Mathematical Proofs
A Transition to Advanced Mathematics

Chapter 1
Sets

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Definition

A **set** is a collection of objects.

Objects in a set are called the **elements** of the set.

It is customary to use

> capital (upper case) letters (such as A, B, C, S, X, Y) to designate sets and
>
> lower case letters (for example, a, b, c, s, x, y) to represent elements of sets.

If a is an element of the set A, then we write $a \in A$; if a does not belong to A, then we write $a \notin A$.

Describing a Set

If a set consists of a small number of elements, then this set can be described by explicitly listing its elements between braces where the elements are separated by commas.

$S = \{1, 2, 3\}$

$X = \{1, 3, 5, \ldots, 49\}$ is the set of all positive odd integers less than 50

$Y = \{2, 4, 6, \ldots\}$ is the set of all positive even integers.

Definition

A set that contains no elements is called the **empty set**, denoted by $\emptyset$.

Describing a Set

Example 1

The set $S = \{1, 2, \{1, 2\}, \emptyset\}$ consists of four elements, two of which are sets, namely, $\{1, 2\}$ and $\emptyset$. If we write $C = \{1, 2\}$, then we can also write $S = \{1, 2, C, \emptyset\}$. ♦

Example 2

The set $T = \{0, \{1, 2, 3\}, 4, 5\}$ also has four elements, namely, the three integers $0, 4$ and 5 and the set $\{1, 2, 3\}$. Even though $2 \in \{1, 2, 3\}$, the number 2 is not an element of T; that is, $2 \notin T$. ♦

Describing a Set

If a set S consists of those elements satisfying some condition or possessing some specified property, then we can define S as

$$S = \{x : \; p(x)\} \text{ or } S = \{x \mid p(x)\}.$$

$S = \{x : \; (x-1)(x+2)(x+3) = 0\} = \{1, -2, -3\}$

$T = \{x : \; |x| = 2\}$

Example 3

Let $A = \{3, 4, 5, \ldots, 20\}$. If B denotes the set consisting of those elements of A that are less than 8, then we can write

$$B = \{x \in A : \; x < 8\} = \{3, 4, 5, 6, 7\}.$$

♦

Describing a Set

Definition

A real number is **rational** if it can be expressed in the form $\frac{m}{n}$, where $m, n \in \mathbf{Z}$ and $n \neq 0$.

$\frac{2}{3}$, $\frac{-5}{11}$ and $17 = \frac{17}{1}$ are rational numbers

Definition

A real number that is not rational is called **irrational.**

$\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{2}$, π and e are known to be irrational.

Every real number is either rational or irrational.

Definition

A **complex number** is a number of the form $a + bi$, where $a, b \in \mathbf{R}$ and $i = \sqrt{-1}$.

$2 + i$, 2 and i are complex numbers.

Describing a Set

Symbols for some sets

symbol	for the set of
N	natural numbers (positive integers)
Z	integers
Q	rational numbers
I	irrational numbers
R	real numbers
C	complex numbers

Describing a Set

Definition

For a set S, the **cardinal number** or **cardinality** of S is the number of elements in S, denoted by $|S|$.

If $A = \{1, 2\}$ and $B = \{1, 2, \{1, 2\}, \emptyset\}$, then $|A| = 2$ and $|B| = 4$. Also, $|\emptyset| = 0$.

Definition

A set S is **finite** if $|S| = n$ for some nonnegative integer n. A set S is **infinite** if it is not finite.

Example 4

Let $D = \{n \in \mathbf{N} : \ n \leq 9\}$, $E = \{x \in \mathbf{Q} : \ x \leq 9\}$, $H = \{x \in \mathbf{R} : \ x^2 - 2 = 0\}$ and $J = \{x \in \mathbf{Q} : \ x^2 - 2 = 0\}$.

(a) Describe the set D by listing its elements.

(b) Give an example of three elements that belong to E but do not belong to D.

(c) Describe the set H by listing its elements.

(d) Describe the set J in another manner.

(e) Determine the cardinality of each set D, H and J.

Example 4 (continued)

Solution.

