac{|p - p_{13}|}{|p|} < \frac{|b_{14} - a_{14}|}{|a_{14}|} \leq 9.0 \times 10^{-5},$$

so the approximation is correct to at least within 10^{-4}.

- The correct value of p to nine decimal places is $p = 1.365230013$



4 iteration(s) of the bisection method applied to
$f(x) = x^3 + 4x^2 - 10$
with initial points $a = 1.25$ and $b = 1.5$
f(b)
f(p4)
f(a)
a
p4
b
f(x)

Theoretical Result for the Bisection Method

Theorem

Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \leq \frac{b-a}{2^n}, \quad \text{when} \quad n \geq 1.$$

Theoretical Result for the Bisection Method

Proof.

For each $n \geq 1$, we have

$$b_n - a_n = \frac{1}{2^{n-1}}(b - a) \quad \text{and} \quad p \in (a_n, b_n).$$

Since $p_n = \frac{1}{2}(a_n + b_n)$ for all $n \geq 1$, it follows that

$$|p_n - p| \leq \frac{1}{2}(b_n - a_n) = \frac{b - a}{2^n}.$$

□

Theoretical Result for the Bisection Method

Rate of Convergence

Because

$$|p_n - p| \leq (b-a)\frac{1}{2^n},$$

the sequence $\{p_n\}_{n=1}^{\infty}$ converges to p with rate of convergence $O\left(\frac{1}{2^n}\right)$; that is,

$$p_n = p + O\left(\frac{1}{2^n}\right).$$

Theoretical Result for the Bisection Method

Conservative Error Bound

- It is important to realize that the theorem gives only a bound for approximation error and that this bound might be quite conservative.
- For example, this bound applied to the earlier problem, namely where

$$f(x) = x^3 + 4x^2 - 10$$

ensures only that

$$|p - p_9| \le \frac{2-1}{2^9} \approx 2 \times 10^{-3},$$

but the actual error is much smaller:

$$|p - p_9| = |1.365230013 - 1.365234375| \approx 4.4 \times 10^{-6}.$$

Theoretical Result for the Bisection Method

Example: Using the Error Bound

Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$.

Solution

- We we will use logarithms to find an integer N that satisfies

$$|p_N - p| \leq 2^{-N}(b - a) = 2^{-N} < 10^{-3}.$$

- Logarithms to any base would suffice, but we will use base-10 logarithms because the tolerance is given as a power of 10.

Theoretical Result for the Bisection Method

Solution (Cont'd)

- Since $2^{-N} < 10^{-3}$ implies that $\log_{10} 2^{-N} < \log_{10} 10^{-3} = -3$, we have

$$-N \log_{10} 2 < -3 \quad \text{and} \quad N > \frac{3}{\log_{10} 2} \approx 9.96.$$

- Hence, ten iterations will ensure an approximation accurate to within 10^{-3}.
- The earlier numerical results show that the value of $p_9 = 1.365234375$ is accurate to within 10^{-4}.
- Again, it is important to keep in mind that the error analysis gives only a bound for the number of iterations.
- In many cases, this bound is much larger than the actual number required.

The Bisection Method

Final Remarks

- The Bisection Method has a number of significant drawbacks.
- Firstly it is very slow to converge in that N may become quite large before $p - p_N$ becomes sufficiently small.
- Also it is possible that a good intermediate approximation may be inadvertently discarded.
- It will always converge to a solution however and, for this reason, is often used to provide a good initial approximation for a more efficient procedure.