(a) $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(b) $\frac{7}{5}$, 0, -3.

(c) $H = \{\sqrt{2}, -\sqrt{2}\}$.

(d) $J = \emptyset$.

(e) $|D| = 9$, $|H| = 2$ and $|J| = 0$. ♦

Example 5

In which of the following sets is the integer -2 an element?

$S_1 = \{-1, -2, \{-1\}, \{-2\}, \{-1, -2\}\}$, $S_2 = \{x \in \mathbf{N} : -x \in \mathbf{N}\}$,

$S_3 = \{x \in \mathbf{Z} : x^2 = 2^x\}$, $S_4 = \{x \in \mathbf{Z} : |x| = -x\}$,

$S_5 = \{\{-1, -2\}, \{-2, -3\}, \{-1, -3\}\}$.

Solution. The integer -2 is an element of the sets S_1 and S_4. For S_4, $|-2| = 2 = -(-2)$. The set $S_2 = \emptyset$. Since $(-2)^2 = 4$ and $2^{-2} = 1/4$, it follows that $-2 \notin S_3$. Because each element of S_5 is a set, it contains no integers. ♦

Subsets

Definition

A set A is called a **subset** of a set B if every element of A also belongs to B.

If A is a subset of B, then we write $A \subseteq B$.

If $X = \{1, 3, 6\}$ and $Y = \{1, 2, 3, 5, 6\}$, then $X \subseteq Y$. .

$\mathbf{N} \subseteq \mathbf{Z}$, $\mathbf{Q} \subseteq \mathbf{R}$ and $\mathbf{R} \subseteq \mathbf{C}$.

Every set is a subset of itself.

$\emptyset \subseteq A$ for every set A.

Example 6

Find two sets A and B such that A is both an element of and a subset of B.

Solution. Suppose that we seek two sets A and B such that $A \in B$ and $A \subseteq B$. Let's start with a simple example for A, say $A = \{1\}$. Since we want $A \in B$, the set B must contain the set $\{1\}$ as one of its elements. On the other hand, we also require that $A \subseteq B$, so every element of A must belong to B. Since 1 is the only element of A, it follows that B must also contain the number 1. A possible choice for B is then $B = \{1, \{1\}\}$, although $B = \{1, 2, \{1\}\}$ would also satisfy the conditions. ♦

Example 7

Let $S = \{1, \{2\}, \{1, 2\}\}$.

(a) Determine which of the following are elements of S:
1, $\{1\}$, 2, $\{2\}$, $\{1, 2\}$, $\{\{1, 2\}\}$.

(b) Determine which of the following are subsets of S:
$\{1\}$, $\{2\}$, $\{1, 2\}$, $\{\{1\}, 2\}$, $\{1, \{2\}\}$, $\{\{1\}, \{2\}\}$, $\{\{1, 2\}\}$.

Solution.

(a) The following are elements of S: 1, $\{2\}$, $\{1, 2\}$.

(b) The following are subsets of S: $\{1\}$, $\{1, \{2\}\}$, $\{\{1, 2\}\}$. ♦

Subsets

Definition

In a typical discussion of sets, we are ordinarily concerned with subsets of some specified set U, called the **universal set.**

For $a, b \in \mathbf{R}$ and $a < b$, the **open interval** (a, b) is the set

$$(a, b) = \{x \in \mathbf{R} : \ a < x < b\}.$$

Therefore, all of the real numbers $\frac{5}{2}, \sqrt{5}, e, 3, \pi$, 4.99 belong to $(2, 5)$, but none of the real numbers $\sqrt{2}, 1.99, 2, 5$ belong to $(2, 5)$.

Subsets

Definition

For $a, b \in \mathbf{R}$ and $a \leq b$, the **closed interval** $[a, b]$ is the set

$$[a, b] = \{x \in \mathbf{R} : \; a \leq x \leq b\}.$$

While $2, 5 \notin (2, 5)$, we do have $2, 5 \in [2, 5]$. The "interval" $[a, a]$ is therefore $\{a\}$. Thus, for $a < b$, we have $(a, b) \subseteq [a, b]$.

Definition

For $a, b \in \mathbf{R}$ and $a < b$, the **half-open** or **half-closed intervals** $[a, b)$ and $(a, b]$ are defined as expected:

$$[a, b) = \{x \in \mathbf{R} : \ a \leq x < b\} \text{ and } (a, b] = \{x \in \mathbf{R} : \ a < x \leq b\}.$$

For $a \in \mathbf{R}$, the infinite intervals $(-\infty, a), (-\infty, a], (a, \infty)$ and $[a, \infty)$ are defined as

$$\begin{array}{ll} (-\infty, a) = \{x \in \mathbf{R} : \ x < a\}, & (-\infty, a] = \{x \in \mathbf{R} : \ x \leq a\}, \\ (a, \infty) = \{x \in \mathbf{R} : \ x > a\}, & [a, \infty) = \{x \in \mathbf{R} : \ x \geq a\}. \end{array}$$

The interval $(-\infty, \infty)$ is the set $\mathbf{R}$.

Note that the infinity symbols ∞ and $-\infty$ are not real numbers; they are only used to help describe certain intervals. Therefore, $[1, \infty]$, for example, has no meaning.

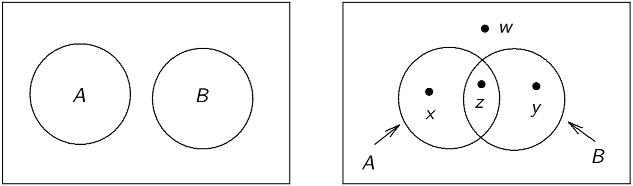
Subsets

Definition

Two sets A and B are **equal**, indicated by writing $A = B$, if they have exactly the same elements.

Subsets

It is often convenient to represent sets by diagrams called **Venn diagrams.** The figure below shows Venn diagrams of two sets A and B.



The diagram on the left represents two sets A and B that have no elements in common; while the diagram on the right is more general. The element x belongs to A but not to B, the element y belongs to B but not to A, the element z belongs to both A and B, while w belongs to neither A nor B.

Definition

A set A is a **proper subset** of a set B if $A \subseteq B$ but $A \neq B$.

If A is a proper subset of B, then we write $A \subset B$ or $A \subsetneq B$.

For example, if $S = \{4, 5, 7\}$ and $T = \{3, 4, 5, 6, 7\}$, then $S \subset T$.

The set consisting of all subsets of a given set A is called the **power set** of A and is denoted by $\mathcal{P}(A)$.

Example 8

For each set A below, determine $\mathcal{P}(A)$. In each case, determine $|A|$ and $|\mathcal{P}(A)|$.

$$\text{(a) } A = \emptyset, \quad \text{(b) } A = \{a, b\}, \quad \text{(c) } A = \{1, 2, 3\}.$$

Solution.

(a) $\mathcal{P}(A) = \{\emptyset\}$. In this case, $|A| = 0$ and $|\mathcal{P}(A)| = 1$.

(b) $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. In this case, $|A| = 2$ and $|\mathcal{P}(A)| = 4$.

(c) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. In this case, $|A| = 3$ and $|\mathcal{P}(A)| = 8$. ♦

If A is any finite set, with n elements say, then $\mathcal{P}(A)$ has 2^n elements.

Subsets

Example 9

If $C = \{\emptyset, \{\emptyset\}\}$, then

$$\mathcal{P}(C) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}.$$

It is important to note that no two of the sets $\emptyset, \{\emptyset\}$ and $\{\{\emptyset\}\}$ are equal. (An empty box and a box containing an empty box are not the same.) For the set C above, it is therefore correct to write

$$\emptyset \subseteq C,\ \emptyset \subset C,\ \emptyset \in C,\ \{\emptyset\} \subseteq C,\ \{\emptyset\} \subset C,\ \{\emptyset\} \in C,$$

as well as

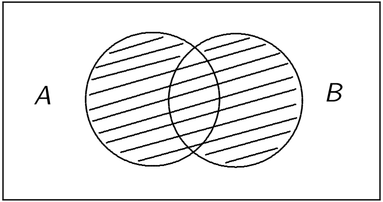
$$\{\{\emptyset\}\} \subseteq C, \{\{\emptyset\}\} \notin C, \{\{\emptyset\}\} \in \mathcal{P}(C).$$ ♦

Definition

The **union** of two sets A and B, denoted by $A \cup B$, is the set of all elements belonging to A or B, that is,

$$A \cup B = \{x : \ x \in A \text{ or } x \in B\}.$$

A Venn diagram for $A \cup B$ is shown in the figure below. The shaded region indicates the set $A \cup B$.



Example 10

For the sets

$$A_1 = \{2, 5, 7, 8\},\ A_2 = \{1, 3, 5\} \text{ and } A_3 = \{2, 4, 6, 8\},$$

we have

$$\begin{aligned} A_1 \cup A_2 &= \{1, 2, 3, 5, 7, 8\}, \\ A_1 \cup A_3 &= \{2, 4, 5, 6, 7, 8\}, \\ A_2 \cup A_3 &= \{1, 2, 3, 4, 5, 6, 8\}. \end{aligned}$$

Also, $\mathbf{N} \cup \mathbf{Z} = \mathbf{Z}$ and $\mathbf{Q} \cup \mathbf{I} = \mathbf{R}$. ♦

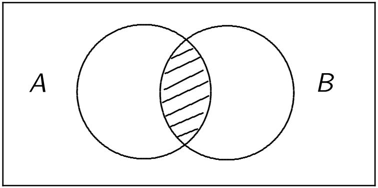
Set Operations

Definition

The **intersection** of two sets A and B is the set of all elements belonging to both A and B. The intersection of A and B is denoted by $A \cap B$. In symbols,

$$A \cap B = \{x : \ x \in A \text{ and } x \in B\}.$$

A Venn diagram for $A \cap B$ is shown in the figure below, again indicated by the shaded region.



Set Operations

Example 11

For the sets

$$A_1 = \{2, 5, 7, 8\},\ A_2 = \{1, 3, 5\} \text{ and } A_3 = \{2, 4, 6, 8\},$$

we have

$$A_1 \cap A_2 = \{5\},\ A_1 \cap A_3 = \{2, 8\} \text{ and } A_2 \cap A_3 = \emptyset.$$

Also, $\mathbf{N} \cap \mathbf{Z} = \mathbf{N}$ and $\mathbf{Q} \cap \mathbf{R} = \mathbf{Q}$. ♦

Definition

If two sets A and B have no elements in common, then $A \cap B = \emptyset$ and A and B are said to be **disjoint**.

Example 12

For the sets

$$A_1 = \{2, 5, 7, 8\},\ A_2 = \{1, 3, 5\} \text{ and } A_3 = \{2, 4, 6, 8\},$$

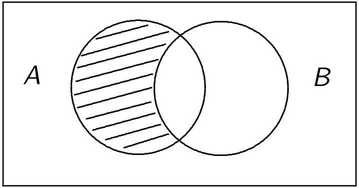
A_2 and A_3 are disjoint; however, A_1 and A_3 are not disjoint since 2 and 8 belong to both sets. ♦

Definition

The **difference** $A - B$ of two sets A and B (also written as $A \setminus B$ by some mathematicians) is defined as

$$A - B = \{x : \ x \in A \text{ and } x \notin B\}.$$

A Venn diagram for $A - B$ is shown in the figure below.



Example 13

For the sets $A_1 = \{2, 5, 7, 8\}$ and $A_2 = \{1, 3, 5\}$, we have

$$A_1 - A_2 = \{2, 7, 8\} \text{ and } A_2 - A_1 = \{1, 3\}.$$

Furthermore, $\mathbf{R} - \mathbf{Q} = \mathbf{I}$. ♦

Example 14

Let $A = \{x \in \mathbf{R} : \ |x| \leq 3\}$, $B = \{x \in \mathbf{R} : \ |x| > 2\}$ and $C = \{x \in \mathbf{R} : \ |x-1| \leq 4\}$.

(a) Express A, B and C using interval notation.

(b) Determine $A \cap B$, $A - B$, $B \cap C$, $B \cup C$, $B - C$ and $C - B$.

Solution.

(a) $A = [-3, 3], B = (-\infty, -2) \cup (2, \infty)$ and $C = [-3, 5]$.

(b) $A \cap B = [-3, -2) \cup (2, 3]$, $A - B = [-2, 2]$,
$B \cap C = [-3, -2) \cup (2, 5]$, $B \cup C = (-\infty, \infty)$,
$B - C = (-\infty, -3) \cup (5, \infty)$ and $C - B = [-2, 2]$. ♦

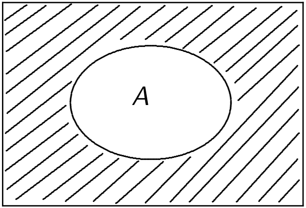
Set Operations

Definition

Suppose that we are considering a certain universal set U, that is, all sets being discussed are subsets of U. For a set A, its **complement** is

$$\overline{A} = U - A = \{x : \ x \in U \text{ and } x \notin A\}.$$

If $U = \mathbf{Z}$, then $\overline{\mathbf{N}} = \{0, -1, -2, \ldots\}$; while if $U = \mathbf{R}$, then $\overline{\mathbf{Q}} = \mathbf{I}$. A Venn diagram for $\overline{A}$ is shown in the figure below.



Example 15

Let $U = \{1, 2, \ldots, 10\}$ be the universal set,

$$A = \{2, 3, 5, 7\} \text{ and } B = \{2, 4, 6, 8, 10\}.$$

Determine each of the following:

$$\text{(a) } \overline{B}, \quad \text{(b) } A - B, \quad \text{(c) } A \cap \overline{B}, \quad \text{(d) } \overline{\overline{B}}.$$

Solution.

(a) $\overline{B} = \{1, 3, 5, 7, 9\}$.

(b) $A - B = \{3, 5, 7\}$.

(c) $A \cap \overline{B} = \{3, 5, 7\} = A - B$.

(d) $\overline{\overline{B}} = B = \{2, 4, 6, 8, 10\}$. ♦

Definition

The **union** of the $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ is denoted by

$$A_1 \cup A_2 \cup \cdots \cup A_n \text{ or } \bigcup_{i=1}^{n} A_i$$

and is defined as

$$\bigcup_{i=1}^{n} A_i = \{x : \ x \in A_i \text{ for some } i, 1 \leq i \leq n\}.$$

Thus, for an element a to belong to $\bigcup_{i=1}^{n} A_i$, it is necessary that a belongs to at least one of the sets $A_1, A_2, \ldots, A_n$.

Example 16

Let $B_1 = \{1,2\}$, $B_2 = \{2,3\}$, ..., $B_{10} = \{10,11\}$; that is,

$$B_i = \{i, i+1\} \text{ for } i = 1, 2, \ldots, 10.$$

Determine each of the following:

(a) $\displaystyle\bigcup_{i=1}^{5} B_i$ (b) $\displaystyle\bigcup_{i=1}^{10} B_i$ (c) $\displaystyle\bigcup_{i=3}^{7} B_i$ (d) $\displaystyle\bigcup_{i=j}^{k} B_i$, where $1 \le j \le k \le 10$

Solution.

(a) $\displaystyle\bigcup_{i=1}^{5} B_i = \{1, 2, \ldots, 6\}$ (b) $\displaystyle\bigcup_{i=1}^{10} B_i = \{1, 2, \ldots, 11\}$

(c) $\displaystyle\bigcup_{i=3}^{7} B_i = \{3, 4, \ldots, 8\}$ (d) $\displaystyle\bigcup_{i=j}^{k} B_i = \{j, j+1, \ldots, k+1\}$. ♦

Definition

The **intersection** of the $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ is expressed as

$$A_1 \cap A_2 \cap \cdots \cap A_n \text{ or } \bigcap_{i=1}^{n} A_i$$

and is defined by

$$\bigcap_{i=1}^{n} A_i = \{x : \ x \in A_i \text{ for every } i, 1 \leq i \leq n\}.$$

Example 17

Let $B_i = \{i, i+1\}$ for $i = 1, 2, \ldots, 10$. Determine the following:

(a) $\displaystyle\bigcap_{i=1}^{10} B_i$ (b) $B_i \cap B_{i+1}$ (c) $\displaystyle\bigcap_{i=j}^{j+1} B_i$, where $1 \le j < 10$

(d) $\displaystyle\bigcap_{i=j}^{k} B_i$ where $1 \le j < k \le 10$

Solution. (a) $\displaystyle\bigcap_{i=1}^{10} B_i = \emptyset$ (b) $B_i \cap B_{i+1} = \{i+1\}$

(c) $\displaystyle\bigcap_{i=j}^{j+1} B_i = \{j+1\}$ (d) $\displaystyle\bigcap_{i=j}^{k} B_i = \{j+1\}$ if $k = j+1$;

while $\displaystyle\bigcap_{i=j}^{k} B_i = \emptyset$ if $k > j+1$ ♦

Indexed Collections of Sets

Definition

For an index set I, suppose that there is a set S_α for each $\alpha \in I$. We write $\{S_\alpha\}_{\alpha \in I}$ to describe the collection of all sets S_α where $\alpha \in I$. Such a collection is called an **indexed collection of sets**. We define the **union** of the sets in $\{S_\alpha\}_{\alpha \in I}$ by

$$\bigcup_{\alpha \in I} S_\alpha = \{x : \ x \in S_\alpha \text{ for some } \alpha \in I\},$$

and the **intersection** of these sets by

$$\bigcap_{\alpha \in I} S_\alpha = \{x : \ x \in S_\alpha \text{ for all } \alpha \in I\}.$$

Hence, an element a belongs to $\bigcup_{\alpha \in I} S_\alpha$ if a belongs to at least one of the sets in the collection $\{S_\alpha\}_{\alpha \in I}$; while a belongs to $\bigcap_{\alpha \in I} S_\alpha$ if a belongs to every set in the collection $\{S_\alpha\}_{\alpha \in I}$.

Example 18

For $n \in \mathbf{N}$, define $S_n = \{n, 2n\}$. For example,

$$S_1 = \{1, 2\}, S_2 = \{2, 4\} \text{ and } S_4 = \{4, 8\}.$$

Then $S_1 \cup S_2 \cup S_4 = \{1, 2, 4, 8\}$. We can also describe this set by means of an index set. If we let $I = \{1, 2, 4\}$, then

$$\bigcup_{\alpha \in I} S_\alpha = S_1 \cup S_2 \cup S_4. \qquad \blacklozenge$$

Example 19

For each $n \in \mathbf{N}$, define A_n to be the closed interval $[-\frac{1}{n}, \frac{1}{n}]$ of real numbers; that is,

$$A_n = \left\{ x \in \mathbf{R} : -\frac{1}{n} \le x \le \frac{1}{n} \right\}.$$

So

$$A_1 = [-1, 1], A_2 = [-\tfrac{1}{2}, \tfrac{1}{2}], A_3 = [-\tfrac{1}{3}, \tfrac{1}{3}]$$

and so on. We have now defined the sets $A_1, A_2, A_3, \ldots$.

Example 19 (continued)

The union of these sets can be written as $A_1 \cup A_2 \cup A_3 \cup \cdots$ or $\bigcup_{i=1}^{\infty} A_i$. Using $\mathbf{N}$ as an index set, we can also write this union as $\bigcup_{n\in\mathbf{N}} A_n$. Since $A_n \subseteq A_1 = [-1, 1]$ for every $n \in \mathbf{N}$, it follows that

$$\bigcup_{n\in\mathbf{N}} A_n = [-1, 1].$$

Certainly, $0 \in A_n$ for every $n \in \mathbf{N}$; in fact,

$$\bigcap_{n\in\mathbf{N}} A_n = \{0\}. \qquad \blacklozenge$$

Definition

A collection $\mathcal{S}$ of subsets of a set A is called **pairwise disjoint** if every two distinct subsets that belong to $\mathcal{S}$ are disjoint.

Let $A = \{1, 2, \ldots, 7\}$, $B = \{1, 6\}$, $C = \{2, 5\}$, $D = \{4, 7\}$ and

$$S = \{B, C, D\}.$$

Then S is a pairwise disjoint collection of subsets of A.

Let $A' = \{1, 2, 3\}$, $B' = \{1, 2\}$, $C' = \{1, 3\}$, $D' = \{2, 3\}$ and

$$S' = \{B', C', D'\}.$$

Although S' is a collection of subsets of A' and $B' \cap C' \cap D' = \emptyset$, the set S' is *not* a pairwise disjoint collection of sets since $B' \cap C' \neq \emptyset$, for example.

Partitions of Sets

Definition

A **partition** of A is a collection $\mathcal{S}$ of subsets of A satisfying the three properties:

(1) $X \neq \emptyset$ for every set $X \in \mathcal{S}$;

(2) for every two sets $X, Y \in \mathcal{S}$, either $X = Y$ or $X \cap Y = \emptyset$;

(3) $\bigcup_{X \in \mathcal{S}} X = A$.

Example 20

Consider the following collections of subsets of the set

$$A = \{1, 2, 3, 4, 5, 6\}:$$

$$S_1 = \{\{1,3,6\},\{2,4\},\{5\}\}; \quad S_2 = \{\{1,2,3\},\{4\},\emptyset,\{5,6\}\};$$
$$S_3 = \{\{1,2\},\{3,4,5\},\{5,6\}\}; \quad S_4 = \{\{1,4\},\{3,5\},\{2\}\}.$$

Determine which of these sets are partitions of A.

Solution. The set S_1 is a partition of A. The set S_2 is not a partition of A since $\emptyset$ is one of the elements of S_2. The set S_3 is not a partition of A either since the element 5 belongs to two distinct subsets in S_3, namely, $\{3,4,5\}$ and $\{5,6\}$. Finally, S_4 is also not a partition of A because the element 6 belongs to no subset in S_4. ♦

Example 21

Let $A = \{1, 2, \ldots, 12\}$.

(a) Give an example of a partition S of A such that $|S| = 5$.

(b) Give an example of a subset T of the partition S in (a) such that $|T| = 3$.

(c) List all those elements B in the partition S in (a) such that $|B| = 2$.

Example 21 (continued)

Solution.

(a) We are seeking a partition S of A consisting of five subsets. One such example is

$$S = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}\}.$$

(b) We are seeking a subset T of S (given in (a)) consisting of three elements. One such example is

$$T = \{\{1, 2\}, \{3, 4\}, \{7, 8, 9\}\}.$$

(c) We have been asked to list all those elements of S (given in (a)) consisting of two elements of A. These elements are $\{1,\ 2\}$, $\{3,\ 4\}$, $\{5,\ 6\}$. ♦

Cartesian Products of Sets

Definition

The **Cartesian product** $A \times B$ of two sets A and B is the set consisting of all ordered pairs whose first coordinate belongs to A and whose second coordinate belongs to B; that is,

$$A \times B = \{(a, b) : \ a \in A \text{ and } b \in B\}.$$

If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

For all finite sets A and B, $|A \times B| = |A| \cdot |B|$.

Example 22

If $A = \{x, y\}$ and $B = \{1, 2, 3\}$, then

$$A \times B = \{(x,1),(x,2),(x,3),(y,1),(y,2),(y,3)\},$$

while

$$B \times A = \{(1,x),(1,y),(2,x),(2,y),(3,x),(3,y)\}.$$

Since, for example, $(x,1) \in A \times B$ and $(x,1) \notin B \times A$, these two sets do not contain the same elements; so $A \times B \neq B \times A$. Also,

$$A \times A = \{(x,x),(x,y),(y,x),(y,y)\}$$

and

$$\begin{aligned} B \times B = \{&(1,1),(1,2),(1,3),\\ &(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}. \end{aligned}$$ ♦